

Supplementary material for “Testing overidentifying restrictions with a restricted parameter space”

A Proofs

Proof of Proposition 1. The proof follows the outline in Section 3 and proceeds in four steps (I-IV). (I) First, we verify that equation (4) holds. (II) Second, we show that, under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–3, $\sqrt{n}(\hat{\theta}_n - \theta_n) = O_p(1)$ or, equivalently, that J_n is asymptotically equivalent to equation (5) times $2n$. (III) Then, we show that

$$nG'_n(\theta_n)\mathcal{W}G_n(\theta_n) - Z'_n\mathcal{I}Z_n \xrightarrow{d} \chi^2(H - p) \quad (6)$$

under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–4. (IV) Last, we show that $q_n(\sqrt{n}(\hat{\theta}_n - \theta_n))$ is asymptotically independent of $nG'_n(\theta_n)\mathcal{W}G_n(\theta_n) - Z'_n\mathcal{I}Z_n$ and that it fails to be $o_p(1)$ under Assumptions 1–5, with $\gamma_n = \gamma_0 \forall n \geq 1$, where γ_0 is given in Assumption 5, such that $\limsup_{n \rightarrow \infty} P_{\gamma_0}(J_n > \chi^2_{1-\alpha}(H - p)) > \alpha$. The desired result then follows, because

$$\alpha < \limsup_{n \rightarrow \infty} P_{\gamma_0}(J_n > \chi^2_{1-\alpha}(H - p)) \leq \limsup_{n \rightarrow \infty} \sup_{\gamma \in \Gamma_0} P_{\gamma}(J_n > \chi^2_{1-\alpha}(H - p)) \equiv \text{AsySz}_J.$$

(I) Note that Assumption 1 implies Assumption 6* in Ketz (2018b), which, in turn, implies Assumption 6 in Ketz (2018b), which states that $\hat{\theta}_n - \theta_n = o_p(1)$, under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$. Then, equation (4) is implied by Assumption 7 in Ketz (2018b) with $DQ_n(\theta) = G'_\theta\mathcal{W}G_n(\theta)$ and $D^2Q_n(\theta) = G'_\theta\mathcal{W}G_\theta$, noting that

$$G'_n(\theta_n)\mathcal{W}_nG_n(\theta_n) - G'_n(\theta_n)\mathcal{W}G_n(\theta_n) = o_p(1/n)$$

given Assumptions 1 and 3 and given that $\hat{\theta}_n - \theta_n = o_p(1)$ under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$. Assumption 7 in Ketz (2018b) is implied by Lemma 10.3 in Andrews and Cheng (2014b), which establishes that Assumptions GMM1, GMM2, and GMM5 in Andrews and Cheng (2014a) imply Assumptions D1–D3 in Andrews and Cheng (2012), where the former “correspond” to Assumptions 1–3 and the latter to Assumptions 2 and 3 in Ketz (2018a) and Assumption 7 in Ketz (2018b). Here, the “correspondence” is such that the assumptions in Ketz (2018a,b) constitute simplified versions of the assumptions in Andrews and Cheng (2012, 2014a) in that they do not allow for lack of identification in some part of the parameter space but instead allow the true parameter vector to be near or at the boundary of the (optimization) parameter space.

(II) Given that Assumptions 2 and 3 in Ketz (2018a) and Assumptions 6 and 7 in Ketz (2018b) are implied by Assumptions 1–3, it follows from Lemma 8 in Ketz (2018b) that $\sqrt{n}(\hat{\theta}_n - \theta_n) = O_p(1)$ under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–3.

(III) To show that equation (6) holds, let $\mathcal{W} = AA'$ and $B = A'G_\theta$ (such that $B'B = \mathcal{I}$). Then, the left hand side of equation (6) satisfies (recall the definition of Z_n)

$$nG'_n(\theta_n)\mathcal{W}G_n(\theta_n) - Z'_n\mathcal{I}Z_n = (\sqrt{n}A'G_n(\theta_n))'M_B(\sqrt{n}A'G_n(\theta_n)) = Y'M_BY + o_p(1),$$

under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–4, where $Y \sim N(0_H, I_H)$ and where M_B is an idempotent matrix of rank $H - p$, and the result follows.

(IV) Take $\gamma_n = \gamma_0 \forall n \geq 1$. Then, given Assumptions 1–5, it can readily be deduced from Theorem 2(f) and Lemma 2 in Andrews (1999) and the continuous mapping theorem that

$$q_n(\sqrt{n}(\hat{\theta}_n - \theta_0)) = \inf_{\lambda \in \Lambda} q(\lambda) + o_p(1),$$

where $q(\lambda) = (\lambda + Z)'\mathcal{I}(\lambda + Z)$ and where $Z = \mathcal{I}^{-1}B'Y \sim N(0_p, \mathcal{I}^{-1})$. Since \mathcal{I} is positive definite (by Assumption 3) and since $\text{cl}(\Lambda)$ is a strict subset of \mathbb{R}^p (by Assumption 5), we have that $\inf_{\lambda \in \Lambda} q(\lambda) \neq o_p(1)$ or, equivalently, that $P(h(Z) > 0) > 0$, where $h(Z) \equiv \inf_{\lambda \in \Lambda} q(\lambda) \geq 0$. Furthermore, M_BY and Z are (jointly) normally distributed and uncorrelated ($EM_BYZ' = 0_{H \times p}$) and, thus, independent. The desired result, then, follows, since

$$\begin{aligned} \limsup_{n \rightarrow \infty} P_{\gamma_0}(J_n > \chi_{1-\alpha}^2(H-p)) &= P(Y'M_BY + h(Z) > \chi_{1-\alpha}^2(H-p)) \\ &= \int P((M_BY)'M_BY + h(z) > \chi_{1-\alpha}^2(H-p) | Z = z) dF_Z(z) \\ &= \int P((M_BY)'M_BY + h(z) > \chi_{1-\alpha}^2(H-p)) dF_Z(z) > \alpha, \end{aligned}$$

where the third equality follows from the independence of M_BY and Z . □

Proof of Proposition 2. Given equation (6), it suffices to proof that

$$J_n^M = nG'_n(\theta_n)\mathcal{W}G_n(\theta_n) - Z'_n\mathcal{I}Z_n + o_p(1) \tag{7}$$

under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–4. In what follows, we omit the phrase “under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$ and Assumptions 1–4” whenever convenient. Note that, by Theorem 6 in Andrews (1999), Assumption 1(ii) implies that

$$G(\theta; \gamma^*) = G(\theta_n; \gamma^*) + G_\theta(\theta - \theta_n) + o(\|\theta - \theta_n\|),$$

since $\|G_\theta(\theta_n; \gamma^*) - G_\theta(\theta^*; \gamma^*)\| = o(1)$; recall that $G_\theta = G_\theta(\theta^*; \gamma^*)$ by definition. This, together with Assumption 2, implies that

$$\sup_{\theta \in \Theta: \|\sqrt{n}(\theta - \theta_n)\| \leq \epsilon} \|G_n(\theta) - G_n(\theta_n) - G_\theta(\theta - \theta_n)\| = o_p(1/\sqrt{n}). \quad (8)$$

Under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$, equation (8), together with Assumption 3 and $\sqrt{n}(\hat{\theta}_n - \theta_n) = O_p(1)$ (which follows from the proof of Proposition 1), implies that $\sqrt{n}G_n(\hat{\theta}_n) = O_p(1)$. This, in turn, implies that

$$J_n^M = nG'_n(\hat{\theta}_n)\mathcal{W}G_n(\hat{\theta}_n) - nG'_n(\hat{\theta}_n)\mathcal{W}G_\theta(G'_\theta\mathcal{W}G_\theta)^{-1}G'_\theta\mathcal{W}G_n(\hat{\theta}_n) + o_p(1)$$

given Assumptions 1 and 3, using Slutsky's and the continuous mapping theorem. By equation (8), the first term of the previous equation satisfies

$$nG'_n(\theta_n)\mathcal{W}G_n(\theta_n) + 2nG'_n(\theta_n)\mathcal{W}G_\theta(\hat{\theta}_n - \theta_n) + (\hat{\theta}_n - \theta_n)G'_\theta\mathcal{W}G_\theta(\hat{\theta}_n - \theta_n) + o_p(1)$$

and the second term satisfies

$$\begin{aligned} nG'_n(\theta_n)\mathcal{W}G_\theta(G'_\theta\mathcal{W}G_\theta)^{-1}G'_\theta\mathcal{W}G_n(\theta_n) + 2nG'_n(\theta_n)\mathcal{W}G_\theta(\hat{\theta}_n - \theta_n) \\ + (\hat{\theta}_n - \theta_n)G'_\theta\mathcal{W}G_\theta(\hat{\theta}_n - \theta_n) + o_p(1). \end{aligned}$$

Combining the last two displays, we obtain equation (7); recall the definitions of Z_n and \mathcal{I} . □

Proof of Corollary 1. The proof follows immediately from Corollary 2.1(c) in Andrews, Cheng, and Guggenberger (2019) taking $h(\lambda) = \lambda$ (using their notation). □

B Verification of Assumptions 1–5 for Example 1

In this section, we verify Assumptions 1–5 for Example 1. For convenience, the following Lemma reproduces Lemma 12.2 in Andrews and Cheng (2014b), which we use repeatedly.

Lemma 1. *Suppose $\{w_i : i \geq 1\}$ is an iid sequence and Θ is compact. Suppose (i) for some function $M(w) : \mathcal{W} \rightarrow \mathbb{R}_+$ (where with an abuse of notation \mathcal{W} denotes the support of w_i) and all $\delta > 0$, $\|s(w, \theta_1) - s(w, \theta_2)\| \leq M(w)\delta$, $\forall \theta_1, \theta_2 \in \Theta$ with $\|\theta_1 - \theta_2\| \leq \delta$ and $\forall w \in \mathcal{W}$ and (ii) $E_\gamma \sup_{\theta \in \Theta} \|s(w, \theta)\|^{1+\epsilon} + E_\gamma M(w) \leq C \forall \gamma \in \Gamma$ for some $C < \infty$. Then, $\sup_{\theta \in \Theta} \|s(w_i, \theta) - E_{\gamma^*} s(w_i, \theta)\| = o_p(1)$ under $\{\gamma_n\} \in \Gamma(\gamma^*)$ and $E_{\gamma^*} s(w_i, \theta)$ is uniformly continuous on Θ for all $\gamma^* \in \Gamma$.*

We first verify Assumption 1 for $\mathcal{W}_n(\theta) = (\frac{1}{n} \sum_{i=1}^n z_i z_i')^{-1}$ to establish consistency of the first-step estimator, $\bar{\theta}_n$. Here and in what follows, we apply Lemma 1 with $\Theta = \bar{\Theta}$ and $\Gamma(\gamma^*) = \Gamma_0(\gamma^*)$. Take $s(w, \theta) = g(\theta, w) = z(y - x'\theta)$. We have that

$$\|s(w, \theta_1) - s(w, \theta_2)\| = \|zx'(\theta_2 - \theta_1)\| \leq \|zx'\| \|\theta_1 - \theta_2\|.$$

Taking $M(w) = \|zx'\|$ and $\delta = \|\theta_1 - \theta_2\|$, condition (i) of Lemma 1 is satisfied. By Hölder's inequality and the conditions in (2), we have $E_\phi M(w_i) = E_\phi \|z_i x_i'\| \leq (E_\phi \|z_i\|^2)^{1/2} (E_\phi \|x_i\|^2)^{1/2} \leq C_1$ for some $C_1 < \infty$. Furthermore, we have

$$\|s(w, \theta)\| = \|zx'(\theta^* - \theta) + zu\| \leq \|zx'(\theta^* - \theta)\| + \|zu\|$$

by the triangle inequality. Thus,

$$\|s(w, \theta)\|^{1+\epsilon} \leq C_2^* (\|zx'(\theta^* - \theta)\|^{1+\epsilon} + \|zu\|^{1+\epsilon})$$

for some $C_2^* < \infty$, using $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $p > 0$ (and $a, b \in \mathbb{R}$). Furthermore,

$$\begin{aligned} \sup_{\theta \in \bar{\Theta}} \|s(w, \theta)\|^{1+\epsilon} &\leq C_2^* (\sup_{\theta \in \bar{\Theta}} \|zx'(\theta^* - \theta)\|^{1+\epsilon} + \|zu\|^{1+\epsilon}) \\ &\leq C_2^* (\sup_{\theta^* \in \Theta} \sup_{\theta \in \bar{\Theta}} \|zx'(\theta^* - \theta)\|^{1+\epsilon} + \|zu\|^{1+\epsilon}) \\ &\leq C_2^{**} (\|zx'\|^{1+\epsilon} + \|zu\|^{1+\epsilon}) \end{aligned}$$

for some $C_2^{**} < \infty$, since $\bar{\Theta}$ and Θ are compact. Then, by Hölder's inequality and the conditions in (2), we have $E_\phi \sup_{\theta \in \bar{\Theta}} \|s(w, \theta)\|^{1+\epsilon} \leq C_2$ for some $C_2 < \infty$. Taking $C = \max(C_1, C_2)$, condition (ii) of Lemma 1 is satisfied. Therefore, $\sup_{\theta \in \bar{\Theta}} \|G_n(\theta) - G(\theta; \gamma^*)\| \xrightarrow{P} 0$ under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$, where $G(\theta; \gamma^*) = E_{\gamma^*} g(\theta, w_i) = E_{\gamma^*} z_i (y_i - x_i' \theta) = E_{\phi^*} z_i x_i' (\theta^* - \theta)$, where the last equality follows since $E_{\phi^*} z_i u_i = 0_H$ for $\phi^* \in \Phi_0$. This verifies the first part of Assumption 1(i) and Assumption 1(ii). Next, take $s(w, \theta) = zz'$. Condition (i) of Lemma 1 is trivially satisfied (taking $M(w) = \|zz'\|$) and condition (ii) of Lemma 1 is satisfied given the conditions in (2). Therefore, $\|\frac{1}{n} \sum_{i=1}^n z_i z_i' - E_{\phi^*} z_i z_i'\| \xrightarrow{P} 0$ under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$. By the continuous mapping theorem and the conditions in (2), $\sup_{\theta \in \bar{\Theta}} \|\mathcal{W}_n(\theta) - \mathcal{W}(\theta; \gamma^*)\| \xrightarrow{P} 0$ under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$, where $\mathcal{W}(\theta; \gamma^*) = (E_{\phi^*} z_i z_i')^{-1}$. This verifies the second part of Assumption 1(i) and Assumption 1(iii). Next, note that $Q(\theta; \gamma^*) = (\theta^* - \theta)' E_{\phi^*} x_i z_i' (E_{\phi^*} z_i z_i')^{-1} E_{\phi^*} z_i x_i' (\theta^* - \theta)$, where $E_{\phi^*} x_i z_i' (E_{\phi^*} z_i z_i')^{-1} E_{\phi^*} z_i x_i'$ is positive definite given the conditions in (2). Since $\bar{\Theta}$ is convex, it follows that θ^* is the unique minimizer, cf. Section 4 in Perlman (1969), which verifies Assumption 1(iv). As Assumption 1 implies that $\|\bar{\theta}_n - \theta_n\| = o_p(1)$, cf. Proof of

Proposition 1, we conclude that, under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$,

$$\|\bar{\theta}_n - \theta^*\| \leq \|\bar{\theta}_n - \theta_n\| + \|\theta_n - \theta^*\| = o_p(1). \quad (9)$$

Next, we verify Assumption 1 for $\mathcal{W}_n \equiv \mathcal{W}_n(\theta) = (\frac{1}{n} \sum_{i=1}^n g(\bar{\theta}_n, w_i)g(\bar{\theta}_n, w_i)')^{-1}$. To verify the second part of Assumption 1(i), we apply Lemma 1 with $s(w, \theta) = g(\theta, w)g(\theta, w)' = zz'(y - x'\theta)^2$; the first part as well as Assumption 1(ii) are the same as above. Using the mean value theorem, we have

$$s(w, \theta_1) - s(w, \theta_2) = 2zz'(y - x'\theta_+)(\theta_1 - \theta_2),$$

where θ_+ “lies between” θ_1 and θ_2 . Plugging in $y = x'\theta^* + u$, we have

$$s(w, \theta_1) - s(w, \theta_2) = 2zz'(x'(\theta^* - \theta_+) + u)(\theta_1 - \theta_2),$$

Taking $\delta = \|\theta_1 - \theta_2\|$, we have, due to compactness of $\bar{\Theta}$ and Θ ,

$$\begin{aligned} \|s(w, \theta_1) - s(w, \theta_2)\| &\leq 2\|zz'(x'(\theta^* - \theta_+) + u)\|\delta \\ &\leq 2(\|zz'x'(\theta^* - \theta_+)\| + \|zz'u\|)\delta \\ &\leq 2(\|zz'\|\|x'(\theta^* - \theta_+)\| + \|zz'\|\|u\|)\delta \\ &\leq 2C_1^*(\|zz'\|\|x\| + \|zz'\|\|u\|)\delta \end{aligned}$$

for some $C_1^* < \infty$. Taking $M(w) = 2C_1^*(\|zz'\|\|u\| + \|zz'\|\|x\|)$, condition (i) of Lemma 1 is satisfied. By Hölder’s inequality and the conditions in (2), we have $E_\phi M(w_i) \leq C_1$ for some $C_1 < \infty$. With a slight abuse of notation, we have

$$\|zz'(y - x'\theta)^2\|^{1+\epsilon} = \|z(y - x'\theta)\|^{2+\epsilon} \leq C_2^*(\|zx'(\theta^* - \theta)\|^{2+\epsilon} + \|zu\|^{2+\epsilon})$$

for some $C_2^* < \infty$. Then, due to compactness of $\bar{\Theta}$ and Θ , we have (with the same abuse of notation)

$$\begin{aligned} \sup_{\theta \in \bar{\Theta}} \|zz'(y - x'\theta)^2\|^{1+\epsilon} &\leq C_2^*(\sup_{\theta \in \bar{\Theta}} \|zx'(\theta^* - \theta)\|^{2+\epsilon} + \|zu\|^{2+\epsilon}) \\ &\leq C_2^*(\sup_{\theta^* \in \Theta} \sup_{\theta \in \bar{\Theta}} \|zx'(\theta^* - \theta)\|^{2+\epsilon} + \|zu\|^{2+\epsilon}) \\ &\leq C_2^{**}(\|zx'\|^{2+\epsilon} + \|zu\|^{2+\epsilon}) \end{aligned}$$

for some $C_2^{**} < \infty$. By Hölder’s inequality and the conditions in (2), we have $E_\phi \sup_{\theta \in \bar{\Theta}} \|s(w, \theta)\|^{1+\epsilon} \leq$

C_2 for some $C_2 < \infty$. Taking $C = \max(C_1, C_2)$, condition (ii) of Lemma 1 is satisfied. Therefore,

$$\sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{i=1}^n z_i z_i' (y_i - x_i' \theta)^2 - E_{\gamma^*} z_i z_i' (y_i - x_i' \theta)^2 \right\| \xrightarrow{p} 0 \quad (10)$$

under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$. Given equations (9) and (10) and continuity of $E_{\gamma^*} z_i z_i' (y_i - x_i' \theta)^2$ (which follows from Lemma 1), we conclude that, under $\{\gamma_n\} \in \Gamma_0(\gamma^*)$,

$$\|\mathcal{W}_n^{-1} - \mathcal{W}^{-1}\| \leq \|\mathcal{W}_n^{-1} - E_{\gamma^*} z_i z_i' (y_i - x_i' \bar{\theta}_n)^2\| + \|E_{\gamma^*} z_i z_i' (y_i - x_i' \bar{\theta}_n)^2 - \mathcal{W}^{-1}\| = o_p(1),$$

where $\mathcal{W} \equiv \mathcal{W}(\theta^*; \gamma^*) = (E_{\phi^*} z_i z_i' u_i^2)^{-1}$. Applying the continuous mapping theorem, this verifies the second part of Assumption 1(i). Furthermore, with \mathcal{W} thus defined, Assumptions 1(iii) and (iv) are also verified, using the same arguments as above.

To verify Assumption 2, we proceed by verifying Assumption 2* in Ketz (2019), which constitutes a sufficient condition. Assumption 2*(i) in Ketz (2019) is clearly satisfied. By Hölder's inequality and the conditions in (2), Lemma 1 applies with $s(w, \theta) = -zx'$, which verifies Assumption 2*(ii) in Ketz (2019).

Assumption 3(i) follows from Lemma 12.3 in Andrews and Cheng (2014b) given the conditions in (2), noting that $G_n(\theta_n) = \frac{1}{n} \sum_{i=1}^n z_i u_i$. Assumptions 3(ii) and (iii) are satisfied given the conditions in (2), with $G_\theta = -E_\phi z_i x_i'$ and $\mathcal{W} = (E_\phi z_i z_i' u_i^2)^{-1}$.

Assumption 4 follows immediately from the conditions in (2), given that the two-step estimator uses \mathcal{W}_n in the second step.

Assumption 5 is verified noting that Φ_0 is not empty: $(u_i, x_i', z_i)'$ may, for example, be jointly normally distributed, with zero mean and appropriately chosen variance matrix.

C Details for Example 2

In this section, we use the subscript t rather than i , where t represents a market. In the random coefficients logit model, $g(w_t, \theta)$ takes the following form

$$g(w_t, \theta) = z_t' \xi(\theta, s_t, x_t),$$

where z_t is a $J \times H$ matrix of instruments, x_t is a $J \times K_1$ matrix of product characteristics (with $p = K_1 + K_2$), s_t is a $J \times 1$ vector of market shares, and $\xi(\theta, s_t, x_t)$ is $J \times 1$ vector of residuals. Here, J denotes the number of products in each market. $\xi(\theta, s_t, x_t)$ is defined as the vector that satisfies

$$s(\theta, \xi(\theta, s_t, x_t), x_t) = s_t,$$

where the j^{th} entry of the $J \times 1$ vector of model implied market shares, $s(\theta, \xi_t, x_t)$, is given by

$$s_j(\theta, \xi_t, x_t) = \int \frac{e^{x'_{jt}\mu + \xi_{jt} + \sum_{k=1}^{K_2} x_{jt,k} \sqrt{\sigma_k^2} u_k}}{1 + \sum_{l=1}^J e^{x'_{lt}\mu + \xi_{lt} + \sum_{k=1}^{K_2} x_{lt,k} \sqrt{\sigma_k^2} u_k}} dF_v(v),$$

where $\xi_t = (\xi_{1t}, \dots, \xi_{Jt})'$, $x_t = (x_{1t}, \dots, x_{Jt})'$ and $x_{jt} = (x_{jt,1}, \dots, x_{jt,K_1})' \forall j \in \{1, \dots, J\}$, $\sigma^2 = (\sigma_1^2, \dots, \sigma_{K_2}^2)'$, $v = (v_1, \dots, v_{K_2})'$, and where $F_v(v)$ denotes the *cdf* of $N(0_{K_2}, I_{K_2})$. We note that $s_j(\theta, \xi_t, x_t)$ and, thus, $g(w_t, \theta)$ are not defined for σ^2 with $\sigma_k^2 < 0$ for some $k \in \{1, \dots, K_2\}$. For more details on the random coefficients logit model, we refer the reader to Section 3 in Ketz (2019).

Note that Assumptions 1–4 are equivalent to Assumption 1–3 in Ketz (2019). The latter are satisfied given the definition of the parameter space $\Phi(\theta)$ in Ketz (2019) (or $\ddot{\Phi}(\theta^*)$ using the notation in that paper), see Appendix D in Ketz (2019). Assumption 5 is satisfied taking, for example, $\gamma_0 = (\theta_0, \phi_0)$ with $\theta_0 = 0_p$ and ϕ_0 equal to an arbitrary element in $\Phi(\theta_0)$, assuming the latter to be non-empty.

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