Q-based Equilibria

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Abstract

In dynamic environments, Q-learning is an adaptative rule that provides an estimate (a Q-value) of the continuation value associated with each alternative. A naive policy consists in always choosing the alternative with highest Q-value. We consider a family of Q-based policy rules that may systematically favor some alternatives over others, for example rules that incorporate a leniency bias that favors cooperation. In the spirit of Compte and Postlewaite [2018], we look for equilibrium biases (or $Qb - equilibria$) within this family of Q-based rules. We examine classic games under various monitoring technologies.

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Contents

1. Introduction

Long-term relationships have been extensively studied in economics, aimed at understanding how repeated interactions can foster cooperation. Until recently, the methodology has been mostly constructive: given a stage game characterized by its payoff and information structures, the repeated game analysis often consisted in finding, within the huge set of feasible strategies of the repeated game, strategies that work well, i.e., *design* strategies that, when played, support a given level of cooperation, say, and also ensure that each player has incentives to play their part all along. This design perspective has been at the heart of the Folk Theorem results in the literature.[1]

Recent work on algorithmic collusion departs significantly from this design perspective. It starts from an *all-purpose strategy* that adapts admirably well to its environment, a reinforcement learning algorithm – Q-learning, and then it examines, across a variety of games, the consequence for mutual play when each player uses such an algorithm. Conditioning behavior on past outcomes is at the heart of cooperation, and since Q-learning does precisely that, Q-learning is a good candidate strategy for enabling collusion or cooperation, as Calvano et al. [2020] and Banchio and Mantegazza [2022] confirm.

This paper takes a middle ground. It starts from the algorithmic perspective and it incorporates a strategic dimension in the $Q-$learning algorithm, taking the form of a (one-dimensional) *bias* favoring some alternatives over others. We call this $Qb-$*learning*, where b could stand for "biased" or "based". Assuming that each player sets its own bias b_i non-cooperatively, we look for an equilibrium bias profile b^* and call it a $Qb-$*equilibrium*.

Specifically, in its *memoryless* version, a Q-learning algorithm uses past experience to calculate a Q-value $Q_i^t(a_i)$ for each action that player i may play at stage t of the game.[2] Calling r_i^t the reward obtained at t, $Q-$values are updated according to:[3]

$$Q_i^{t+1}(a_i) = (1-\alpha)Q_i^t(a_i) + \alpha((1-\delta)r_i^t + \delta \max_{x_i} Q_i^t(x_i)) \tag{1}$$

when a_i is played, and

$$Q_i^{t+1}(a_i) = Q_i^t(a_i) \text{ otherwise.}$$

As time goes by, these Q-values are updated many times for all a_i because the agent is assumed

[1] See Fudenberg and Maskin [1986], Abreu, Pearce, and Stacchetti [1990], Fudenberg, Levine, and Maskin [1994], Sugaya [2022], or Mailath and Samuelson [2006] for an overview of the literature.

[2] A more sophisticated algorithm could compute the Q-values $Q_{i,h_i}^t(a_i)$, *conditional on the recent history* h_i *observed by* i. Althouth we restrict attention to memoryless algorithms, our definitions could apply to these more general versions of Q-learning.

[3] We use the normalization $(1-\delta)r + \delta \max Q$ in order to normalize Q-values to stage-game values, as is standard in the repeated game literature. Note that we assume no updating of $Q_i(a)$ when a is not played. This is called asynchronous Q-learning: players do not attempt to draw inferences about the payoffs they would have obtained had they played differently nor use these inferences to update Q values.

to experiment with positive probability in every period: in its $\varepsilon-$greedy version, experimentation occurs with probability ε in every period and any feasible action is then selected with same probability.

The Q-value $Q_i^t(a_i)$ can be interpreted as the *subjective* evaluation of choosing a_i at t, and it is often thought of a proxy for the continuation value associated with playing a_i at t. Given Q-values, the "naive" Q-learner then chooses (unless she experiments) an action a_i^t which maximizes the $Q-$value. Instead, we allow the agent to select an action that maximizes a *biased criterion*:

$$a_i^t \in \arg\max_a \; Q_i^t(a_i) + b_i G_i(a_i),$$

where b_i is a one dimensional bias and $G_i(a_i)$ is an exogenous distortion that may reflect the agent 's broad understanding of the structure of the game, or some natural ordering on strategies possibly based on broad characteristics of the payoff structure of the game, we shall come back to that in Section 5.[4]

Our motivation for introducing biased $Q-$learning is that in strategic environments, but not only (as we shall see), Q-values may provide a biased estimate of the benefits of choosing a particular action, so the variable b_i can actually be seen as an instrument, which, if appropriately set, may allow a player to *de-bias* Q-values, to some extent.

Given this option, we shall examine an equilibrium in biased learning algorithms, where each b_i is assumed to be set optimally (ex ante) to maximize player i's *objective* (rather than subjective) long-run payoff. Practically, we will approximate ex ante long-run payoffs by running simulations over *a fixed horizon*, starting from an initial condition which we will in most case choose to be *unfavorable* to cooperation. Note that we won't be modelling how agents learn to adjust their bias optimally. For that, we keep the traditional game-theoretic short-cut, which consists in assuming that some unmodeled evolutionary/learning process eventually drives each player to their optimal or near-optimal choice of bias, given the biases chosen by others.[5]

How can Q-values be biased? Consider a *single* agent facing an economic environment where payoffs are *state-contingent* and where the evolution of the state is governed by the actions played by the agent. This is typically what happens in strategic situations, where one's own play (and own past play) affects the behavior of others.

When the evolution of the state depends on the action played, $Q-$values do not necessarily give adequate guidance on which actions to play: the agent may be stuck in what we call a Q*-trap*, where Q-values suggest using actions that keep the dynamic of evolution in low-rewarding states, with occasional exploration not permitting exit from Q-traps. Higher experimentation levels ε combined with higher speed of adjustment α may permit exits, but these higher ε and α may be conducive

[4] In games with only two actions $a_i \in \{0, 1\}$, the shape of the distorsion plays no role, one can set $G_i(a_i) = a_i$.

[5] An alternate route would consist in defining a more sophisticated Q-learning process which would also evaluate policies b_i on a different time scale.

to lower welfare and faster return to the $Q-$trap. Section 2 provides a simple class of examples of this kind.

Strategic situations. In games such as the repeated prisoner's dilemma or more generally repeated games of price or quantity competition, a Q-trap may arise (though not always) when all players choose a non-cooperative action (defection, low price or high quantity): cooperation may be sustained for some (possibly long time) when initial conditions are favorable (as in Banchio and Mantegazza [2022]), but if initial conditions are not favorable, learning to recoordinate on cooperation may be hard, with non-cooperative phases becoming the preponderant ones.[6]

In such circumstances we obtain $Qb-$equilibria that are more cooperative than the outcome generated naive $Q-$learning. Note that our quest for "equilibria" is important here. There is always the possibility of sustaining joint cooperation with policy rules strongly biased in favor of cooperation. However, if one's opponent is strongly biased in that way (and, say, cooperates always independently of others' behavior), then a naive $Q-$learner will do better: through experimentation, she will eventually learn to take advantage of her highly biased opponent.

Also note that since the choice of bias is supposed to be strategic in our framework, there is no guarantee that $Qb-$equilibria leads to more cooperation. As a matter of fact, the opposite could be true, as biasing one's own policy towards *less* cooperation could be profitable: if this more severe attitude does not undermine too much the sustainability of cooperation, it could in principle lead to more gains.

Intuitively, Qb-learning helps for the following reason. Under naive $Q-$learning, a typical path of play consists of an alternation between *high*$-Q$ *and low*$-Q$ *phases.* Low $Q-$phases are traps where players mostly defects (unless they experiment). High-Q phases consists of (relatively frequent) alternations between *jointly cooperative phases* (which boost Q-values) and *disorganized phases* that mostly consist of CD's and DD's (which depresses Q values of both players). The role of these disorganized phases is to realign Q-value differences across players, to facilitate simultaneous re-coordination on cooperation. Sometimes these disorganized phases are too long and players end up in a low-Q phase, from which exit takes time and requires simultaneous experimentations.

Biased-Q learning distorts decisions in high-Q phases (and this distortion can be costly – both privately and socially), but it reduces the chance of falling in a low-Q phase (and shortens them if they arise). The latter effect can be strong enough to provide incentives for each player to bias their decision rule in favor of cooperation.

Monitoring technology. There is a long-tradition in economics of examining repeated games according to their monitoring technology, distinguishing between *perfect* (observable action profiles), *imperfect public* (observable public signals correlated with actions), or *imperfect private*

[6] We emphasize players starting from initially unfavorable conditions not only because defection may arise endogenously. One can also see that as a simple modelling device to capture economic environments subject to exogenous and persistent shocks, where players have to learn when they should restart cooperating.

(private signals correlated with actions – and typically independent conditional on the actions) monitoring. This distinction has been useful in structuring the study of these games as the monitoring technology shapes the strategies available and the easiness with which dynamic programming techniques can be used to solve them (Abreu et al. [1990]). Both perfect and public monitoring allow the use of public strategies where computing continuation values after any history can be done (by the analyst) perfectly.

With Q-learning, this distinction seems irrelevant. Q-values provides a *private* summary statistic of one's past payoffs, so Q-based strategies are not public – they use private information. Furthermore, it would seem that the nature of monitoring should not matter much: whether monitoring is perfect, imperfect, public or private, some averaging of past payoffs is going on, so this should not affect much behavior: it merely adds some randomness into $Q-$values.

What our analysis reveals however is that this randomness affects the evolution and co-evolution of $Q-$values across players in ways that affect the chance of falling and staying in a $Q-$trap. In particular, shocks on payoffs deteriorate the ability of players to sustain cooperation durably, but most of all, *private* shocks deteriorate the ability to exit from traps (when players use naive Q-learning). Exit from traps requires coordinated moves, and these are more difficult to generate when shocks are private.

Given this difficulty, the option to bias $Q-$learning has great potential for helping players sustain cooperation, and this is what Section 4 confirms. We obtain $Qb-$equilibria that sustain an equivalently high level of cooperation whether payoffs are deterministic or not, and whether shocks are correlated or private.

Finally, in Section 5, we apply our analysis to the oligopoly set up introduced by Calvano et al. [2020] and obtain collusive (memoryless) Qb-equilibria where players mostly play higher than Nash prices, with occasional disorganized phase where many different price profiles are played.

Related literature.

Our paper is inspired by a rapidly growing body of work that examines how effective algorithmic strategies based on reinforcement learning can be at supporting collusion or cooperation (Calvano et al. [2020, 2021], Klein [2021], Banchio and Mantegazza [2022], Asker, Fershtman, and Pakes [2022], Banchio and Skrzypacz [2022] and Hansen, Misra, and Pai [2021].[7] Except for the later paper, which examines tacit collusion induced by the use of UCB-learning (Auer [2002]),[8] the others consider, like us, Q-learning algorithms (Watkins and Dayan [1992]).

Our main departure from these studies is that (i) we allow for *biased policy rules*; (ii) we examine the degree to which players manage to *re-coordinate* on cooperation or collusion after prolonged

[7] The more general underlying economic motivation is whether the use of algorithms fosters supra-competitive prices, a question that has also been examined by Brown and MacKay [2021] without focusing specifically on reinforcement learning algorithms.

[8] UCB stands for uniform confidence bound. It builds an independent tract record for each alternative (unlike $Q-$learning, which computes $Q(a)$ using $\max Q$ hence the values $Q(a')$ computed for actions different from a).

defective phases – or when starting from *initially unfavorable* conditions; (iii) we allow for payoffs being subject to common (as in Calvano et al. [2021]) or *private shocks.*

There are other differences. Calvano et al. [2020, 2021] assume vanishing experimentation and memory-based $Q-$values. Memory is important in this work because when experimentation vanishes, memoryless Q-values are ineffective in supporting collusion. Supporting collusion then requires some conditioning on others' prices.[9] In contrast, Banchio and Mantegazza [2022] consider experimentation that does not vanish and, under some conditions, obtain cooperation even with memoryless Q-values. One difference with our work is that BM characterizes the *continuous limit* of the $Q-$value updates when each player independently updates at randomly distributed times (according to a Poisson distribution), which we do not. Under this continuous limit, BM find that exit from bad initial conditions are impossible, so if, for whatever reason, current payoff conditions call for prolonged defection, players may be unable to eventually recoordinate on cooperation.[10]

Our work is also related to Compte and Postlewaite [2015 and 2018, Ch. 14] who examine repeated prisoner's dilemma with stochastic signals or payoffs, where players are constrained to choosing a strategy within a *limited family of stochastic automata.* The automata perform reasonably well for a class of bandit problems where the risky arm has state-dependent benefits and where the state has some persistence: once in a while, it pays to check whether the risky arm turns out to be profitable again. The *frequency* with which one checks is then conceived as *a strategic variable.* In the context of the prisoners' dilemma, cooperation is the risky arm, and higher frequency of checking whether cooperation is profitable means more leniency. Like here, lenient strategies facilitate recoordination on cooperation, and CP find equilibrium leniencies.

The approach adopted in Compte and Postlewaite and here consists in looking for equilibria *within a limited set of rules.* Which strategy *set* should be considered is a modelling choice. In contexts where $Q-$learning would seem to be a reasonable hypothesis, $Q-$based learning seems to be a plausible enough added sophistication,[11] to the extent that players can at least approximately compare the performance of various biases. We do not model the process leading to equilibrium biases, but an evolutionary or reinforcement learning process framed under a different clock (allowing longer review periods or longer batches) is a natural candidate. At a broad level, our exercise has the flavor of Axelrod [1984]: within an ecology of reinforcement-learning rules (parameterized by their leniency biases), what leniency biases can we expect to eventually survive?

The method departs from traditional game theory/Bayesian analysis in that it does not check

[9] Klein [2021] examines a similar duopoly setup. There is no memory but moves are sequential, so effectively, Q-values are assumed to be contingent on the rival's price.

[10] The reason is that under the continuous limit examined, feedback is instantaneous and this precludes simultaneous experimentation. Keeping the continuous time assumption, alternative assumptions about the duration of experimentation necessary to obtain reliable feedback would yield different conclusions regarding the possibility of recoordination (as sufficiently synchronous experiments could then arise).

[11] Other/alternative departures might be considered, where players strategically adjust the speed α or the extent of experimentation ε.

for all a priori feasible strategies (both in terms of candidate equilibrium profiles and in terms of available deviations). It also departs from most of the bounded rationality literature, which like us, aims to introduce limits on sophistication, but often does so by introducing misspecifications (Esponda and Pouzo [2016]), biased interpretation of the environment or feedbacks (Jehiel [2005]), or biased estimates of the alternatives (Osborne and Rubinstein [1998]), with players maximizing a subjective criterion which may not be congruent with their own true welfare. Here, Q-values provide a misspecified estimate of continuation values and the naive Q-learning rule corresponds to subjective maximization. In contrast, when examining *Qb-equilibria*, the performance of the various Q-based strategies available is assumed to be correct. The limit on sophistication comes from the (assumed) inability to evaluate all strategies: only some strategies are considered, not all of them. This is in the spirit of Simon [1955], who advocated the notion of *considered set*.

Regarding collusion, our analysis shows that even if at first sight, collusion would be difficult to sustain, in particular when payoffs are stochastic and subject to idiosyncratic shocks, players may have incentives to bias their decision rule, and that appropriately biased Q-learning may be both particularly efficient at sustaining collusion and incentive compatible, and thus posing a challenge to regulatory bodies, as these biases may be hard to detect for an outsider.

In terms of cooperative skills, it is worth noting that players may not be able to learn to adjust their bias and realize that there could gain from being more lenient. There may even be, sometimes, several equilibrium leniencies. This suggests that having an exogenous inclination for leniency may be important in sustaining or fostering cooperative outcomes, in particular in environments where payoffs are stochastic and private, as, under naive $Q-$learning, the ability to sustain cooperation in such environment is more limited.

In Section 2, we consider a single agent problem where the action of the agent affects some (unobservable) underlying state. In such environments (i.e., where the state is partially observable), it is well-known Q-learning may fail to approach the optimal strategy (see Singh, Jaakkola, and Jordan [1994] or more recently Barfuss and Mann [2022]). Our example is pedagogical, illustrating that, even when facing an exogenously given (and relatively simple) stochastic tit-for-tat strategy, a Q-learner may fail to approach the optimal strategy, *whatever the exploration level* ε *and speed of adjustment* α *assumed*.

Another related paper is Karandikar, Mookherjee, Ray, and Vega-Redondo [1998], who consider players who hold a slow-moving aspiration level (which geometrically aggregates own past gains), and independently switch action with some probability when the current gain is below the aspiration. There is no experimentation but *exogenous shocks* on the aspiration level lead to exogenous changes of actions. Karandikar et al. [1998] show that when aspirations are (sufficiently-) slow moving, convergence to CC is much easier than exits from CC: cooperation is the only stable outcome in the long-run. Intuitively, starting from CC and aspiration levels above the value of mutual defection, DC creates a disappointment for player 2, eventually leading to DD, which in turn creates, since aspiration are only moving slow, joint disappointment and repeated pressures back to CC. Inversely, starting from DD and low aspiration levels, a shock on say player 1's aspiration level induces many CD's and DD's, which raises the aspiration level of player 2 as well, hence eventually, as explained above, repeated pressures back to CC whenever DD is played.

The mechanism by which cooperation is sustained in Karandikar et al. [1998] bears some similarity with that induced by Q-learning. One can view Q-values as *action-specific aspiration levels,* noting however that given Equation (1) and the term maxQ, these levels cannot lie far apart from one another when experimentation is frequent enough. Cooperation can be sustained with these strategies because actions that are not currently $Q-$optimal have only slow-changing (and nearby) aspiration levels: whenever defection becomes optimal for say player 1, her $Q-$value associated with cooperation only moves slowly;[12] the defection is quickly matched by player 2 (because his aspiration levels across actions are nearby), and when this happens, cooperation becomes attractive again.[13] One difference with Karandikar et al. is that there is no easy way to exit from DD under Q-learning, because CD just raises the Q-value of defection for player 2, hence raises the barrier to cooperation for player 2, while in Karandikar et al., CD raises the aspiration level of player 2, facilitating a subsequent switch to cooperation for player 2.

The paper is organized as follows. In Section 2, we present a dynamic decision problem in which $Q-$learning performs poorly. Section 3 considers the repeated prisoner's dilemma, explaining first the dynamics of play and the various phase alternations (cooperative, disorganized and defective), and then the role of biases in mitigating the occurrence of defective phases. Section 4 introduces the two monitoring technologies we consider, while Section 5 applies our analysis to the oligopoly setup studied in Calvano et al. [2020]. Section 6 concludes.

2. Q-traps

In this Section, we consider a single agent with two actions available, $a \in A = \{1, 2\}$, aiming to maximize long-run gains against an *exogenously given automaton,* which transits between two states $\theta \in \{1, 2\}$. We consider a "naive" $Q-$learner who revises $Q^t(a)$ for $a \in \{1, 2\}$ according to (1), experiments with probability ε in any period, and otherwise chooses the naive policy, i.e., chooses action 1 whenever

$$\Delta^t \equiv Q^t(1) - Q^t(2) \geq 0$$

A critical assumption made here is that $Q-$learning does not evaluate $Q^t(a, \theta)$ for each state θ, for example because the state is not observable. We explain below why the Q-learning algorithm may fail to maximize long-run gains, and keep the agent trapped in a "bad" state for long durations.

[12] The role of these differentiated speeds of adjustment has been emphasized by Asker et al. [2022] and Banchio and Mantegazza [2022]. Competitive outcomes are more likely when Q-learning is synchronous (as opposed to asynchronous). When synchronous, players use observations to make inferences about the payoff they would have obtained had they played differently, using these inferences to update all Q-values at once.

[13] In general, there is no reason to expect that simultaneous recoordination on cooperation will be immediate, and disorganized play may arise. But as already mentionned, disorganized play induces an attunement of Q-values propitious to recoordination on cooperation.

The automaton. We assume that the automaton has stationary transition probabilities that are contingent on the current action of the agent. Specifically, we let $\pi^a_{\theta\theta'} = \Pr_a(\theta \to \theta')$ denote the probability to transit from θ to θ' when the agent plays a, and to fix ideas, we assume

$\pi^a_{\theta\theta'}$	$1 \to 2$	$2 \to 1$
$a = 1$	0.01	0.05
$a = 2$	0.05	0

These transitions corresponds to a form of stochastic TIT-for-TAT: interpreting action 1 as cooperation and action 2 as defection, the automaton switches to the (bad) state 2 with a small probability (0.01) when facing an agent that cooperates, and a larger probability (0.05) when the agent defects. Once in state 2, the automaton switches back to (good) state 1 with positive probability (0.05), *but only when the agent cooperates.*

Payoff structure. Regarding payoffs we assume

$a\backslash\theta$	1	2
1	2	y
2	x	1

where $y < 1$. So state 1 is the favorable state for the agent. Given the transition probabilities, if the agent plays $a = 1$ always, he induces a distribution over states that puts weight $q = 0.83 = 0.05/(0.01+0.05)$ on state 1, and thus enjoys $2q+(1-q)y$ in the long-run. If he plays $a = 2$ always, he generates a distribution that puts weight $q = 0$ on the state 1, and thus enjoys 1. So long as $y > -4$, choosing $a = 1$ is thus preferable. As a matter of fact, if x is not too large, choosing $a = 1$ always is the optimal strategy even if the agent could condition his action on the state.[14]

When $x < 2$, the payoff structure of the one-stage interaction resembles a *coordination* or *bandit* problem: with exogenous changes in the underlying state, the issue would be to track the state and *match it*. When $x > 2$, the payoff structure suggests that 2 is a *dominant* action, which it would be if variations in the underlying state were exogenous. With endogenous state changes however, neither matching the state nor playing the seemingly dominant strategy are good strategies.

The dynamics of Q-learning. $Q-$learning is well-adapted to problems where changes in the underlying state are *exogenous*, allowing one to track which alternative is currently the best one. Tracking cannot be perfect, as it requires some experimentation (which is a source of inefficiencies when the state has not changed), and not-too-fast (nor too-slow) adaptation (α has to be well-tuned to the timing of exogenous changes in state). But, to the extent that changes of state are not too frequent, $Q-$learning will work well for many combinations (α, ε).

[14] If the agent could condition his behavior on the state, another option would be to play $a = 2$ in state 1 and $a = 1$ in state 2. For large enough x, this would be the optimal state-contingent strategy: it would yield $0.5x + 0.5y$, to be compared with $\max(1, 2q + (1 - q)y)$.

In our setup, changes in the underlying state are endogenous (i.e., they depend on the action of the agent) and starting from unfavorable conditions, $Q-$learning works badly for all combinations (α, ε).[15] The issue is that the agent may be trapped in the bad state $(a, \theta) = (2, 2)$ for long durations. Escaping the trap is possible with enough experimentation and/or high enough adjustment speed α (we explain why below), but whatever facilitates exits from the trap (i.e., high α and ε) also facilitates adjustments away from the desirable long-run outcome $(a, \theta) = (1, 1)$ –back to the trap.

Figure 1 below describes the dynamics of Q-values when $x = 1$ and $y = -0.5$ under low speed and low experimentation ($\alpha = 0.1$ and $\varepsilon = 0.1$), starting from favorable initial conditions ($Q^0(1) = 1.5$, $Q^0(2) = 1.4$ and $\theta = 1$).

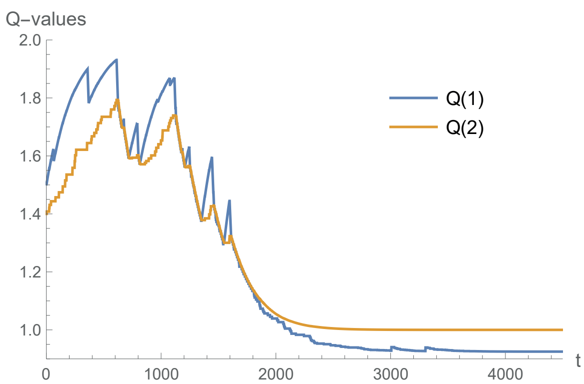

Figure 1: Evolution of Q-values

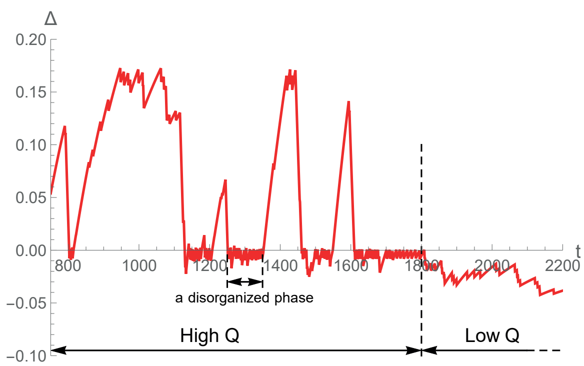

Figure 2: Evolution of Δ

Under low experimentation, for the path considered, the agent keeps staying in the favorable state during about 600 periods, until the state changes durably enough to $\theta = 2$ so that $Q^t(1)$ drops below $Q^t(2)$, making action $a = 2$ more attractive than $a = 1$.

Then follows what we shall call a *disorganized phase* where $\theta = 2$ and the agent quickly alternates between his two actions. This alternation is not triggered by experimentation, but the fact that Q-values are both high, with each $Q(a)$ decreasing, *but only when a is played*. As a result, the difference $\Delta^t = Q^t(1) - Q^t(2)$ remains small and its sign is repeatedly changing, as illustrated in Figure 2.

Eventually, the occasional choices of action $a = 1$ may trigger a change of state (recall that $\pi^1_{21} > 0$), in which case a prolonged favorable phase may restart, with $Q-$values rising again. However, it may also happen that the state $\theta = 2$ is persistent enough to drive $Q^t(1)$ below 1. When this happens, occasional experimentation does not help: it just confirms that $Q^t(1)$ is a worse action, and the agent remains trapped playing $a = 2$ for a long duration.

With higher experimentation and higher speed ($\varepsilon = 0.3$ and $\alpha = 0.3$), we obtain $Q-$values

[15] Starting from favorable conditions where $Q(1) > Q(2)$, $\varepsilon = 0$ yields the optimal strategy. So we shall examine welfare levels obtained over a given (long) horizon, starting from $Q(1) < Q(2)$.

that adjust more quickly to changes in states, so variations in Q-values are larger. The dynamic is similar in the sense that eventually, the agent enters a phase where $Q-$values drop below 1. The main difference is that escaping (relatively quickly) from *low-Q* traps is now feasible:

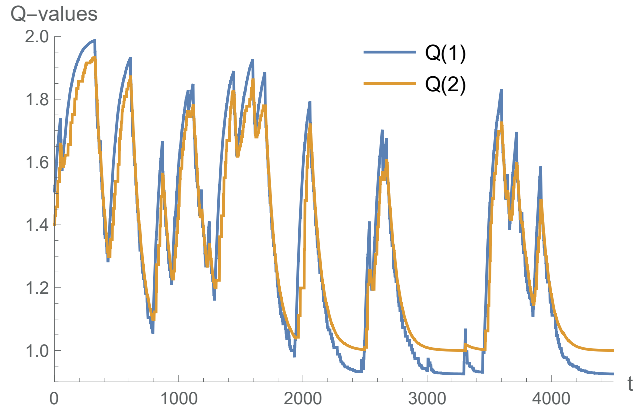

Figure 3: Evolution of Q-values with $\varepsilon = 0.3$ and $\alpha = 0.3$

Intuitively, escaping the trap requires a conjunction of several events: (i) a change of state from $\theta = 2$ to $\theta = 1$ (induced by experimentation of $a = 1$); (ii) some persistence of state $\theta = 1$, and (iii) sufficiently many periods of experimentation of $a = 1$ within that lapse of time to enable a significant rise of $Q^t(1)$.

A large ε helps for (i) and (iii) and a large α allows for a fast rise of $Q^t(1)$, so both high ε and high α are conducive to exits from the trap. Unfortunately however, these exits are not long-lasting. High ε and α facilitate exits from traps, but they also induce fast adaptation whenever the state changes back to $\theta = 2$, so exits are generally not durable.[16]

Welfare levels.

Table 1 reports welfare levels for different values of α and ε, for three different pairs (y, x):[17]

Table 1: Welfare levels

(a) $(y, x) = (-0.5, 1)$

$a \backslash \varepsilon$	0.1	0.2	0.3	0.4
0.1	0.93	0.88	0.92	0.93
0.3	0.96	1.02	1.02	1.
0.5	1.	1.02	1.05	1.02

(b) $(y, x) = (0.5, 1)$

$a \backslash \varepsilon$	0.1	0.2	0.3	0.4
0.1	1.41	1.45	1.38	1.3
0.3	1.25	1.32	1.27	1.25
0.5	1.22	1.27	1.28	1.26

(c) $(y, x) = (0.5, 2.5)$

$a \backslash \varepsilon$	0.1	0.2	0.3	0.4
0.1	1.1	1.16	1.19	1.23
0.3	1.07	1.17	1.22	1.25
0.5	1.08	1.17	1.23	1.26

In all cases, welfare falls much below the maximal feasible welfare $(2q + (1 - q)y)$, though the loss is less pronounced when $y = 0.5$ and $x = 1$. Intuitively, when y is higher, the trap is less deep

[16] Furthermore, high experimentation levels may produce many mismatches $(a, \theta) = (1, 2)$ or $(2, 1)$, which eventually hurts welfare (when $x < 2$).

[17] Welfare levels are obtained by looking at the last 80.000 periods of a 100.000 period simulation, starting from initially unfavorable conditions ($Q^0(1) = 0.9 < Q^0(2) = 1$ and $\theta = 2$).

and exits are easier: there are values of α and ε which avoid too-fast adaptation (hence preserve long favorable high-Q phases) and still allow relatively fast exits from traps.

With $x = 2.5$, reaching high welfare levels is difficult *even when y is high*. The reason is the following. A disorganized phase ends with a change of state (back to $\theta = 1$). At that date, it is possible that $\Delta^t < 0$ (so 2 is played), but Q-values are close to one another and when $x = 1$, Δ^t increases, the agent thus soon plays $a = 1$, leading to a fast and prolonged rise in $Q^t(1)$. When $x > 2$, the opposite happens: if 2 is played, this leads to a fast rise in $Q^t(2)$ (see Appendix). This makes action 2 the better strategy for some time, hastening the return to $\theta = 2$: overall, the agent only rarely plays action 1 in state 1.

3. The Prisoner's Dilemma

We now examine the interaction of two Q-learners, each trying to determine whether to cooperate or defect based on the Q-values of cooperation and defection. Payoffs have the same structure as before, with $x > 2$ and $y < 1$:

$a_1 \backslash a_2$	C	D
C	2	y
D	x	1

We further define $\overline{Q}_i^t \equiv \max_{a_i \in \{C,D\}} Q_i(a_i)$ and

$$\Delta_i^t = Q_i^t(C) - Q_i^t(D) \text{ and } \rho_i^t = \Delta_i^t - \Delta_i^{t-1}.$$

3.1. Naive Q-learning

While players do not face a simple two-state automaton, their behavior bears some resemblance with it: confronted with a player that mostly defects, occasional experimentation will confirm that D is the best option, and be conducive to further defection. As a result players may be trapped in DD for some time.

Exiting DD is possible, but the mechanism differs slightly from that of Section 2. It requires *simultaneous* experimentation (hence sufficiently frequent experimentation on each side), so as to induce some CC's which, if adaptation α is not too small, will induce a rise of Δ^t for both that may propel the interaction into a cooperative phase. Unfortunately, while high experimentation levels and/or high adaptation allow exits, they also hurt the sustainability of cooperation, as in our previous automaton example. Table 2 reports expected welfare levels over a 10.000-period horizon, starting from unfavorable conditions, for different values of α and ε, assuming $x = 2.5$ and $y = -0.5$:[18]

[18] For $y = -0.5$ and symmetric strategies, the welfare above 1 coincides with the fraction of the time players cooperate. Payoffs are obtained by running 90 simulations of 10000 periods, starting from $Q_i(C) = 0.95$ and $Q_i(D) = 1$ for both players, and computing the mean payoff obtained.

Table 2: Welfare levels

$a\backslash\varepsilon$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
0.1	1.	1.	1.	1.	1.	1.01	1.01	1.01
0.3	1.	1.03	1.03	1.06	1.06	1.04	1.04	1.04
0.5	1.02	1.11	1.18	1.16	1.16	1.1	1.08	1.07

We also report the chances of exit from defection over this 10.000-period horizon:

Table 3: Probabilities of exit from DD

$a\backslash\varepsilon$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
0.1	0.	0.	0.	0.	0.	0.	0.	0.
0.3	0.	0.06	0.1	0.2	0.29	0.47	0.63	0.86
0.5	0.04	0.27	0.67	0.92	0.98	1.	1.	1.

To illustrate the dynamics of Q-values, let us consider a path of Q-values obtained when setting $\alpha = 0.5$ and $\varepsilon = 0.1$. For these ε and α, exits from defection are relatively frequent (over the time horizon shown), but joint cooperation does not last very long:[19]

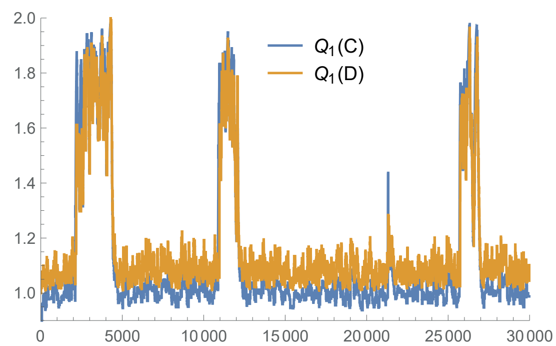

Figure 4: Evolution of Q-values for $\alpha = 0.5$ and $\varepsilon = 0.1$

The occasional peaks correspond to phases where cooperation occurs (hence phases where Q-values rise), but on a closer look, these peaks involve relatively rapid alternations of **jointly cooperative phases** (where $\Delta_i^t > 0$ for both) and **disorganized phases** (composed mostly of $CD^{'}$s and DD's). In **disorganized phases**, Q-values remain high (suggesting to players that some cooperation should be viable), but, as it turns out, both Δ_i^t remain close to 0.

Figure 5 plots the dynamics of Δ's between $t = 2000$ and $t = 2500$, starting from a defection trap (with low Q's). At $t = 2120$, players simultaneously jump to a prolonged cooperative phase which increases durably Q-values. Within this high-Q phase, alternations between disorganized and cooperative phases occur.

Figure 6 shows the end of the high-Q phase, which terminates with a prolonged disorganized phase. The duration of the disorganized phase is long enough to induce a drop in $Q-$values that

[19] We plot the Q-values of player 1. $Q_1(C)$ is the blue curve, $Q_2(D)$ is the orange curve.

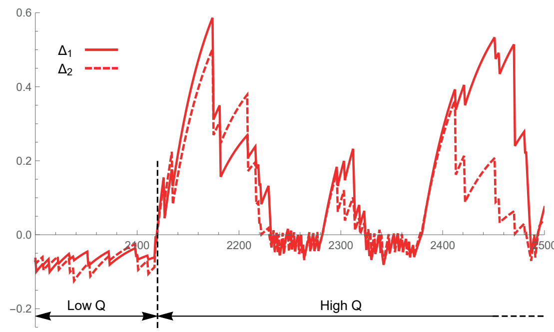

Figure 5: Alternation of cooperative and disorganized phases

makes prospects of cooperation slim ($Q-$values drop below 1), and players are trapped again in a (long) defection phase.

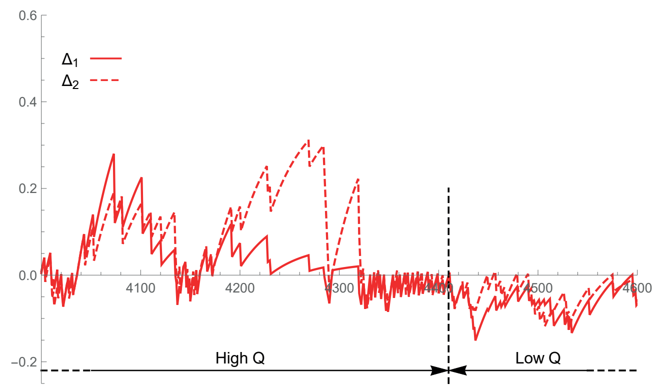

Figure 6: Alternation ending in defection

The role of disorganized phases.

Consider a (relatively-)high $Q-$phase, where $\overline{Q}_i^t \in (1,2)$. When either CC or DD is played, both Δ_i's must rise (because $1 < \overline{Q}_i^t$ so playing DD decreases the value of defection, and because $\overline{Q}_i^t < 2$ so CC increases the value of cooperation). When CD or DC is played, both Δ_i must decrease. Thus, for any action profile played, *the signs of* ρ_1 *and* ρ_2 *are identical.* Regarding magnitudes however, if players start from exact same initial conditions, magnitudes are identical so long as either CC or DD are played, but the magnitudes of ρ_1 and ρ_2 generically differ when CD is played. This implies that in general, the $\Delta_i' s$ do not cross the $0-$frontier at the same time: so we must have $\Delta_i^t > 0 > \Delta_j^t$ and a disorganized phase must start.

Figure 7 illustrates one such instance, where we force initial conditions to be sufficiently asymmetric:[20] Disorganized phases consists of three sub-phases. First, CD's put downward pressures on the Δ_i's, until $\Delta_2 < \Delta_1 < 0$. At this point, Δ_1 is close to 0 and a **spread-reduction phase** starts where, unless experimentations occur, player 2 plays D and player 1 alternates between Cs'

[20] We choose $\Delta_1 = 0.3$ and $\Delta_2 = -0.01$ (with $\max Q_1 = 1.4$ and $\max Q_2 = 1.32$). We force the intial asymmetry to be large to highlight the presence of the CD's and spread reduction phases on the figure.

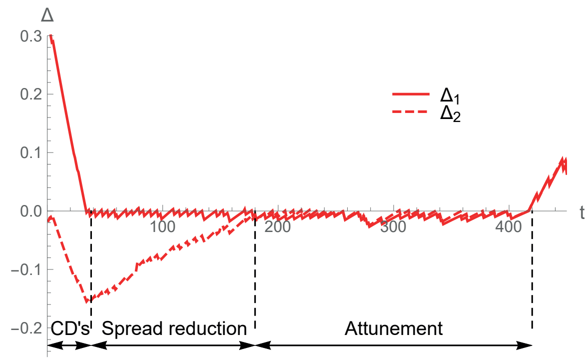

Figure 7: The anatomy of a disorganized phase

and Ds',[21] until the point where both Δ's are close to 0. In the last phase, which (by chance) may be very short (but can sometimes be long), players alternate between CD's DC's and DD's until both simultaneously switch to $\Delta_i > 0$, with cooperation then driving up both Δ's and Q's. We call this phase an **attunement phase**.

Sometimes attunement can be very long and the disorganized phase then terminates in a defection phase. Intuitively, only DD's put pressures upward on Δ's, but DD's necessarily reduce Q's values. Furthermore, as Q values become smaller, the upward pressure becomes weaker,[22] and eventually, the Δ_i's remain persistently below 0.

In summary, the role of a disorganized phase is to reduce the spread $|\Delta_i - \Delta_j|$ and keep both Δ's close to 0, which allows for the possibility of a simultaneous recoordination on cooperation.

3.2. Q-based learning

We now investigate the consequence of agents adopting biased policy rules, whereby a player chooses to cooperate whenever

$$Q_i(C) + b_i > Q_i(D)$$

Obviously, if $b_i > 0$, this seems conducive to more cooperation for player i, but it is not clear that this would be a good policy: in case Q-value are good proxies for continuation values, a biased policy can only hurt; and worse, being more cooperative might actually have the drawback of making defection more attractive to one's opponent, hence to generate more joint defections eventually. When Q-traps are an issue however, being more lenient may help and foster joint cooperation beneficial to both.

We start with our previous example where $x = 2.5$ and $y = -0.5$. Note that in this example, only CC yields a Pareto superior payoff, so with symmetric strategies, expected gains exactly reflect the extent of cooperation. We set $\alpha = 0.5, \delta = 0.95$ and $\varepsilon = 0.1$, chosen so that players experience

[21] Along this phase, player 1 keeps alternating between $C's$ and D's because Δ_1 remains close to 0 as CD's put pressures downward while DD's put pressures upward.

[22] Δ_i increases when DD is played insofar as $\max Q_i > 1$.

an alternation being all phases (traps, cooperative and disorganized).

Given these parameters, we consider a finite grid of possible biases $b_i \in B_i$, and assume $b_i = \kappa_i \varpi$ for $\kappa_i \in \{0, 1, .., K\}$, for some fixed increment ϖ, which we set at $\varpi = 0.02$.[23] For each policy profile (κ_1, κ_2), we simulate 90 paths of play, each of length 10.000 periods, starting from initially unfavorable conditions (where $Q_i(C) = 0.95$ and $Q_i(D) = 1$ for both players). We compute the average gains $v_i(\kappa_i, \kappa_j)$ obtained over 90 such paths.[24] Since the draws may by chance favor one player, we further anonymized payoffs and compute $\overline{v}(\kappa, \kappa') = (v_i(\kappa, \kappa') + v_j(\kappa, \kappa'))/2$. We set $K = 4$ and $\varpi = 0.02$ and report (see Table 4) the gain matrix $\overline{v}$, as well as, for each pair $\kappa = (\kappa_1, \kappa_2)$, the fraction of the time joint cooperation occurs:

Table 4: Biased Q-learning

(a) Gains

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4
0	1.16	1.41	1.56	1.66	1.89
1	1.38	1.61	1.71	1.77	1.84
2	1.45	1.63	**1.74**	1.8	1.87
3	1.4	1.57	1.69	**1.81**	1.88
4	0.5	1.19	1.49	1.69	1.87

(b) Frequencies of CC

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4
0	0.16	0.39	0.5	0.53	0.19
1	0.39	0.61	0.67	0.67	0.52
2	0.5	0.67	0.74	0.74	0.68
3	0.53	0.67	0.74	0.81	0.78
4	0.19	0.52	0.68	0.78	0.87

For example, $\overline{v}(2, 0) = 1.45$ provides the payoff of *player 1* when she biases her policy rule by $2\varpi = 4\%$ while player 2 does not. The payoff of player 2 is obtained by looking at the entry $\overline{v}(0, 2) = 1.56$. The frequency of cooperation for the profile $\kappa = (2, 0)$ is 50%.

The bias 2ϖ by player 1 generates a payoff asymmetry because the lenient policy of player 1 modifies the distribution of CD's and DC's at her expense. Nevertheless, the effect on the occurrence of cooperation is strong, as it increases from 16% to 50%, and this provides each player with incentives to bias their policy away from the naive policies $\kappa = (0, 0)$.

From Table 4a, it is immediate to check that there are two pure strategy equilibria $b^* = 2\varpi$ and $b^* = 3\varpi$.

The tables also show that the stronger the bias, the more the opponent benefits, as $\overline{v}(\kappa_1, \kappa_2)$ is rising in κ_2. Interestingly however, the reason why player 1 benefits depends on the magnitude of κ_2. Player 1 and player 2 both benefit from a small rise in κ_2 away from 0 because this strongly increases the chance of joint cooperation. For $\kappa_2 > 3$ however, the first line of Table 4b shows that the biased policy rule of player 2 *decreases* the chance of joint cooperation: it makes defections more attractive to player 1 and at $\kappa_2 = 4$, the chance of joint cooperation drops down to 19% – player 1 mostly benefits from player 2's generous policy by frequently defecting (and enjoying 2.5).

To summarize the discussion above, Table 5 compares the distribution over action profiles at $\kappa = (0, 0)$, $\kappa = (2, 0)$, $\kappa = (2, 2)$ and $\kappa = (4, 0)$:

[23] The relevant magnitude is ϖ/Q. Since Q-values are normalized, and typically equal to 1 in a long defection phase, each increment corresponds to a 2% bias.

[24] For robustness check, we report in the Appendix simulations run over longer horizons (100000 periods).

$\kappa \backslash \%$	CC	CD	DC	DD
$(0,0)$	16	6	6	73
$(2,0)$	50	9	5	35
$(2,2)$	74	7	7	12
$(4,0)$	19	50	4	26

Table 5: $Pr(a|\kappa)$

To conclude, we plot biased Q-values at a Qb-equilibrium ($b^* = 2\varpi$), as well as the differences

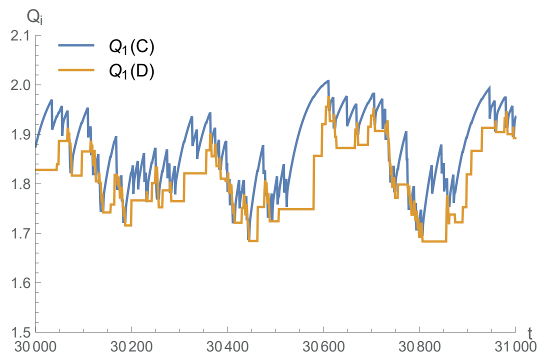

Figure 8: Evolution of Q-values under biased Q-learning

$\Delta_i = Q_i(C) + b_i^* - Q_i(D)$:

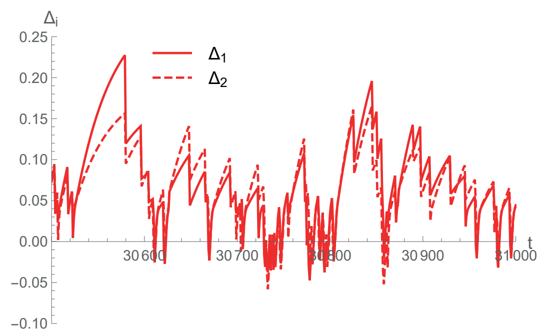

Figure 9: Evolution of Δ under biased Q-learning

In equilibrium, the effect of biases is to eliminate defection phases: players alternate between cooperative phases and disorganized phases, and disorganized phases are short enough to avoid a drop in $Q-$values.

Further examples and qualifications.

Under naive Q-learning, players alternate between high-Q phases and low-Q phases. The role of the bias is to facilitate transitions to (and the persistence of) high-Q phase. Each player has individual incentives to do so because, without bias, low-Q phases tend to last long, under the parameters chosen. We describe below two situations where individual incentives to depart from $b = (0, 0)$ would be weaker.

Lower speed α. The speed of alternation between these phases depends on the parameters of Q-learning. With a smaller speed of adjustment α, there may be no alternation over the horizon considered: starting from high-Q's, conditions (most likely) remain favorable, starting from low-Q's, conditions (most likely) remain unfavorable. Said differently, the gap $|\Delta|$ remains (most likely) bounded away from 0. Then, individual incentives to depart from $b = (0, 0)$ may be much weaker, or there may be multiple Qb-equilibria. We report such cases in the Appendix.

Higher y. When y is closer to 1, Q-traps are shallower and exits are easier. As a matter of fact, over a long horizon, high Q phases may become preponderant even when there is no bias. In such cases, there may be no incentives to distort the policy rule. We illustrate this by considering $y = 0.5$ and $\alpha = 0.5$. For simulations running 10000 periods, there are still incentives to distort, but for 100000-period simulations, incentives to distort disappear: $b^* = (0, 0)$ is the unique equilibrium.

4. Stochastic payoffs

We have so far assumed that payoffs are obtained with certainty. We now assume that payoffs are *stochastic*, meaning that conditional on the action profile a played, each player i obtains a random payoff z_i. In light of the traditional approach to repeated games, this payoff can be interpreted as an imperfect *signal* correlated with the other's behavior, so this assumption moves us to realm of games with *imperfect monitoring.* Furthermore, if the payoffs are drawn independently (conditional on actions played), monitoring is *private.*

The distinction between perfect/imperfect and public/private monitoring has been useful in the traditional approach. With perfect or imperfect public monitoring, *public information* is available. This allows perfectly coordinated play where public information is used by players to finely tune their behavior to others'. Under private monitoring, no such fine tuning is possible.

With $Q-$based strategies, the distinction seems less relevant. Play is only based on own payoff experience, i.e. own Q-values. So, even if actions are perfectly observable, this information is not used to tune one's behavior to others'. As a matter of fact, as soon as players experiment and play differently (either CD or DC) – or if payoffs are asymmetric, players get *different payoff histories*, and potentially, Δ_1^t and Δ_2^t then differ from one another. As we have seen, these differences eventually result in disorganized phases where play is *not perfectly coordinated* with frequent alternation between $CD's$ and DD's. Nevertheless, and although coordination is not assumed, play turns out to be reasonably well coordinated under Q-learning, with an alternation between cooperative high-Q phases, defective phases and disorganized phases. Disorganized phases are the ones that involve mis-coordinations, but these phases cannot last long, otherwise they lead to mutual defection.

What happens when payoffs are stochastic? One effect is to increase the variability of the payoffs obtained. So (for given α), this potentially induces larger variations in $Q-$values. Stochastic payoffs may also modify the co-evolution of the pair (Δ_1^t, Δ_2^t), in particular if payoff shocks are independent. We shall see is that large variations in $Q-$values may help coordinated exits from traps, *to the extent* *that payoff shocks are correlated.* We shall then investigate the incentives to bias the policy rule.

4.1. Payoff structure.

We consider a very simple stochastic payoff structure with only two payoff realizations $z_i \in \{0, V\}$. Cooperating costs L, but it increases the probability of a good outcome. We let p_k denote the probability of a good outcome when k players cooperate. The expected payoff matrix for player 1 is thus

$a_1 \backslash a_2$	C	D
C	$p_2V - L$	$p_1V - L$
D	p_1V	p_0V

Any expected payoff matrix examined in the previous Section (parameterized by x and y) can be generated by setting $V > 2 + x - y$ and adjusting p and L.[25] So in what follows, we shall keep referring to x and y, and mention the value V considered.

4.2. Correlated payoffs

We start with the case where payoff realizations are perfectly correlated across players ($z_1 = z_2 = z$). So when a is either CC or DD, realized gains are identical. When $a = CD$ or DC, payoff realizations are identical, but gains differ: if CD has been played, then player 1 gets $z - L$ while player 2 gets z.

The main difference with the previous case is that Q-values may be subject to large variations induced by the randomness in payoff realizations. Prolonged sequence of bad or good draws create exogenous variations in the Q-values. Given these variations, a sequence of bad draws when CC is played may shorten a cooperative phase and trigger a defection phase. But a sequence of bad draws when DD is played may also allow for a simultaneous drop in the $Q-$values of defection (hence a joint increase in the Δ's), which may trigger a cooperative phase.

We illustrate this assuming $y = 0$ and $x = 2.5$, and $V = 5$. We further set $\alpha = 0.1, \varepsilon = 0.1$ and $\delta = 0.95$.[26] In case payoffs are not random, this low speed of adjustment makes it hard to exit from Q-traps, and it also makes cooperative phases very resilient. The evolution of Q values is drawn in Figure 10 below: starting from favorable conditions, Q values remain high. There are episodic disorganized phases, but they do not last long and players soon return to cooperation.

In case payoffs are random (and correlated) (see Figure 11), Q values are much more variable, and one observes transitions between mostly-cooperative phases and mostly-defective phases, independently of initial conditions, and these transitions arise for different Q-values (in the case of non-random payoff, defective phases were only arising for low-Q's).

For the random-payoff case, we report below (see Table 6) the occurrence of each pair CC,CD, DC

[25] For a given x, y, with $V > 2 + x - y$, we let $p_0 = 1/V$, $p_1 = x/V$, $L = x - y$, and $p_2 = (2 + x - y)/V$.

[26] We choose a low α to highlight the difference with the non-random case and the role of random payoffs.

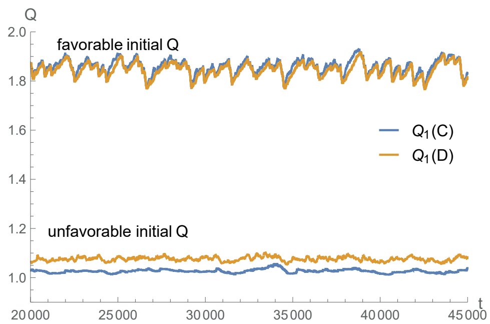

Figure 10: Q-values with no randomness

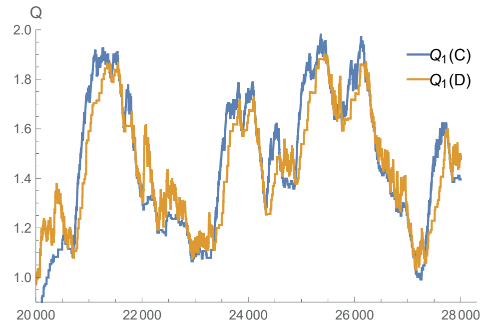

Figure 11: Q-values with correlated shocks

and DD for $\alpha = 0.1$ and various specifications of ε, over draws of 200.000 periods each, as well as expected gains.

Table 6

ε	gains	%CC	%CD	%DC	%DD
0.05	1.51	0.48	0.07	0.07	0.39
0.1	1.28	0.23	0.1	0.1	0.57
0.15	1.21	0.15	0.12	0.12	0.61
0.2	1.15	0.08	0.14	0.14	0.64
0.25	1.14	0.06	0.15	0.15	0.63

There are two notable facts about this table. The alternation between joint cooperation and joint defection obtains *even when experimentation is small.* Small experimentation levels actually *help* sustain higher welfare. As we shall see, high experimentation levels hurt because they shorten cooperative phases without facilitating exits from defective phase.

Let us first give some intuition as to why exits from low-Q traps become feasible with (correlated) payoff-shocks. Intuitively, in the correlated case, either $z = V$ and gains are simultaneously high for both players, or $z = 0$ and gains are simultaneously low for both players. This implies that over the range of Q's that are spanned through the process, and so long as the players choose the same action, the differences

$$\rho_i^t = \Delta_i^t - \Delta_i^{t-1}$$

either *simultaneously rise* or *simultaneously decrease.* This simultaneity facilitates exits from defection phases even when Q values are small for both players, because a simultaneous rise may occur several periods in a row. With non-random payoffs, escaping low-Q traps required *joint experimentation* (hence high experimentation levels). Joint experimentation is no longer necessary.

As a matter of fact, high experimentation levels induce many CD's and DC's that contribute to raise the difference $|\Delta_1 - \Delta_2|$, which makes simultaneous exit from defection less likely (because larger differences make events $\Delta_i^t > 0 > \Delta_j^t$ more likely), explaining the poor performance observed for high ε.[27]

[27] Under events $\Delta_1^t > 0 > \Delta_2^t$, CD is played, and this decreases both Δ_i's on average, making exits difficult –

4.3. Independent payoffs

We turn to the case where payoffs are drawn independently. There are two issues: how easily do players exit from defection traps, how easily do players manage to sustain long cooperative phases? Figure 12, drawn for $\alpha = 0.1$ and $\varepsilon = 0.1$ shows that exits from defection are still possible when payoffs are drawn independently:

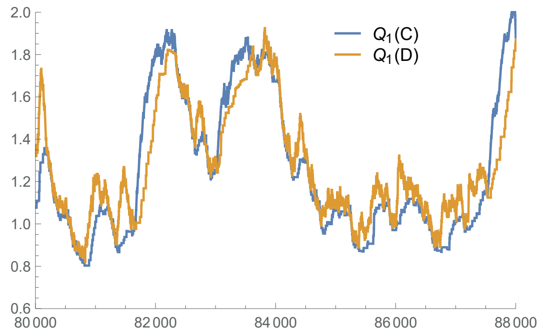

Figure 12: Evolution of Q-values with independent shocks

However, with independent payoff realizations, such exits are very rare and players mostly defect: Table 7 below reports action profile occurrences for $\alpha = 0.1$ and various specifications of ε, as well as the overall gains. The table shows a significant difference with the correlated case.

Table 7

ε	gains	%CC	%CD	%DC	%DD
0.05	1.08	0.05	0.06	0.06	0.82
0.1	1.13	0.08	0.1	0.1	0.73
0.15	1.15	0.09	0.12	0.12	0.67
0.2	1.14	0.07	0.14	0.14	0.65
0.25	1.14	0.06	0.16	0.16	0.62

Cooperation levels now remain small at all experimentation levels. In order to better understand the difference with the correlated case, we simulate (for each ε) 25 draws of 10.000 periods and obtain the median duration of a cooperative phase starting from an exogenously fixed favorable initial condition. We do the same to obtain the median duration of a defective phase, starting from a fixed unfavorable initial condition.[28] We plot these median durations in Figure 13 below.

The figure reveals that whether shocks are independent or correlated, the durations of cooperative phase are similar, with more experimentation shortening the duration of cooperative phases. The main effect of independence is to substantially raise the length of defection phases, in particular when experimentation is small.

though not impossible.

[28] For the favorable conditions we choose $Q_i(C) = 1.5$ and $Q_i(D) = 1.4$ for both players and consider the first date for which $\Delta_i < 0$ for two consecutive periods for both players. For the unfavorable condition we choose $Q_i(C) = 1.2$ and $Q_i(D) = 1.25$ and look for the first date for which $\Delta_i > 0$ for two consecutive periods for both periods.

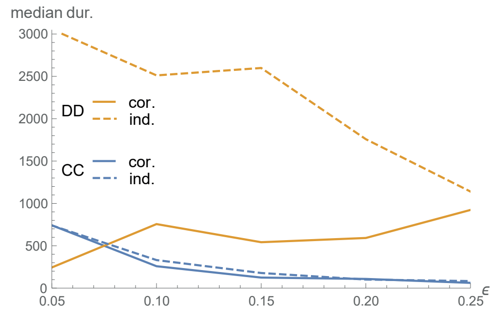

Figure 13: Median exit durations

Intuitively, whether in a defection or cooperative phase, $\rho_i > 0$ after a good payoff realization and $\rho_i < 0$ after a bad payoff realization, as in the correlated case. However, the chance that *both* get a good draw is smaller when shocks are independent: if p is the probability of a joint good payoff realization in the correlated case, this probability drops down to p^2 when shocks are independent. So the ρ_i^t's more rarely rise together, in particular when DD is played (as p is then smaller). Worse, the ρ_i^t's often have different signs, so events where the Δ_i^t's get simultaneously above 0 are rarer.

As a matter of fact, starting from unfavorable conditions, the time it takes for *either* Δ_1 or Δ_2 to rise above 0 is *shorter* under independent shocks[29]: the issue is that when this happens, the other player is most likely defecting. Independence generates a higher gap $|\Delta_1 - \Delta_2|$.

To capture the exit difficulty from a different angle, we plot below the empirical frequency of the event $\Delta_2 > 0$ as a function of Δ_1, for two different values of ε ($\varepsilon = 0.1$ and $\varepsilon = 0.05$, continuous and dashed curves respectively):[30,31]

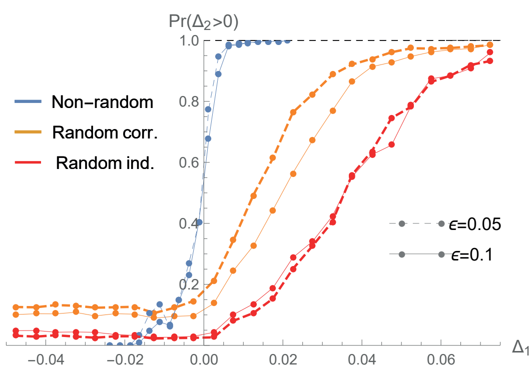

Figure 14: $Pr(\Delta_2 > 0|\Delta_1)$

[29] At $\varepsilon = 0.1$, the median and expected durations are respectively 23 and 36 periods under correlated shocks, while they are only 17 and 23 under independent shocks (obtained with 100 simulations). Similar comparisons obtain for different values of ε, though the gap narrows with higher ε.

[30] We consider intervals of the form $I_k = (k\omega, k\omega + \omega)$ for $k \in \{-n, ..., n\}$, with $\omega = 0.005$, and compute $f_k = \#\{\Delta, \Delta_1 \in I_k \text{ and } \Delta_2 > 0\}/\#\{\Delta, \Delta_1 \in I_k\}$, over a long horizon T, and we set $T = 200000$ and $\omega = 0.05$ to ensure that the sets $\#\{\Delta, \Delta_1 \in I_k\}$ have at least 500 elements.

[31] For the non-random payoff case, we only consider the favorable condition case, as otherwise, the Δ_i's consistently stay below 0.

When payoffs are not random, Δ_1 is a very good predictor of the other's behavior. When payoffs are random, Δ_1 is no longer a good predictor. Still, when Δ_1 is positive and small, there is a non negligible chance that Δ_2 is positive as well when shocks are correlated (and even more so when ε is smaller). However, this chance becomes tiny when shocks are independent.

4.4. Q-based learning

We adopt the same methodology as in Section 3.2. We set $\alpha = 0.1, \delta = 0.95$ and $\varepsilon = 0.1$ and consider a finite grid of possible biases $b_i \in B_i$, where $b_i = \kappa_i \varpi$ for $\kappa_i \in \{0, 1, .., K\}$, for an increment $\varpi = 0.02$. We report below (see Table 8) the anonymized payoff matrix $\overline{v}(\kappa_1, \kappa_2)$ obtained for $K = 6$ for the correlated case and the independent case respectively.

Table 8: Gains under biased Q-learning

(a) Correlated shocks

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4	5	6
0	1.27	1.33	1.49	1.54	1.71	1.94	2.2
1	1.28	1.44	1.54	1.65	1.72	1.93	2.17
2	1.32	1.43	**1.6**	1.72	1.81	1.92	2.07
3	1.19	1.38	1.6	**1.77**	1.86	1.9	1.97
4	1.	1.16	1.46	1.74	**1.87**	1.91	1.93
5	0.71	0.85	1.25	1.66	1.86	1.9	1.92
6	0.49	0.61	1.02	1.57	1.84	1.91	1.92

(b) Independent shocks

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4	5	6
0	1.12	1.19	1.31	1.51	1.68	1.96	2.22
1	1.13	1.31	1.44	1.57	1.72	1.97	2.17
2	1.15	1.33	**1.56**	1.69	1.81	1.94	2.08
3	1.14	1.27	1.54	**1.78**	1.88	1.93	1.97
4	0.96	1.11	1.46	1.75	**1.88**	1.92	1.93
5	0.7	0.84	1.25	1.69	1.86	1.91	1.92
6	0.48	0.64	1.04	1.61	1.85	1.91	1.92

These matrices exhibit properties that are similar to those found earlier. In each case, starting from naive Q-learning there are (here, weak) private incentives to bias towards higher b levels. The independent and correlated cases both have several pure Nash equilibria at $b^* = 2\varpi$, $b^* = 3\varpi$ and $b^* = 4\varpi$. These equilibria are conducive to payoffs of similar magnitude whether shocks are independent or correlated, and to high levels of cooperation.

As in the non-random case, a player benefits from the other choosing a more lenient strategy. For low leniency levels, the gain comes from a higher chance of cooperation (and this benefits both). For higher leniency levels, the asymmetry in leniency levels hurts cooperation but it benefits the less lenient player because of the prevalence of asymmetric CD play (favoring the one who defects more often – the less lenient one, at the expense of the more lenient one).

5. Duopoly

In this Section, we revisit Calvano et al. [2020] in light of our approach. Calvano et al. study a duopoly game where each player has finitely many price alternatives $p_i \in A_i$, with profits $\pi_i(p)$ depending on the profile of prices $p = (p_i, p_j)$. Profits are determined by

$$\pi_i(p_i, p_j) = (p_i - c) \frac{\exp(\frac{d_i - p_i}{\mu})}{1 + \exp(\frac{d_i - p_i}{\mu}) + \exp(\frac{d_j - p_j}{\mu})}$$

where μ is an index of horizontal differentiation (the limit case $\mu \simeq 0$ corresponding to perfect substitutes) and d_i an index of vertical differentiation. In simulations below, we set $d = 2, \mu = 1/6$ and $c = 1$. We also assume a finite number of prices, $p^k = 1.4 + k0.1$ with $k \in \{0, ..., 6\}$. We report the payoff matrix in the Appendix. The Nash equilibrum of this game has $p^* = 1.4$ (i.e., $k_1^* = k_2^* = 0$), yielding 1.97 to each player, and joint profits are maximized for $p^{**} = 1.9$, (i.e., $k_1^{**} = k_2^{**} = 5$), yielding 3.53 to each player.[32]

In the spirit of our previous exercise, we study the long-run properties of naive policy rules where agents take the action with highest Q value (unless they experiment).[33] We then move to $Qb-$equilibria where players use biased policy rules. To fix ideas, we use biased policy rules which take the action that maximizes the biased criteria

$$Q_{i,b_i}(p_i) = Q_i(p_i) + b_i \pi_i(p_i, p_i)$$

where $\pi_i(p_i, p_i)$ is the profit that i would obtain if the other player was adopting the same price. We analyze the game where each player sets optimally their own bias b_i, among a finite number of possible biases, setting $\alpha = 0.1, \delta = 0.95$ and $\varepsilon = 0.1$. Furthermore, to compute expected gains, we run n simulations of length T, starting from unfavorable initial conditions,[34] and compute the average gain over these simulations. We set $n = 8$ and $T = 100000$.

5.1. Naive Q-learning.

We find that when players adopt naive Q-learning (with $b = 0$), players are trapped 60% of the time in the Nash profile (p^0, p^0), with long-run payoff approximately equal to 2.25. Payoffs are above Nash levels, partly because experimentation improves welfare, partly because (p^3, p^3) gets some (small) weight (5%). Figure 15 reports the dynamics of $Q_i(p_0)$ and $\max_{k>0} Q_i(p_k)$ for player 1:

On occasions, player 1 finds that some alternative strategy p^k with $k > 0$ has higher $Q-$value, but, as it turns out, this is rarely concomitant with player 2 finding that some $p^{k'}$ with $k' > 0$ has higher Q-value. As a matter of fact, whenever player 1 prefers some $p^k > p^0$, this reinforces the attraction of player 2 for p^0: the joint experience of some pair $(p^k, p^{k'})$ which would yield gains Pareto superior to Nash gains is not frequent enough.

To illustrate graphically this phenomenon, we plot the differences

$$\Delta_i \equiv Q_i(p_0) - \max_{k>0} Q_i(p_k)$$

[32] As explained in the Introduction, one difference between Calvano et al. [2020] and our work is that we study a setup where experimentation does *not* vanish. Under non-vanishing experimentation, some cooperation is feasible even under Q-learning: with two prices, the game typically has the structure of a prisonner's dilemma, for which we know that some cooperation is feasible under naive Q-learning.

[33] Note that when players experiment, we assume uniform experimentation over all available strategies. An alternative assumption could be that experimentation is only on "nearby" strategies, or driven by relative Q-values.

[34] $Q_i(p^0) = 2$ and $Q_i(p^k) = 1.8$ for all $k > 0$.

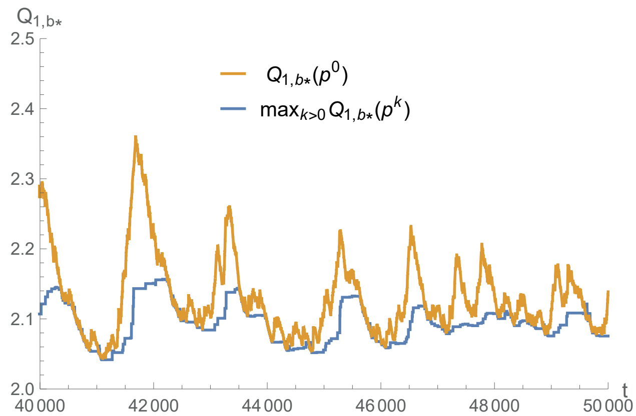

Figure 15: Evolution of Q-values under naive Q-learning

for each player. On the lapse of time shown, these differences are not simultaneously positive, and players remain trapped in low prices.

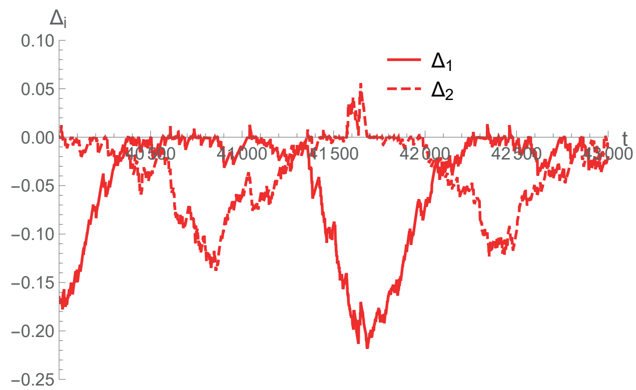

Figure 16: Co-evolution of Δ under naive Q-learning

5.2. Qb-learning.

We compute expected gains when players use biased policy rules with $b_i = \kappa_i \varpi$ where $\varpi = 0.01$, for $\kappa_i \in \{0, ..., 5\}$. We report the values associated with each profile of biases considered in the Appendix, and find that players have individual incentives to bias their policy rule in favor of higher prices (positive b) and, given our grid assumption, we find that $\kappa^* = (3, 3)$ is an equilibrium, yielding an expected gain of 3, substantially larger than the Nash payoffs (but below the efficient level 3.53). In equilibrium, after an initial phase where (p^0, p^0) is frequently played because we start from unfavorable conditions, the profile of prices most frequently played are (p^4, p^4), (p^5, p^5) and (p^3, p^3).[35] To illustrate the dynamic of Q-values in this $Qb-$equilibrium, we plot below the evolution of $\max_{p<p^3} Q^t_{i,b^*}(p)$ and $\max_{p\geq p^3} Q^t_{i,b^*}(p)$:

[35] During the last 80000 periods, the shares are respectively 40%, 20% and 6%. The expected gains reported include the intial phase so these payoffs are lower than they would be if we only reported the average over the last 80000 periods.

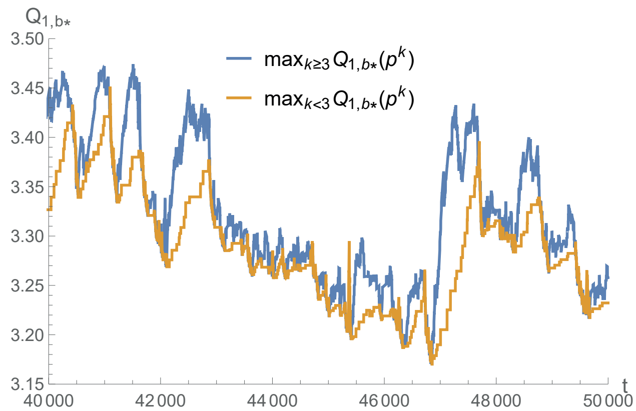

Figure 17: Evolution of Q-values under b^*

We observe that $Q-$values remain significantly above Nash profits yet below the maximum joint profits (3.53). Prices mostly remain at levels above or equal to p^3. To better assess the dynamics when $Q-$values of collusive and non-collusive prices get close to one-another, we define

$$\Delta_i = \max_{p \geq p^3} Q_{i,b^*}(p) - \max_{p < p^3} Q_{i,b^*}(p).$$

When Δ_i is positive, player i plays a price above or equal p^3, and when Δ_i is negative, player i plays a price equal to p^0, p^1 or p^2. Figure 18 plots the co-evolution of Δ_1 and Δ_2.

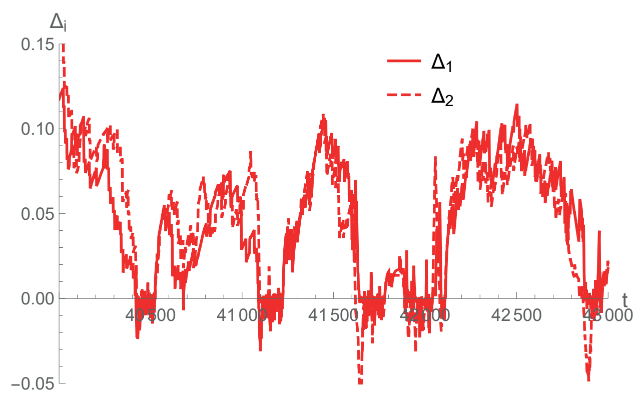

Figure 18: Co-evolution of Δ under b^*

This co-evolution is analogous to that obtained for the prisoner's dilemma when Q-values are high: there are collusive phases when both Δ's are high and these phases contribute to a simultaneous surge of $\max Q$. But there are episodic *disorganized phases* where the Δ's remain close to 0 and players alternate between prices above and below p^3. At the Qb equilibrium however, these disorganized phases are not long enough to trigger price wars where both would durably play $p < p^3$, while in the absence of biases, these disorganized phases would lead to a long price war where p^0 would be preponderant.

6. Conclusion

Collusive or cooperative behavior requires some form of history-dependent behavior, whereby the use of competitive strategies by one player are punished, for example by the use of more competitive strategies by others, to some extent. A natural motivation for history-dependent behavior is *learning*. If there is some *uncertainty* about what others do, and some persistence in what others do, then it may be worth using past data to better adapt one's behavior to others'.

The source of the uncertainty may be some exogenous (and persistent) shifts in the value of cooperation, as one then has to identify *when* cooperation is worthy. The source of uncertainty may also be endogenous, because it can be that others find, given their observations, and possibly erroneously, that cooperation is not reciprocated hence being cooperative is not worthy.

This *learning* motivation for history-dependent strategies has rarely been the focus of the repeated game literature,[36] mostly for technical reasons: finding optimal strategies is easier when there is no exogenous and persistent uncertainty that affects the payoff structure; equilibria are also simpler to design when the signals about payoffs are public and one focuses on public strategies. History-dependence is obtained in equilibrium, but mostly through a well-designed coordination of continuation strategies, rather than as a by-product of individual learning from past data.[37]

Once one departs from the classic approach to repeated games, *incorporating uncertainty* so as to make learning a central dimension of behavior and *preventing* the design of nicely *coordinated* strategies, it becomes natural to think of players (and model them) behaving as if they had few arms available and trying to learn which arm is best, as they would in a bandit problem.

This work, as well as others cited in the Introduction, gives a prominent place to learning. It does so (i) by incorporating some exogenous uncertainty – assuming an exogenous probability of experimentation at any stage, and (ii) by precluding the use of perfectly coordinated strategies – assuming that behavior may only be conditioned on private statistics about one's own past payoffs.

One drawback of course is that the latter restriction (the dependence on private summary statistics of past payoffs) makes the characterization of optimal strategies particularly difficult. One way to bypass this difficult is to forget about optimality altogether and assume that players use strategies (such as Q-learning) that are known to be well-adapted to simple bandit problems where player's behavior have no influence on the others' behavior.[38] This has been the path followed so far by the algorithmic literature, with the major finding that although naive Q-learning is generally suboptimal when one's behavior actually has a long-run influence on others', it may

[36] This is unlike the reputation literature of course, where learning is central (Fudenberg and Levine [1989]).

[37] With private monitoring, there has been attempts to construct (belief-based) equilibria where players effectively use signals to learn about other's past behavior (Compte [1994, 2002] and Sekiguchi [1997]). However, most of the literature has followed the belief-free path set by (Piccione [2002] and Ely and Välimäki [2002]) to construct finely designed strategies where at coordinated times, most of past history can be ignored (see also Sugaya [2022]). These papers rely on player' sharing a common clock to coordinate play and adjust incentives.

[38] Or where the influence is directly observable and summarized by a public state variable.

nevertheless help players sustain significant levels of cooperation.

The path followed in this work, as in Compte and Postlewaite [2015, 2018], has been to incorporate some sophistication, taking the form of a restricted family of behavior rules (here a restricted family of biased Q-learning rules) and allowing each player to select the one that works best for her. For one, this added sophistication allows players to choose a strategy that mostly ignores past private data, so it does *not* assume history-dependence. This enables us to check that cooperation is not just a by-product of players being constrained to use a suboptimal strategy (i.e., naive Q-learning).

Second, it permits us to see that with added sophistication, cooperation more easily prevails. Intuitively, when the process that generates the benefits associated with each arm depends on one's own behavior, for example, if being more cooperative triggers more cooperation from the other side, one may be individually better off biasing one's policy by including a cooperative bias into the policy.

More generally, with biased policies, cooperation may obtain even in environments where payoff signals are stochastic and drawn from conditionally independent distributions, contrasting with the case where only naive Q-learning would be considered. Specifically, our examples suggest that while correlated shocks may foster exits from defection trap even when Q-learning is naive, persistently sustaining cooperation with independent shocks is possible but only when Q-learning is biased. They also suggest that when players are sophisticated enough to bias (optimally) their policy, in equilibrium, the classic distinction between perfect, imperfect public and imperfect private monitoring may be less relevant than generally thought.

Regarding the scope of collusion, we find that tacit collusion is easier to sustain than usually thought, as one need not observe or condition behavior on others to sustain it – a clear challenge to regulatory bodies. Our finding that cooperation or collusion is robust to the horizon over which average payoffs are computed also suggests that one could introduce additional persistent shocks on the worthiness of collusion or cooperation (affecting the horizon over which cooperation has value) without changing our basic insights. Furthermore, our examples also suggest that one may have to be careful in comparing the degree to which collusion can be sustained across various institutional designs (such as first price versus second price, as in Banchio and Skrzypacz [2022]): with Qb-equilibria, the differences between formats could possibly be smaller or starker. We leave this for further research.

Finally, while we have focused on Q-based learning, other reinforcement learning algorithms could be considered (for example in the spirit of UCB, where each arm is evaluated independently). One feature of Q-learning is that it keeps a somewhat tight relationship between Q-values through the whole process, while this is not the case when arms are evaluated independently. It would be interesting to investigate whether this property of Q-learning helps sustain cooperation, or if similar levels of cooperation would obtain under UCB-*based* equilibria.

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Appendix

Q-traps in decision problem. The effect of $x > 2$

The figure below provides a sample path starting from initially favorable conditions when $x = 2.5$.

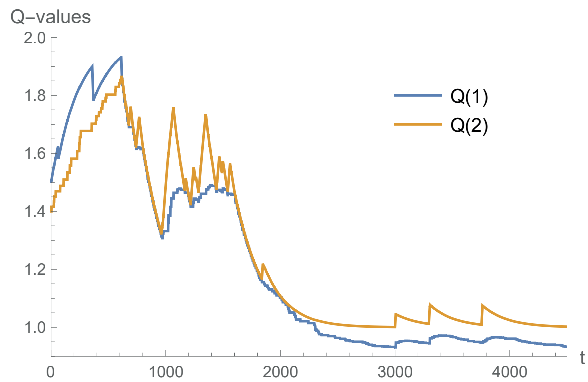

Figure 19: Evolution of Q-values for $x = 2.5$ and $y = 0.5$

Compared to the case where $x = 1$, disorganized phases end with a surge in $Q^t(2)$ while they were ending with a surge in $Q^\tau(1)$ when $x = 1$. When this occurs, action 2 is played, driving up $Q^t(2)$ so long as the state remains equal to $\theta = 1$. A change in the state then drives $Q^t(2)$ back to $Q^t(1)$. Another disorganized phase starts, but each disorganized phase ends similarly, with action 2 being eventually played, until Q values eventually get below 1 persistently. As a consequence, the agent more often plays action 2, and this adversely affect welfare (because $\pi^2_{12} > \pi^1_{12}$).

Qb-learning for prisoners' dilemma

(a) Case $\alpha = 0.5$, $\varepsilon = 0.1$ and $y = -0.5$. We report in Table 9 the payoff matrix obtained for a longer horizon, with simulations made over 100.000-period horizons. We also check for possible biases that include negative values, i.e. $b_i = \kappa_i \varpi$ with $\kappa_i \in \{-1, 0, .., 4\}$. We have:

Table 9: Long horizon, $\alpha = 0.5, y = -0.5$

$\kappa_1 \backslash \kappa_2$	-1	0	1	2	3	4
-1	1.01	1.02	1.04	1.1	1.18	2.02
0	1.02	1.2	1.52	1.69	1.74	1.89
1	1.02	1.48	**1.68**	1.76	1.8	1.83
2	1.04	1.55	1.67	**1.77**	1.83	1.87
3	1.02	1.46	1.6	1.71	1.81	1.88
4	0.13	0.49	1.23	1.52	1.68	1.87

As in the case examined in the main text, there are strong up-ward pressures away from $\kappa = (0, 0)$: the gain increases from 1.2 to 1.55 by individually setting $\kappa_i = 2$. There are now two pure equilibria $\kappa = (1, 1)$ and $\kappa = (2, 2)$, each yielding significantly higher gains than naive Q learning.

(b) For $\alpha = 0.2$, $\varepsilon = 0.1$ and $y = -0.5$, we report payoffs in Table 10. There are still incentives to depart from naive Q-learning but individual incentives are very weak.

Table 10: Long horizon, $\alpha = 0.2, y = -0.5$

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4
0	1.	1.	1.01	1.11	1.9
1	1.	1.14	1.19	1.4	1.84
2	1.	1.16	1.6	1.73	1.9
3	1.01	1.24	1.67	**1.89**	1.9
4	0.37	0.83	1.61	1.88	1.9

One can boost cooperative behavior by using a large bias ($b_1 = 3\varpi$), but when $\alpha = 0.2$, this only raises the chance of joint cooperation from 0 to 6% (compared to from 16% to 53% when $\alpha = 0.5$). In addition, the large asymmetry in decision rules creates an imbalance between CD's and DC's (respectively 8% and 5%) which hurts the lenient player.

(c) For $\alpha = 0.1, \varepsilon = 0.1$ and $y = -0.5$, we obtain:

Table 11: Long horizon, $\alpha = 0.1, y = -0.5$

$\kappa_1 \backslash \kappa_2$	-1	0	1	2	3	4
-1	1.002	1.002	1.002	1.003	1.024	2.002
0	1.002	**1.002**	1.002	1.003	1.024	1.891
1	1.002	1.002	1.002	1.034	1.21	1.871
2	1.002	1.002	1.028	1.622	1.791	1.909
3	0.982	0.982	1.117	1.75	**1.901**	1.903
4	0.135	0.381	0.961	1.786	1.9	1.902

Payoffs have been obtained under the 100000 period horizon. When $\alpha = 0.1$ and if Q learning is naive, there are no exits from the defection Q-trap even under this large horizon assumption, and there are no individual incentives to raise κ_i. Naive Q-learning and $\kappa = (3, 3)$ are both Nash equilibria of this game. Note that a risk dominance argument would likely select the later equilibrium.

(d) For $\alpha = 0.5$, $\varepsilon = 0.1$ and $y = 0.5$, we examine payoffs obtained for 10.000 and 100.000 period simulations respectively. In the former case, $\kappa = (0, 0)$ is not an equilibrium (but $\kappa = (1, 1)$ is), while in the latter case, $\kappa = (0, 0)$ is the only equilibrium.

Table 12: Gains when $\alpha = 0.5$ and $y = 0.5$

(a) Short horizon

$\kappa_1 \backslash \kappa_2$	-1	0	1	2	3
-1	1.05	1.05	1.16	2.36	2.4
0	1.05	1.28	1.57	2.34	2.4
1	1.03	1.38	**1.69**	2.04	2.35
2	0.62	0.64	1.27	1.89	1.99
3	0.6	0.61	0.74	1.82	1.95

(b) Long horizon

$\kappa_1 \backslash \kappa_2$	-1	0	1	2	3
-1	1.41	1.71	1.65	2.21	2.37
0	1.57	**1.83**	1.87	2.12	2.36
1	1.38	1.73	1.85	1.92	2.29
2	0.74	0.92	1.71	1.91	2.02
3	0.63	0.67	0.9	1.56	1.94

Intuitively, when $y = 0.5$ and the horizon is long, defection phases are rare compared to the other phases, and biases cannot play the role of decreasing the impact of defection phases. An upward bias away from (0,0) then hurts the more lenient player because it creates an asymmetric distribution over CD's and DC's at the expense of the more lenient player. A downward bias hurts too because it substantially increases the occurrence of defection phases.

Duopoly

Our duopoly example assumes $p = 1.4 + k0.1$ for $k \in \{0, ..., 6\}$. We report here the payoff matrix, where one can check that $k = (0, 0)$ (i.e., $p = 1.4$) is the unique Nash equilibrium, as undercutting by 0.1 always yield a profitable gain. Joint profits are maximized for $k = (5, 5)$, i.e., $p_i = 1.9$ for both players.

Table 13: Profits $\pi_1(p^{k_1}, p^{k_2})$

$k_1 \backslash k_2$	0	1	2	3	4	5	6
0	**1.97**	2.54	3.01	3.35	3.58	3.71	3.79
1	1.74	2.44	3.13	3.7	4.11	4.38	4.55
2	1.36	2.06	2.87	3.66	4.31	4.78	5.08
3	0.97	1.56	2.34	3.23	4.08	4.77	5.26
4	0.65	1.09	1.73	2.56	3.48	4.32	4.99
5	0.42	0.72	1.18	1.85	2.67	**3.53**	4.29
6	0.26	0.45	0.77	1.24	1.88	2.62	3.33

Tables 14a and 14b report expected gains when players use biased policy rules with $b_i = \kappa_i \varpi$ where $\varpi = 0.01$, for $\kappa_i \in \{0, ..., 5\}$. In Table 14a, expected gains are obtained by taking the average of 8 simulations, each running for 100000 periods, starting from unfavorable initial conditions. This biased-policy rule game has a unique equilibrium, $\kappa_1^* = \kappa_2^* = 3$, hence a 3% weight, with an expected payoff 3.01.

Table 14: Gains when $\alpha = 0.5$ and $y = 0.5$

(a) 100000 period horizon

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4	5
0	2.25	2.58	2.61	2.85	2.93	3.1
1	2.56	2.54	2.7	2.77	3.	3.11
2	2.53	2.66	2.83	2.96	3.	3.12
3	2.72	2.67	2.91	**3.01**	3.02	3.07
4	2.67	2.8	2.87	2.96	2.96	3.04
5	2.51	2.62	2.77	2.88	2.95	3.01

(b) Last 80.000 periods

$\kappa_1 \backslash \kappa_2$	0	1	2	3	4	5
0	2.2	2.57	2.59	2.96	3.07	3.23
1	2.54	2.78	2.73	2.98	3.13	3.24
2	2.52	2.69	**3.14**	3.12	3.14	3.22
3	2.83	2.88	3.08	3.11	3.14	3.19
4	2.8	2.93	3.01	3.09	3.11	3.15
5	2.62	2.73	2.85	2.99	3.08	3.13

As it takes some time for collusion to build up, we also check (Table 14b) the payoff matrix obtained when averaging over the last 80000 periods. The payoffs obtained under naive $Q-$learning are similar (around 2.2), while as soon as both use a bias $\kappa_i \geq 1$, payoffs are significantly larger. Equilibrium biases are $\kappa_1^* = \kappa_2^* = 2$, with an expected payoff 3.14. Expected payoffs are larger than in the previous case because they do not incorporate the time necessary to build up collusion starting from unfavorable conditions.