Supplementary material for "Subvector inference when the true parameter vector may be near or at the boundary"

E Sufficient conditions

We present sufficient conditions for equation (1) and Assumptions 1 and 4. They are taken from Andrews (1999) (A1) with appropriate notational adjustments to allow for drifting sequences of true parameters.

The following assumption corresponds to Assumption 1 in A1.

Assumption 6. Under $\{\gamma_n\} \in \Gamma(\gamma^*)$, $\hat{\theta}_n - \theta_n = o_p(1)$.

The following assumption is sufficient for Assumption 6 and corresponds to Assumptions 1^*(a) and 1^*(b*) in A1.

Assumption 6*.

(i) Under $\{\gamma_n\} \in \Gamma(\gamma^*)$, $\sup_{\theta\in\Theta} |Q_n(\theta) - Q(\theta;\gamma^*)| = o_p(1)$ for some non-stochastic real-valued function $Q(\theta;\gamma^*)$.

(ii) $Q(\theta;\gamma^*)$ is continuous on Θ $\forall \gamma^* \in \Gamma$.

(iii) $Q(\theta;\gamma^*)$ is uniquely minimized by θ^* $\forall \gamma^* \in \Gamma$.

(iv) Θ is compact.

The following assumption corresponds to Assumption 2^* in A1 and is sufficient for equation (1). Here and in what follows, "for all $\epsilon_n \to 0$" stands for "for all sequences of positive scalar constants $\{\epsilon_n : n \geq 1\}$ for which $\epsilon_n \to 0$."

Assumption 7. Under $\{\gamma_n\} \in \Gamma(\gamma^*)$,

$$\sup_{\theta\in\Theta:\|\theta-\theta_n\|\leq\epsilon_n} \frac{|nR_n(\theta)|}{(1+\|\sqrt{n}(\theta-\theta_n)\|)^2} = o_p(1)$$

for all $\epsilon_n \to 0$.

The following assumption is sufficient for Assumption 7 (cf. Lemma in A1) and corresponds to Assumption 2^{2*} in A1, adapted to Θ as defined at the beginnnig of Section 3.

Assumption 7*.

(i) $Q_n(\theta)$ has continuous left/right (l/r) partial derivatives of order two on Θ $\forall n \geq 1$ with probability 1.

(ii) Under $\{\gamma_n\} \in \Gamma(\gamma^*)$, for all $\epsilon_n \to 0$,

$$\sup_{\theta \in \Theta : \|\theta - \theta_n\| \leq \epsilon_n} \left\| \frac{\partial^2}{\partial\theta' \partial\theta} Q_n(\theta) - \frac{\partial^2}{\partial\theta' \partial\theta} Q_n(\theta_n) \right\| = o_p(1),$$

where $(\partial/\partial\theta)Q_n(\theta)$ and $(\partial^2/\partial\theta'\partial\theta)Q_n(\theta)$ denote the $J \times 1$ vector and $J \times J$ matrix of l/r partial derivatives of $Q_n(\theta)$ of orders one and two, respectively.

We note that Assumption 7* is also sufficient for Assumption 4. The following Lemma corresponds to Theorem 1 in A1 and shows that Assumptions 2 and 3 together with Assumptions 6 and 7 are sufficient for Assumption 1.

Lemma 8. *Under $\{\gamma_n\} \in \Gamma(\gamma^*)$ and Assumptions 2, 3, 6, and 7, $\sqrt{n}(\hat{\theta}_n - \theta_n) = O_p(1)$.*

The proof of Lemma 8 is obtained by adapting the proof of Theorem 1 in A1 to accommodate drifting sequences of true parameters. Details are omitted.

F Asymptotic distribution of constrained estimator

We reproduce Theorem 3(b) in A1 with Θ as defined at the beginning of Section 3 and with notational adjustments to allow for drifting sequences of true parameters. Without loss of generality, assume that the last $0 \leq J_2 \leq J$ elements of θ are restricted below by 0, i.e., $\Theta = [-c, c]^{J_1} \times [0, c]^{J_2}$, where $J_1 = J - J_2$. Let $\theta = (\theta_1, \theta_2)$ and $\theta_n = (\theta_{n,1}, \theta_{n,2})$ be conformable partitions. Define

$$\Gamma(\gamma^*, h) = \{\{\gamma_n\} \in \Gamma(\gamma^*) : \sqrt{n}\theta_{n,2} \to h \in (\mathbb{R}_+ \cup \{\infty\})^{J_2}, \theta^* \in \ddot{\Theta}\},$$

where $\ddot{\Theta} = [-\ddot{c}, \ddot{c}]^{J_1} \times [0, \ddot{c}]^{J_2}$ with $\ddot{c} < c$.[1] We use the terminology "under $\{\gamma_n\} \in \Gamma(\gamma^*, h)$" to mean "when the true parameters are $\{\gamma_n\} \in \Gamma(\gamma^*, h)$ for any $\gamma^* \in \Gamma$ with $\theta^* \in \ddot{\Theta}$ and any $h \in (\mathbb{R}_+ \cup \{\infty\})^{J_2}$." Let $\mathcal{J}_n \equiv D^2 Q_n(\theta_n)$ and $\mathcal{Z}_n \equiv -\mathcal{J}_n^{-1}\sqrt{n} D Q_n(\theta_n)$ and note that by Assumptions 2 and 3, we have that under $\{\gamma_n\} \in \Gamma(\gamma^*)$

$$\mathcal{Z}_n \xrightarrow{d} \mathcal{Z}(\gamma^*).$$

[1] This definition of $\Gamma(\gamma^*, h)$ ensures that boundary effects only occur at 0.

The objective function can be written as

$$Q_n(\theta) = Q_n(\theta_n) - \frac{1}{2n}\mathcal{Z}_n'\mathcal{J}_n\mathcal{Z}_n + \frac{1}{2n}q_n(\sqrt{n}(\theta - \theta_n)) + R_n(\theta),$$

where

$$q_n(\lambda) = (\lambda - \mathcal{Z}_n)'\mathcal{J}_n(\lambda - \mathcal{Z}_n).$$

The remainder term, $R_n(\hat{\theta}_n)$, is asymptotically negligible under $\{\gamma_n\} \in \Gamma(\gamma^*, h)$ and Assumption 1 such that the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_n)$, as formalized in Proposition 1 below, is given by the distribution of

$$\hat{\lambda}_h = \arg\min_{\lambda \in \Lambda_h} q(\lambda),$$

where

$$q(\lambda) = (\lambda - \mathcal{Z}(\gamma^*))'\mathcal{J}(\gamma^*)(\lambda - \mathcal{Z}(\gamma^*))$$

and

$$\Lambda_h \equiv (\mathbb{R} \cup \{\pm\infty\})^{J_1} \times [-h_1, \infty] \times \cdots \times [-h_{J_2}, \infty]$$

with $h = (h_1, \ldots, h_{J_2})$.

Proposition 1. *Under $\{\gamma_n\} \in \Gamma(\gamma^*, h)$ and Assumptions 2, 3, 6, and 7 (or Assumptions 1-3), $\sqrt{n}(\hat{\theta}_n - \theta_n) \xrightarrow{d} \hat{\lambda}_h$.*

The proof of Proposition 1 is obtained by adapting the proof of Theorem 3(b) in A1 to accommodate drifting sequences of true parameters. Details are omitted.