DOCUMENT

DE TRAVAIL

N° 301

INCOMPLETE MARKETS, LIQUIDATION RISK,

AND THE TERM STRUCTURE OF INTEREST RATES

Edouard Challe, François Le Grand and Xavier Ragot

October 2010

BANQUE DE FRANCE

EUROSYSTÈME

DIRECTION GÉNÉRALE DES ÉTUDES ET DES RELATIONS INTERNATIONALES

INCOMPLETE MARKETS, LIQUIDATION RISK,

AND THE TERM STRUCTURE OF INTEREST RATES

Edouard Challe, François Le Grand and Xavier Ragot

October 2010

Incomplete markets, liquidation risk,

and the term structure of interest rates[*]

Edouard Challe   François Le Grand   Xavier Ragot[†]

[†]Challe: Ecole Polytechnique and Banque de France; edouard.challe@polytechnique.edu. Le Grand: EMLyon Business School; legrand@em-lyon.com. Ragot: Banque de France and PSE; xavier.ragot@banque-france.fr.

Abstract

We analyze the term structure of real interest rates in a general equilibrium model with incomplete markets and borrowing constraints. Agents are subject to both aggregate and idiosyncratic income shocks, which latter may force them into early portfolio liquidation in a bad aggregate state. We derive a closed-form equilibrium with limited agent heterogeneity (despite market incompleteness), which allows us to produce analytical expressions for bond prices and returns at any maturity. The attractiveness of bonds as liquidity makes aggregate bond demand downward-sloping, so that greater bond supply raises both the level and the slope of the yield curve. Moreover, time-variations in liquidation risk are shown to help explain the rejection of the Expectations Hypothesis.

Keywords: incomplete markets; yield curve; borrowing constraints.

JEL codes: E21; E43; G12.

Résumé

Nous analysons la structure par terme des taux d'intérêt dans un modèle d'équilibre général dans lequel les marchés financiers sont incomplets et où les agents font face à des contraintes de crédit. Les agents font face à la fois à des risques agrégés et à des risques idiosyncratiques non assurables. Nous dérivons une solution en forme fermée, dans un équilibre avec hétérogénéité limitée, malgré l'incomplétude des marchés. Ceci nous permet de dériver des solutions analytiques pour les prix et les rendements des obligations de différentes maturités. Nous trouvons que la demande d'obligation décroit avec le prix de celles-ci, de sorte qu'un accroissement du volume de dette publique augmente à la fois le niveau et la pente de la courbe des taux. On montre par ailleurs, que l'incomplétude des marchés contribue au rejet d'une prime de terme constante pour chaque maturité.

Mots-clés : Marchés incomplets, courbe des taux, contrainte de crédit

Codes JEL : E21; E43; G12.

This paper proposes a tractable general equilibrium model of the (real) term structure in which financial markets are incomplete and where government bonds are held as a buffer stock against uninsurable labor income shocks. In contrast with the complete-market framework, in which Ricardian equivalence holds and hence non-distortionary changes in the public debt leave the yield curve unchanged, we find that the supply of bonds, the pervasiveness of individual income risk, and the way in which this latter interacts with the business cycle all affect the shape of the yield curve. Our basic assumption, which we share with much of the incomplete-market literature, is that agents cannot issue state-contingent securities or debt instruments and thus have a specific "precautionary" motive for holding assets (See Bewley (1983), Scheinkman and Weiss (1986), Huggett (1993) and Aiyagari (1994), among others). The key novelty of our approach is the construction of an equilibrium that allows for an analytical characterization of bond prices at all maturities when idiosyncratic and aggregate labor income risks interact and where active asset trading takes place in equilibrium. Since our primary interest is in the way these sources of risk jointly affect the demands for specific maturities, our analysis focuses on the simple class of zero-coupon real bonds, but other assets involving additional sources of risk (e.g., asset income risk, inflation risk, etc.) could also be studied within this framework.

Our analysis yields four sets of results. We first study the effect on the yield curve of a change in the net supply of government bonds, financed by non-distortionary taxes. This change can be regarded as an exogenous variation in the amount of "aggregate liquidity", defined as the quantity of assets available to self-insure against idiosyncratic income shocks (see below for further discussion of the liquidity concept used here). While this change would not alter the yield curve under complete markets, aggregate bond demand is downward-sloping in our model: increasing the supply of bonds of any maturity lowers the price of all bonds, i.e., it raises the entire yield curve. This is easily understood from the liquidity role played by government bonds in our economy. In the presence of both idiosyncratic income risk and trade restrictions (i.e., debt limits), high-income agents hold bonds of any maturity for precautionary purposes. In this context, more liquidity reduces the attractiveness of bonds and their equilibrium price. Since bonds of various maturities are imperfect substitutes for each other, raising the supply of one particular type of bond will lower the price of all bonds.

Our second result is that a larger bond supply steepens the yield curve by affecting relative prices, i.e., the risk premia associated with bonds of different maturities. In our model, risk premia

differ across bonds because agents may be forced to liquidate assets before maturity, when their selling price is low (due to a bad realization of aggregate uncertainty). Since the risk of early liquidation increases with the maturity of the bond, long bonds command a greater premium than comparatively shorter bonds. Following an increase in the supply of bonds, the desirability of additional liquidity instruments decreases and the premium required to hold long, risky bonds rises more than that on short bonds. Hence both the level and the slope of the yield curve increase with the supply of bonds.

Both results are consistent with a number of recent empirical findings that are at odds with the Ricardian equivalence property implied by the frictionless, complete-markets framework. While early empirical work (such as summarized by Elmendorf and Mankiw (1999)) failed to reach a consensus about the relationship between interest rates and the supply of government bonds, recent studies have been more conclusive. For example, Laubach (2009) reports that an increase in both public debt and fiscal deficits significantly raises interest rates on government bonds. Similarly, Krishnamurthy and Vissing-Jorgensen (2008) find that the size of public debt negatively affects the spread between corporate and Treasury bond yields, and explain this effect by a liquidity-based demand for government bonds. In a related contribution, Longstaff (2004) measures the liquidity premium on U.S. Treasury bonds prices and finds the supply of such bonds to be the most significant source of variation in the liquidity premium. While less work has specifically addressed the relationship between the quantity of government bonds and the slope of the yield curve, findings there are consistent with the basic predictions of our model. For example, Reinhart and Sack (2000) find that (projected) government surpluses are significantly negatively correlated with the slope of the yield curve in OECD countries. Similarly, Dai and Philippon (2006) show that an increase in the government deficit-to-GDP ratio raises long yields more than short ones.

Our third result pertains to the identification of the channels through which market incompleteness contributes to the rejection of the Expectation Hypothesis. More specifically, we show that time-variations in idiosyncratic risk generate or amplify time-variations in term premia, one of the most basic regularities in the empirical term-structure literature (e.g., Campbell and Shiller (1991), Donaldson, Johnsen, and Mehra (1990)).

Last, we derive some of the welfare implications of our liquidity-constrained model. We first show that increasing the quantity of government bonds raises welfare (both ex post and ex ante) only if agents are sufficiently patient; if they are not, the aggregate welfare gains associated with higher

liquidity may be offset by the (potentially large) fall in utility that some agents suffer from higher taxes. Second, while more generous unemployment insurance always increases ex ante welfare, an increase in social contributions incurred by currently employed agents may lower their utility if they are not sufficiently patient to value the future utility gains from the associated insurance scheme.

As far as we are aware, our framework is the first "liquidity-based" general equilibrium asset pricing model in which the entire yield curve, including bonds of arbitrarily long maturities, can be priced. One potential explanation for the lack of such a framework is the inherent complexity of infinite-horizon, incomplete markets models with a large number of assets. On the one hand, market incompleteness implies that agents' wealth and optimal decisions depend on the whole history of idiosyncratic shocks that each agent has faced, so that infinitely many agent types asymptotically co-exist in the economy; this usually precludes the derivation of analytical expressions and general conclusions regarding asset prices. On the other hand, computational methods, when applied to economies hit by aggregate shocks, can only handle a small number of assets, typically one or two (e.g., Den Haan (1996), Heaton and Lucas (1996), Krusell and Smith (1997), Heathcote (2005)). Our framework allows us to get around this issue by endogenously generating finite-dimensional cross-sectional distributions of wealth states and agent types. The central simplifying feature of our analysis is that we focus on equilibria with "full asset liquidation", i.e., where agents immediately face a binding borrowing constraint when their current income falls. Consequently, at this corner solution agents endogenously choose to liquidate their bond portfolio and thus no longer affect asset prices. Note that while we focus on the yield curve implications of this framework here, our hope is that it is flexible enough to be applicable to a much broader range of issues pertaining to general equilibrium-based asset pricing and, more generally, to the macroeconomics of heterogeneous agents.

After a brief discussion of the literature, we introduce our framework in Section 2. Section 3 describes the full asset liquidation equilibrium, and establishes the associated existence conditions. Section 4 studies the impact of changes in bond supplies on the level and the slope of the yield curve. Section 5 analyzes the effects of time-variations in idiosyncratic risk for the shape of the yield curve and the cyclicality of bond premia. The welfare properties of the model are then derived in Section 6, while Section 7 concludes.

1 Related literature

The notion of "liquidity" is not devoid of ambiguity, so it is important to differentiate clearly the definition used here from other common uses in the asset-pricing literature. In our model, liquidity is made up of all assets that allow agents to transfer wealth across time to meet future and uninsurable spending needs (earlier models making use of this liquidity concept include Woodford (1990), Holmström and Tirole (1998) and (2001), Kehoe and Levine (2001), and Fahri and Tirole (2008)). By construction, in frictionless markets the demand for store of values by agents with transitorily high incomes is adequately met by the supply of "inside liquidity" (i.e., private debt) issued by agents with transitorily low incomes. In markets with frictions, however, private asset issuances are restricted and "outside liquidity" (here government bonds) partly substitutes for the missing financial instruments; then, the supply of aggregate liquidity typically constrains competitive allocations and has first-order implications for the desirability and price of liquidity instruments. This approach thus differs from recent work that refers to liquidity as the ease with which agents may trade assets in decentralized markets (e.g. Duffie, Garleanu, and Pedersen (2005), Lagos and Rocheteau (2007) and (2009), Vayanos and Weil (2008)). It also differs from that in work which identifies "illiquidity" with limited asset-market participation arising from margin constraints (as in Brunnermeier and Pedersen (2009), and Chowdhry and Nanda (1998)) or entry costs (e.g., Pagano (1989)).

The idea that markets incompleteness can help explain asset-pricing puzzles was first explored in finite horizon economies. Following the lead of Mankiw (1986) and Weil (1992), who focused on stock returns, Heaton and Lucas (1992), and more recently Holmström and Tirole (2001), have used three-period models to analyze the effects of interactions between idiosyncratic and aggregate risks on the yield curve. These models provide important insights into these interactions but leave open the question of how they affect the yield curve over a long horizon.

There is a key class of infinite-horizon, incomplete-markets models where analytical expressions for the price of long assets can be obtained: those where the no-trade equilibrium prevails. Such is the case in Constantinides and Duffie's (1996) model of the equity premium. A more recent contribution is Krussel, Mukoyama, and Smith (2008), who study asset prices in the autarkic equilibrium of a liquidity-constrained economy. In their model agents value assets (including bonds of different maturities) for their ability to transfer wealth across periods and smooth out idiosyncratic income shocks, but do not trade in equilibrium. In contrast, since our focus in on how the quantity of

liquidity available in the market allows this intertemporal smoothing to take place, and thereby affects the desirability of bonds, our results require active trading of positive net bond supplies by agents following idiosyncratic income shocks.

In order to organize, and put structure on, their empirical findings summarized above, Krishnamurthy and Vissing-Jorgensen (2008) develop a theoretical model of liquidity demand that generates a downward-sloping demand for government bonds, as in our model. The central difference between the two approaches is that their aggregate demand for liquidity is based on the assumption that government bonds directly enters agents' utility, while our model seeks to provide micro-foundations to the liquidity motive for holding bonds based on financial frictions.

Because of their intrinsically non-Ricardian nature, overlapping generations (OLG) models are natural competitors to the asset-pricing framework with infinitely-lived agents that we develop below. For example, OLG models usually have the property that increasing the stock of government bonds can raise the equilibrium interest rate when agents are constrained by the supply of stores of value in the economy (Barro (1974)). However, in our model the risk premia on various maturities are related to the risk of having to liquidate assets (following an unfavorable idiosyncratic income shock) precisely when the economy is in recession (when the aggregate shock is also unfavorable). Aside from the fact that their time scale is ill-suited to the study of business cycle phenomena, basic OLG models with two-period life-spans cannot generate such premia because liquidation occurs with certainty in later-life (and hence there is no liquidation risk). On the other hand, while multi-period OLG models have realistic time scales and can in principle allow for random liquidation before death, they typically cannot be solved in closed form and thus the number of assets under scrutiny must remain small (e.g., Storesletten, Telmer, and Yaron (2007), Gomes and Michaelides (2008)).

Finally, a popular approach in interest rate modeling is to assume the absence of arbitrage and directly consider an exogenous pricing kernel to price bonds of various maturities (see Dai and Singleton (2006), for an overview). Some recent papers following this partial equilibrium tradition introduce macroeconomic factors as determinants of the pricing kernel (see Ang and Piazzesi (2003), on monetary policy, and Dai and Philippon (2006), on fiscal policy). Other papers assume an ad hoc demand for each maturity, the so-called "clientele effect" (e.g., Vayanos and Vila (2007)). In contrast, our approach here is to derive the demand for, and equilibrium price of, bonds from utility maximization.

2 The economy

We consider a discrete-time economy populated by infinitely-lived households who face two sources of risk: an aggregate technology shock that affects the productivity of employed households (Section 2.1); and an uninsurable individual shock that causes households to switch idiosyncratically between employment and unemployment (Section 2.2). Agents may trade real riskless bonds of different maturities (Section 2.3) and use them to self-insure against the income variability induced by changes in employment status (Section 2.4). In equilibrium, the total supply of bonds equals the economywide demand by heterogenous households (Section 2.5).

2.1 Aggregate states

The economy is characterized at every date $t = 0, 1, \ldots$ by an aggregate state h_t, where $h_t = h$ if this state is "high" and $h_t = l$ if it is "low". Let $h^t = \{h_0, \ldots, h_t\}$ denote the history of aggregate states from date 0 to date t and H^t the set of all possible histories. The aggregate state evolves according to a first-order Markov chain with the following transition matrix:

$$T = \begin{bmatrix} \pi^h & 1 - \pi^h \\ 1 - \pi^l & \pi^l \end{bmatrix}.$$

We denote by $\eta^h \equiv \left(1 - \pi^l\right) / \left(2 - \pi^l - \pi^h\right)$ and $\eta^l \equiv 1 - \eta^h$ the unconditional fractions of time spent in state h and l, respectively. We also make the following assumption:

Assumption A $\pi^h + \pi^l > 1$.

Assumption A requires that aggregate states are sufficiently persistent; while not necessary for the derivation of most of our results, it allows us to avoid discussing uninteresting cases arising from rapidly alternating states.[1]

The invariant distribution associated with the transition matrix T is denoted $\Phi = [\Phi_l \ \Phi_h]$, and we assume that the probability distribution across both aggregate states at date 0 is Φ. We denote by ν_t the probability measure over histories up to date t, consistent with the transition matrix T and the initial distribution Φ: $\nu_t : H^t \to [0, 1], t = 0, 1, \ldots$.

---

[1]Estimated Markov switching models are consistent this assumption. For example, Hamilton ((1994), chap 22) finds $\pi^h + \pi^l = 1.65$ at a quarterly frequency for the US economy.

2.2 Individual states

The economy is populated by a continuum $I = [0, 1]$ of agents, with mass 1. In every period, each agent can be in either of two states, "employed" or "unemployed". Let e_t^i denote the status of agent i at date t, where $e_t^i = 1$ if the agent is employed and $e_t^i = 0$ if the agent is unemployed. Each agent's employment status evolves independently according to a first-order Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \alpha & 1 - \alpha \\ 1 - \rho & \rho \end{bmatrix}, \ (\alpha, \rho) \in (0, 1)^2,$$

where $\alpha \equiv \Pr(e_{t+1}^i = 1 \,|\, e_t^i = 1)$ and $\rho \equiv \Pr(e_{t+1}^i = 0 \,|\, e_t^i = 0)$. Note that our baseline specification for Π implies that changes in individual status are not affected by the aggregate state h^t. We explicitly introduce such a dependence in Section 5, with the natural motivation that business cycle shocks may affect probabilities to transit into and out of unemployment.

The initial probability distribution is represented by a row vector $\omega_0 = [\ \omega_0^e \ \ \omega_0^u \ ]$, i.e. ω_0^e (respectively ω_0^u) is the probability at date 0 that agent i is employed (unemployed). Given this simple Markovian structure, the probability distribution at date t, which is $\omega_0 \Pi^t$, converges for $t \to \infty$ to the invariant distribution $\omega = [\omega^e \ \omega^u]$, where $\omega^u \equiv (1 - \alpha) / (2 - \rho - \alpha)$ is the asymptotic unemployment rate and $\omega^e \equiv 1 - \omega^u$ is the employment rate. To simplify the exposition, we assume that the unemployment rate equals its asymptotic level from date 0 on, i.e., $\omega_0 = \omega$.

The history of individual shocks up to date t is denoted by $e^{i,t}$, where $e^{i,t} = \{e_0^i, \dots, e_t^i\} \in \{0, 1\}^t = E^t$. E^t is the set of all possible individual histories up to date t, and $\mu_t^i : E^t \to [0, 1]$, $t = 0, 1, \dots$ denotes the probability measure of individual histories, consistent with the transition matrix Π and the initial probability distribution ω. For example, $\mu_t^i (e^{i,t})$ is the probability that agent i experiences the history $e^{i,t}$ at date t.

The individual and aggregate states affect the economy as follows. Employed agents freely choose their labor supply and produce $z_t = z^l$ or $z_t = z^h$ units of goods per unit of labor in states l and h, respectively, with $z^h \geq 1 \geq z^l > 0$. Unemployed agents get a fixed quantity of "home production" of $\delta > 0$. The following assumption ensures that along our equilibrium the unemployed consume less (and thus enjoy higher marginal utility) than the employed in both aggregate states.

Assumption B $1/z^l < u'(\delta)$.

2.3 Assets and market structure

The only assets that agents may trade are riskless, zero-coupon government bonds that pay off one good unit at maturity. Bond maturities vary from 1 to $n \geq 1$, where n may be arbitrarily large. A bond of maturity $k > 1$ at date t becomes a bond of maturity $k - 1$ at date $t + 1$, and eventually yields 1 at date $t + k$. The price of this bond at date t is $p_{t,k}(h^t)$, and we define the price of a bond of maturity 0 by its payoff, i.e., $p_{t,0}(h^t) = 1$.

Our assumption that government bonds are the only tradable assets has three significant implications. First, there is no asset providing a payoff contingent on agents' idiosyncratic employment status; unemployment risk is thus entirely uninsurable. Second, agents are prevented from issuing securities in both aggregate states, so the quantity of available securities does not depend on the aggregate state. Third, there is no security offering a payoff contingent on the aggregate state.[2]

There is no public consumption, so government expenditures exactly equal payoffs owed to holders of bonds reaching maturity. At a given date t, the n bonds issued at $t - 1$, $t - 2$,..., $t - n$ with respective maturities 1, 2,..., n mature. Bond payoffs are financed by both new bond issuances and taxes. At each date t, a quantity $A_{t,k}$ of bonds paying 1 at date $t+k$ is issued at price $p_{t,k}(h^t)$. The government levies a lump sum tax $\tau_t(h^t)$ on all agents.[3] Since there is a continuum of agents of mass 1, the government budget constraint is:

$$\sum_{k=1}^{n} p_{t,k}(h^t) A_{t,k} + \tau_t(h^t) = \sum_{k=1}^{n} A_{t-k,k}. \tag{1}$$

The aggregate supply of securities of a given maturity is composed of newly-issued bonds of that maturity *plus* longer bonds issued earlier and coming closer to maturity. At date t, a total quantity $B_{t,k}$ of bonds reaching maturity at date $t + k$ is available in the market, where

$$B_{t,k} \equiv \sum_{j=0}^{n-k} A_{t-j,k+j}. \tag{2}$$

For simplicity we assume that the quantity of bonds of a given maturity is constant (i.e. $B_{t,k} = B_k$, $\forall t \geq 0$), which is equivalent to constant issuances (i.e. $A_{t,k} = A_k$, $\forall t \geq 0$). It implies that the

---

[2]These properties are central in the literature on liquidity-constrained economies since the seminal work of Bewley (1980). See also Kehoe and Levine (2001), and the references therein.

[3]We are thus assuming that unemployed agents also pay taxes. This ensures that the government does not provide income insurance via the tax system (i.e., by limiting the consumption fall suffered by agents hit by a bad idiosyncratic shock), but only via its control of the stock of outside liquidity. Assuming that these agents do not pay taxes does not affect our results.

lump sum tax satisfying (1) is:

$$\tau_t(h^t) = \sum_{k=1}^{n} \left(p_{t,k-1}(h^t) - p_{t,k}\left(h^t\right) \right) B_k.\qquad(3)$$

2.4 Agents' behavior

Each agent $i \in I$ has preferences over consumption and labor that are described by the subjective discount factor $\beta \in (0,1)$ and the instant utility function $u\left(c\right) - l$, where l is labor supply and u is a C^2 function satisfying $u'\left(.\right) > 0$, and $u''\left(.\right) < 0$ (this follows Scheinkman and Weiss (1986)). We denote the quantity of k-period bonds held by agent i at the end of period t by $b^i_{t,k}$, and the corresponding bond holdings at the beginning of period 0 by $b^i_{-1,k}$ (specific assumptions about initial bond holdings will be made in Section 3.3). Agent i's problem consists in choosing the sequences of consumption, $c^i_t\left(h^t, e^{i,t}\right)$, labor supply $l^i_t\left(h^t, e^{i,t}\right)$, and bond holdings $\left(b^i_{t,k}\left(h^t, e^{i,t}\right)\right)_{1 \leq k \leq n}$, defined over $H^t \times E^t$, so as to solve:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in H^t} \nu_t\left(h^t\right) \sum_{e^{i,t} \in E^t} \mu^i_t\left(e^{i,t}\right) \left(u\left(c^i_t\left(h^t, e^{i,t}\right)\right) - l^i_t\left(h^t, e^{i,t}\right)\right),\qquad(4)$$

$$\text{s.t.}\quad c^i_t\left(h^t, e^{i,t}\right) + \tau_t(h^t) + \sum_{k=1}^{n} p_{t,k}\left(h^t\right) b^i_{t,k}\left(h^t, e^{i,t}\right) = \sum_{k=1}^{n} p_{t,k-1}\left(h^t\right) b^i_{t-1,k}\left(h^{t-1}, e^{i,t-1}\right)$$

$$+ e^i_t z_t l^i_t\left(h^t, e^{i,t}\right) + \left(1 - e^i_t\right)\delta,\qquad(5)$$

$$b^i_{t,k}\left(h^t, e^{i,t}\right) \geq 0, \qquad \text{for } k = 1, \ldots, n,\qquad(6)$$

$$c^i_t\left(h^t, e^{i,t}\right), \quad l^i_t\left(h^t, e^{i,t}\right) \geq 0,\qquad(7)$$

$$\lim_{t \to \infty} \beta^t u'\left(c_t\left(h^t, e^{i,t}\right)\right) b^i_{t,k}\left(h^t, e^{i,t}\right) = 0, \qquad \text{for } k = 1, \ldots, n.\qquad(8)$$

Equation (5) is agent i's budget constraint at date t: total income is made up of the sale value of the bond portfolio as well as labor income if $e^i_t = 1$ and home production if $e^i_t = 0$; this income is used to pay taxes and purchase consumption goods and bonds of various maturities. Inequality (6) reflects the fact that agents cannot issue securities. Finally, conditions (7) and (8) are the non-negativity and transversality conditions, which are always satisfied in the equilibrium we consider.

Let $\varphi^i_{t,k}$ be the Lagrange multipliers associated with the borrowing constraints (6), which are positive functions defined over $H^t \times E^t$. From the Lagrangian function, the first-order conditions

of this problem are:

$$
\begin{cases}
u'\left(c_t^i\left(h^t, e^{i,t}\right)\right) & = & 1/z_t & \text{if} & e_t^i = 1, \\
l_t^i\left(h^t, e^{i,t}\right) & = & 0 & \text{if} & e_t^i = 0,
\end{cases}
\tag{9}
$$

$$
u'\left(c_t^i\left(h^t, e^{i,t}\right)\right) p_{t,k}\left(h^t\right) = \beta E_t\left[u'\left(c_{t+1}^i\left(h^{t+1}, e^{i,t}\right)\right) p_{t+1,k-1}\left(h^{t+1}\right)\right] + \varphi_{t,k}^i\left(h^t, e^{i,t}\right).
\tag{10}
$$

Equation (9) describes the agent's optimal labor supply: when the agent is employed $(e_t^i = 1)$, the marginal utility of consumption is equal to the marginal disutility of labor over the marginal product of labor, while labor supply is zero when the agent is unemployed $(e_t^i = 0)$. The Euler equation (10) sets the marginal cost of acquiring one unit of bonds of each maturity today equal to the marginal gain associated with its payoff tomorrow (here the $E_t[\cdot]$ operator denotes expectations over both aggregate and idiosyncratic states, conditional on the information available at date t, i.e., h^t and $e^{i,t}$). When the shadow cost of the borrowing constraint is positive, meaning that the constraint is binding $(\varphi_{t,k}^i\left(h^t, e^{i,t}\right) > 0)$, the agent i would increase his expected utility and to issue k-period bonds (but is prevented from doing so, by assumption).

2.5 Market clearing and equilibrium definition

We denote by $\Lambda_t : \left(\mathbb{R}^+\right)^n \times E \to [0,1]$ the probability measure describing the distribution of agents across individual wealth and productivity in period t. For example, $\Lambda_t\left(b_1, \ldots, b_n, 1\right)$ denotes the measure of agents who are employed $(e_t^i = 1)$ and hold the portfolio $b_1, \ldots, b_n$ This measure depends on the history of shocks and the initial distribution of agents, denoted Λ_0. The market-clearing condition sets the aggregate demand for bonds equal to the exogenous supply of bonds for all maturities, i.e.,

$$
\int_{(b_1, \ldots, b_k, \ldots, b_n, e) \in (\mathbb{R}^+)^n \times E} b_k\, d\Lambda_t\left(b_1, \ldots, b_k, \ldots, b_n, e\right) = B_k, \quad \forall k = 1, \ldots, n.
\tag{11}
$$

By Walras Law, the good market clears when all bond markets clear. We are now in a position to define the equilibrium in our economy.

Definition 1 *For an initial distribution of bond holdings and employment status Λ_0, an equilibrium consists of individual choices $\{c_t^i\left(h^t, e^{i,t}\right), \left(b_{k,t}^i\left(h^t, e^{i,t}\right)\right)_{k=1,\ldots,n}, l_t^i\left(h^t, e^{i,t}\right)\}_{t=0,\ldots,\infty}$ and prices $\{(p_{t,k})_{k=1,\ldots,n}\}_{t=0,\ldots,\infty}$ such that:*

1. *Given prices, individual choices solve the agents' problem (i.e., equations (4)-(8) hold);*

2. Λ_t *evolves consistently with individual policy rules and transition matrices for individual and aggregate states;*

3. *All bond markets clear at all dates (i.e., equation (11) holds).*

3 Equilibrium with full asset liquidation

One implication of our particular market structure is that government bonds serve as a buffer allowing agents to partially offset the lack of full credit and insurance markets. Many models of this class imply smooth portfolio-rebalancing in equilibrium: high-income agents gradually build up their asset wealth, while low-income agents gradually decumulate assets (e.g. Scheinkman and Weiss (1986), or Aiyagari (1994)). Since we focus on the implications of liquidation risk for bond pricing, we construct our equilibrium in such a way that agents liquidate assets when a bad idiosyncratic income shock hits (i.e. $b_{t,k}^i\left(h^t, e^{i,t}\right) = 0$ for $k = 1, \ldots, n$ if $e_t^i = 0$). All unemployed agents therefore face a binding borrowing constraint (i.e., their fall in labor income is not entirely offset by the liquidation value of the portfolio), while only employed agents participate in bond markets and thus affect bond prices. This focus on an equilibrium with full liquidation drastically reduces the number of agent types in the economy, thereby allowing us to study bond pricing analytically for an arbitrarily large number of maturities.

Our equilibrium is obtained by construction: we first conjecture, and then derive, a sufficient condition for the existence of an equilibrium along which employed agents are never borrowing-constrained (i.e. they are willing to end the period with positive asset holdings), while unemployed agents always are (i.e. they would like to *borrow*, rather than save). This joint conjecture can formally be written as, for all $k = 1, \ldots, n$:

$$e_t^i = 1 \Rightarrow \varphi_{t,k}^i\left(h^t, e^{i,t}\right) = 0 \quad \text{and} \quad e_t^i = 0 \Rightarrow \varphi_{t,k}^i\left(h^t, e^{i,t}\right) > 0. \tag{12}$$

From now on, we simplify notation by omitting the references to aggregate and individual histories when no ambiguity arises from doing so.

3.1 Consumption levels and the pricing kernel

We first consider the consumption of an unemployed agent in period t. If the agent was employed in the previous period, then from the budget constraint (5) and conjecture (12) the agent earns δ as well as the liquidation value of his portfolio. Since the agent is borrowing-constrained, he will consume his entire income net of taxes, i.e.,

$$c_t^i = \sum_{k=1}^{n} p_{t,k-1} \, b_{t-1,k}^i + \delta - \tau_t \; (> 0). \tag{13}$$

Currently unemployed agents who were already unemployed in the previous period will already have liquidated their assets. Their consumption, which is identical for all such agents and denoted c_t^{uu}, is simply:

$$c_t^{uu} = \delta - \tau_t \; (> 0). \tag{14}$$

We now turn to employed agents. From the intratemporal optimality conditions (equation (9)), their consumption is identical for all such agents and given by:

$$c_t^e = u'^{-1} \left(1/z_t \right) \; (> 0). \tag{15}$$

If an employed agent is employed in the next period, which occurs with probability α, then his marginal utility of consumption will be $1/z_{t+1}$ (see (9)). If the agent moves into unemployment next period, then his marginal utility will be $u' \left(c_{t+1}^i \right)$, where by construction c_{t+1}^i is given by (13). Then, substituting these marginal utilities into (10) under conjecture (12), the Euler equations characterizing optimal bond holdings by employed agents are, for $k = 1, \ldots, n$:

$$\frac{p_{t,k}}{z_t} = \alpha \beta E_t \left[\frac{p_{t+1,k-1}}{z_{t+1}} \right] + (1-\alpha) \beta E_t \left[u' \left(\sum_{j=1}^{n} p_{t+1,j-1} \, b_{t,j}^i + \delta - \tau_{t+1} \right) p_{t+1,k-1} \right]. \tag{16}$$

We restrict our attention to symmetric equilibria where bond holdings of all maturities are identical across employed agents. From (16), the bond demands $b_{t,j}^i$ are functions of aggregate variables only (and thus identical across employed agents in symmetric equilibrium), and we denote by $b_{t,k}^e$ the quantity of k–period bonds held by any employed agents at date t. Since the total supply of such bonds is B_k, market clearing requires that $b_{t,k}^e = B_k/\omega^e$, $k = 1, \ldots, n$. Then, using (3) we

14

may rewrite (16) as follows:

$$p_{t,k} = E_t \left[m^e_{t+1} \, p_{t+1,k-1} \right],\tag{17}$$

where the (unique) pricing kernel generating bond prices is:

$$m^e_{t+1} = \alpha \, \beta \, \frac{z_t}{z_{t+1}} + (1 - \alpha) \, \beta \, z_t \, u' \left(\delta + \frac{1 - \omega^e}{\omega^e} \sum_{j=1}^{n} p_{t+1,j-1} \, B_j + \sum_{j=1}^{n} p_{t+1,j} \, B_j \right).\tag{18}$$

Equations (17)–(18) pin down the price of k–period bonds as a function of the current and next aggregate states, all future prices, and, crucially, aggregate bond supplies (note that E_t is now, by a slight abuse of notation, the expectation over aggregate uncertainty only). The pricing kernel (18) is the sum of two distinct terms that encompass the two possible employment states of employed agents in the next period. If the agent stays employed, which occurs with probability α, then labor supply adjusts until the marginal utility of consumption equals $1/z_{t+1}$; this would be the only term to appear were markets to be complete and were agents fully able to smooth out their idiosyncratic income shocks. The second term in the right-hand side of (18) reflects the liquidation risk that is associated with the possibility that the agent be hurt by an unfavorable change in employment status. Bond quantities directly affect prices through their effect on the value of the liquidated portfolio, which in turn feeds back into current equilibrium prices.

3.2   Conjectured price structure

We focus on the equilibrium where bond prices only depend on the realization of aggregate shocks. From the literature on asset pricing with finite state space (e.g., Mehra and Prescott (1985)), we conjecture the following expression for bond prices:

$$\forall t \geq 0, \forall k \in \{0, \ldots, n\}, \forall s \in \{h, l\} \quad p^s_k = C^s_k \, z^s,\tag{19}$$

where the C^s_ks are constants, and where $C^s_0 = 1/z^s$ from our definition of a zero-maturity bond price by its payoff (see Section 2.3). This price structure entails a form of stationarity, since bond prices depend only on their maturity and the current aggregate state. In consequence, there are two yield curves, one for each value of the aggregate state. Our existence proof will consist in showing that such a stationary equilibrium exists.

The yield-to-maturity of a bond with maturity $k = 1, \ldots, n$ in state $s = h, l$ can be defined by

the usual logarithmic expression: $r_k^s = -k^{-1} \ln p_k^s$. The average yield curve is generated by average yields, i.e., the sum of yields-to-maturity in each aggregate state weighted by their unconditional frequency: $r_k = \eta^h r_k^h + \eta^l r_k^l$.

3.3 Existence of the equilibrium

The existence of the full asset liquidation equilibrium is proved in two steps. We first derive our existence result in an economy with bonds in zero net supply (so that no trade takes place) and without aggregate shocks. We then show that the yield curve is continuous with respect to the introduction of small, positive bond supplies and a small degree aggregate uncertainty, so that our existence result directly extends to this more general case.

3.3.1 Existence conditions

The stationary distribution with four agent types was constructed under the conjecture that unemployed agents are always borrowing-constrained, while no employed agent is. We now derive the conditions under which this holds.

Conditions on agents' initial wealth. In order to avoid the unnecessary complications implied by the transitional adjustment of agents' wealth towards the invariant cross-sectional distribution, we assume that at the beginning of period 0 employed agents hold an initial quantity of bonds $b_{-1,k}^e = B_k/\omega^e$ with probability α, and hold no bonds with probability $1 - \alpha$. Unemployed agents, on the other hand, hold no bonds with probability ρ, and $b_{-1,k}^e = B_k/\omega^e$ bonds with probability $1 - \rho$. As a result, from an *ex ante* point of view, agents are employed with positive bond holdings with probability $\alpha \omega^e$, and unemployed with positive bond holdings with probability $(1 - \rho)\,\omega^u$. The initial joint distribution of employment status and bond holdings is thus identical to the stationary distribution.

Conditions on parameter values. We now derive the conditions ensuring that all unemployed agents are borrowing-constrained and hence do not participate in bond markets. Agents who are unemployed at both dates $t - 1$ and t consume $\delta - \tau_t$ (see (14)). If they become employed in the next period, which occurs with probability $1 - \rho$, then their marginal utility of consumption will be $1/z_{t+1}$ (see (15)). However, if they remain unemployed, which occurs with probability ρ, then from (14) their marginal utility of consumption will be $u'(\delta - \tau_{t+1})$. Hence condition (12), which

requires that the borrowing constraint bind for such agents, holds if and only if, for all $k = 1, \ldots, n$:

$$p_{t,k}\, u'\left(\delta - \tau\left(h_t\right)\right) > \beta\left(1 - \rho\right) E_t\left[\frac{p_{t+1,k-1}}{z_{t+1}}\right] + \beta\rho E_t\left[p_{t+1,k-1} u'\left(\delta - \tau_{t+1}\right)\right], \tag{20}$$

where τ_t is given by (3). On the other hand, agents who were employed at date $t - 1$ and who become unemployed at date t consume their home production δ plus the liquidation value of their bond portfolio. Again from equation (12), these agents face a binding borrowing constraint if and only if, for all $k = 1, \ldots, n$:

$$p_{t,k}\, u'\left(\delta + \sum_{j=1}^{n} p_{t,j-1}\frac{B_j}{\omega^e} - \tau_t\right) > \beta(1 - \rho)E_t\left[\frac{p_{t+1,k-1}}{z_{t+1}}\right] + \beta\rho E_t\left[p_{t+1,k-1} u'\left(\delta - \tau_{t+1}\right)\right]. \tag{21}$$

Since (21) implies (20), we only need to check that the equilibrium satisfies (21).

3.3.2   Existence of a no-trade equilibrium without aggregate shocks

If assets are in zero net supply, then there is no trade between agents and both the liquidation value of the portfolio and taxes will equal zero. Without aggregate uncertainty $z^h = z^l = 1$, equation (19) becomes $p_{t,k} = C_k$ (i.e., bond prices only depend on their maturity). Then, substituting (19) into (18) and (21) and rearranging, condition (21) becomes:

$$\left(\alpha + (1 - \alpha)u'\left(\delta\right)\right) u'\left(\delta\right) > 1 - \rho + \rho\, u'\left(\delta\right). \tag{22}$$

Since $u'\left(\delta\right) > 1$ by assumption B, the right hand side of (22) is maximum at $\rho = 1$, in which case (22) remains true for any feasible value of α; hence the no-trade equilibrium exists in the economy without aggregate risk. Finally, note that in the no-trade steady state the consumption levels of employed agents and agents falling into unemployment are $u'^{-1}\left(1\right)$ and δ, respectively (see (13)–(15)). By assumption B, the former is always greater than the latter, although they can be made arbitrarily close to each other; this is also the case when both aggregate uncertainty and the supply of bonds are sufficiently small.

3.3.3   Continuity of the yield curve w.r.t. bond supplies and aggregate shocks

We now introduce the following notation: $B \equiv \left[B_n \ldots B_1\right]^\top$ is the vector of bond quantities for the n maturities, $Z \equiv \begin{bmatrix} z^l & z^h \end{bmatrix}^\top$ the vector of productivities, and $C \equiv \begin{bmatrix} C_n^h & C_n^l \ldots C_0^h & C_0^l \end{bmatrix}^\top$ the

vector of price coefficients in both aggregate states and for the n maturities (see (19)). 1_n and 0_n are vectors of length n containing respectively only ones and zeros. We then have the following proposition:

Proposition 1 (Regularity of the yield curve) *i) If B is in the neighborhood of 0_n and Z in the neighborhood of 1_2, then C is a $\mathcal{C}^1$ function of B and of Z. ii) The equilibrium exists under the condition that both aggregate uncertainty and bond supply be small.*

All proofs are in the Appendix. The first part of Proposition 1 essentially states that, starting from a no uncertainty/zero net supply situation (i.e., where (22) holds), a gradual increases in aggregate risk or bond supplies does not cause the yield curve to jump. The second part of the proposition is a direct implication of this continuity result: as the equilibrium exists in the zero volume, no aggregate uncertainty case, the equilibrium also exists when volumes and aggregate risk are sufficiently small (that is, (21) holds). From now on, all our results are derived in the neighborhood of $B = 0_n$ and $Z = 1_2$. Moreover, and as indicated in the relevant propositions, several of our results are derived under the assumption that idiosyncratic uncertainty is small, in the sense that α is close to 1.[4]

4 Shape of the yield curve and the supply of bonds

This section analyses how the supply of bonds of different maturities affects the level and the slope of the yield curve when agents are exposed to liquidation risk. We first provide a simple example based on two bond maturities, i.i.d. aggregate shocks and quadratic utility. While the "long yield" (i.e., the yield on long-maturity bonds) is not properly defined in this setup, the latter allows us to illustrate the main workings of the liquidity-based demand for bonds in a particularly simple way. We then analyse the general case where bonds of arbitrarily long maturities co-exist with short-maturity bonds.

4.1 A simple example

Let us momentarily i) set the supply of bonds of maturity greater than two periods to zero, ii) restrict the structure of aggregate shocks so that $z = z^h = 1 + \varepsilon$ with probability $1/2$, $z = z^l = 1 - \varepsilon$

---

[4]This latter assumption is borne out by the data. For example, estimating transition rates between employment and unemployment from the U.S. Panel Study of Income Dynamics, Engen and Gruber (2001) find $\alpha = 0.97$ at quarterly frequency. Carrol *et al.* (2003) construct annual job-loss probabilities from the Current Population Survey and find a value for the average household of 0.02 (i.e., $\alpha = 0.98$).

with complementary probability, where ε is small, and iii) assume that instant marginal utility takes the form $u'(c) = u_1 - u_2 c$ (i.e., the utility function is quadratic), with $u_1, u_2 > 0$ and:

$$u_1 - u_2 \delta > 0.$$

The latter condition ensures that households who only consume their home production income enjoy positive marginal utility. Under our maintained assumption of small bond supply, the condition implies that the utility function is increasing and concave over the relevant range of individual consumption levels.

Calling $p_k = \left(p_k^l + p_k^h \right)/2$ the mean (across aggregate states) price of bonds of maturity k, we show in Appendix B that in the vicinity of zero bond supply ($B_1 = B_2 = 0$) we have:

$$\frac{\partial p_k}{\partial B_i} < 0, \ k, i = 1, 2. \tag{23}$$

The equation (23) states that an increase in the quantity of bonds of maturity 1 or 2 lowers the average price of both bonds. Alternatively, we have $\partial r_k / \partial B_i > 0$ (with $r_k = \left(r_k^h + r_k^l \right)/2$ here), i.e., an increase in the supply of bonds of either maturity raises the mean yield curve. This effect of bond supplies on the level of the yield curve directly follows from the liquidity role of bonds in our economy. Employed agents, who earn high labor income, wish to self-insure against unemployment risk; available bonds of either maturity will serve precisely this purpose. A smaller aggregate supply of bonds makes this liquidity support more valuable and produces higher bond prices (lower bond yields), relative to the situation where bonds are more abundant. Conversely, an increase in the supply of any type of bonds raises total liquidity, lowers the (expected) marginal utility associated with higher bond holdings and hence lowers the price of both bonds (raises both yields). In short, incomplete markets coupled with borrowing constraints make aggregate bond demand downward-sloping.

Moreover, the slope of the average yield curve is found to be:

$$S = \frac{1}{2} \left(\beta (1 - \alpha) u_2 (B_1 + B_2) + \beta \alpha \frac{1}{p_1} - \frac{1}{2} \right) \varepsilon^2 + o \left(\varepsilon^2 \right), \tag{24}$$

where $o(\varepsilon^2)$ is a function verifying $\lim_{\varepsilon \to 0} o(\varepsilon^2)/\varepsilon^2 = 0$. The slope S is composed of three terms, scaled by the variance of the aggregate risk ε^2. The first term, $\beta (1 - \alpha) u_2 (B_1 + B_2)/2$, represents

the *liquidation risk premium* that agents require for holding long bonds. In contrast to one-period bonds, which pay 1 for sure in the next period (i.e., regardless of the agent's employment status), long bonds have to be resold at an uncertain price in the next period if the agent becomes unemployed. The premium thus reflects the fact that agents care about the future joint realization of a bad individual income shock forcing them to sell the bond and a bad aggregate shock driving the bond price downwards. The premium increases with the individual risk of having to liquidate assets in the next period, $1 - \alpha$ (and disappears completely when individual risk is absent or fully insured, i.e., when $\alpha = 1$). This premium is also an increasing function of the total supply of bonds, $B_1 + B_2$, because the entire portfolio affects an agent's marginal utility and hence the benefit from liquidation; as the supply of bonds increases, the marginal value of additional bonds (the "price of liquidity") falls and agents' relative demand for long bonds, who have greater liquidation risk, falls as well.

The second term, $\beta\alpha/(2p_1)$, represents the risk premium for holding a long bond rather than a short bond until the next period in the case where the agent remains employed (which occurs with probability α). This premium moves negatively with the mean resell price of long bonds, p_1: the larger this price, the smaller this risk and implied premium.[5] The third term, $-1/4$, comes out of Jensen inequality.

We have derived these properties under extremely restrictive assumptions about agents' utility, the aggregate shock process and the number of bond maturities available in the economy. One particularly unpleasant feature of our example is that the "long yield" is not properly defined: it is identical to a two-period yield that fluctuates substantially with the aggregate state. We now show that our results hold in the context of the general framework developped in the preceeding sections, and in which we define the long yield as that of infinite-maturity bonds.

4.2 The general case

The following proposition summarizes some general properties regarding the shape of the yield curve in either aggregate states.

Proposition 2 (Ranking and monotonicity of yield curves) *Assume that α is close to 1. Then, i) The yield curve in the good aggregate state is increasing in maturity and lies strictly*

---

[5]Note that in the complete-market case we have $\alpha = 1$ (since there is no liquidation risk) and $p_1 = \beta$, so that $\beta\alpha/p_1$ is equal to 1 and is thus unresponsive to changes in the supply of bonds. In contrast, when $\alpha < 1$ the term $\beta\alpha/p_1$ depends on p_1 and hence on the B_is.

below that in the bad aggregate state, which is decreasing in maturity. ii) Yields in both states converge to a common limit r^{lim}.

The ranking of yield curves essentially results from the fact that employed agents in the good aggregate state earn higher incomes, and thus wish to save more and drive yields down, relative to the bad aggregate state. The monotonicity property follows from the stationary Markovian structure of aggregate shocks: conditionally on being in the good state, long bonds are riskier than short bonds because they are more likely to be traded after a move into the bad state has taken place; conversely, conditionally on being in the bad state long bonds bring the possibility of resale in the good state before having reached maturity. Finally, bonds of infinite maturity can be seen as bonds which pay one unit of goods in the (unconditional) mean aggregate state; hence the difference in yields across states for these bonds is zero.

We can now state our main results regarding the impact of bond supplies on the shape of the yield curve. Defining the slope of the yield curve $r^{\text{lim}} - r_1$ as the difference between the long and the average short yield, we have:

Proposition 3 (Impact of bond supplies on the shape of the curve) *i) Increasing the net supply of bonds of any maturity raises all bond yields. ii) Assuming that α is close to 1, increasing the supply of bonds of any maturity raises the slope of the yield curve.*

The first statement in Proposition 3 establishes that a greater bond supply of any maturity decreases the prices of bonds of *all* maturities (including the price of arbitrarily long bonds) in both aggregate states, and hence shifts the average yield curve upwards. This results hold whenever $\alpha < 1$; when $\alpha = 1$ no agent is ever constrained and this effect of bond supply on prices vanishes. The second statement, which applies to the difference between infinite-maturity and one-period bonds, relates to the change in relative bond prices induced by a change in the total supply of bonds. As the total quantity of bonds increases, agents are better able to self-insure, leading to lower bond prices. However, bonds of different maturities are imperfect substitutes for each other here, as their probability of being liquidated before maturity (due to a bad idiosyncratic state) differs, with early liquidation implying substantial business cycle risk (i.e., the risk of being sold at low price due to a bad aggregate state). In this context, an increase in the supply of bonds favors safer, shorter bonds.

Let us illustrate (but clearly without quantitative ambition) the effect of bond supplies on the shape of the yield curve by means of the following example. We proceed in two steps. First, we calibrate the model so that it generates a realistic average yield curve and a realistic marginal impact of bond supplies on the level of the yield curve. In so doing, we rely on Laubach's (2009) recent estimate according to which a one percentage point increase in the public debt to GDP ratio raises the level of the yield curve by three or four basis points. Second, we compute the implied effect of this marginal increase on the slope of the yield curve. We assume that $u\left(c\right) = Ac^{1-\sigma}/\left(1-\sigma\right)$, with $A = 0.4$ and $\sigma = 2.5$, $\beta = 0.94$, $\alpha = 0.9995$, $\rho = 0.5$, and $\delta = 0.2$, $\left(z^{l}, z^{h}\right) = \left(0.35, 0.6\right)$ and $\left(\pi^{l}, \pi^{h}\right) = \left(0.5, 0.8\right)$.[6] In this example a very small amount of liquidation risk (i.e., a value of α close to one) is sufficient to obtain a realistic effect of changes in public debt on the level of the yield curve. We look at average yields on one-year, fifteen-year and thirty-year bonds, after having checked numerically that our equilibrium exists for the parameters and bond supplies under consideration. Table 1 summarizes our results.

Interest rates	r_1	r_{15}	r_{30}
Benchmark economy (%)	2.815	5.440	5.544
Economy after a debt increase (%)	2.850	5.478	5.582
Yield variation after the debt increase (bp)	3.50	3.79	3.81

Table 1: Effect of debt increase on the yield curve

The first line of Table 1 provides mean yields under zero bond supply, our benchmark economy. The second line displays the equilibrium values of the same yields after debt-to-GDP ratio has been raised by one percentage point. The difference in yields, expressed as basis points (bp), appears in the third line. Note that the slope of the yield curve (i.e., the difference between yields on thirty-period and one-period bonds) is raised by 0.3 bp after the increase in bond supplies, which is roughly one tenth of the level effect under our calibration.

---

[6]In this numerical illustration we relax our normalizing assumption $z^{h} > 1 > z^{l}$. This does not affect our results.

5   Time-varying liquidation risk and the Expectations Hypothesis

We have thus far restricted our attention to equilibria where the aggregate state only affects labor productivity. One salient feature of business cycles is that aggregate shocks also affect the unemployment rate and thus the probabilities of transiting into and out of unemployment. We now study how time-variations in unemployment probabilities and the implied liquidation risk affect the yield curve. To focus on the empirically relevant case, we assume in this section that $\pi^h > \pi^l$, that is, relatively long booms are broken by shorter recessions.

The following structure allows us to keep the analysis tractable. Assume that the individual probability of staying employed in the next period only depends on the current aggregate state: $\alpha^s \equiv \Pr(e^i_{t+1} = 1 \,|\, e^i_t = 1, h_t = s)$, $s = l, h$. This transition rate will affect the employment rate in the next period, and we assume that this latter rate can only take two values, denoted $\omega^{e,s}$, $s = l, h$. If, for instance, the aggregate state is h at date t, then $\alpha_t = \alpha^h$ and $\omega^e_{t+1} = \omega^{e,h}$. A natural assumption is that the bad aggregate state is associated with both higher unemployment and greater unemployment risk, i.e., $\omega^{e,h} > \omega^{e,l}$ and $\alpha^h > \alpha^l$.

Simple flow accounting implies that the probabilities of exiting unemployment consistent with this joint assumption only depend on the current s and past aggregate state κ and are given by the solution to the following system, given $\alpha^s, \omega^{e,\kappa}$ where $s, \kappa = l, h$:

$$\omega^{e,h} = \omega^{e,h}\alpha^h + \left(1 - \rho^{hh}\right)\left(1 - \omega^{e,h}\right), \qquad \omega^{e,h} = \omega^{e,l}\alpha^h + \left(1 - \rho^{lh}\right)\left(1 - \omega^{e,l}\right),$$
$$\omega^{e,l} = \omega^{e,l}\alpha^l + \left(1 - \rho^{ll}\right)\left(1 - \omega^{e,l}\right), \qquad \omega^{e,l} = \omega^{e,h}\alpha^l + \left(1 - \rho^{hl}\right)\left(1 - \omega^{e,h}\right),$$

where $\rho^{\kappa\zeta}$ $(\kappa, \zeta = l, h)$ is the probability of remaining unemployed in the next period when the current aggregate state is ζ and the aggregate state in the previous period was κ. For example, the first equation states that an economy that was in state h at dates t and $t-1$ has an employment rate of $\omega^{e,h}$ at date t and will have the same employment rate at date $t+1$ (the left-hand side); given that a share α^h of currently employed agents will stay so in the next period, it must be the case that a share ρ^{hh} of currently unemployed agents will transit into employment, in order to produce an employment rate of exactly $\omega^{e,h}$ at date $t+1$. By assumption, the same employment rate will prevail at date $t+1$ if $h_t = h$ but $h_{t-1} = l$, as stated in the second equation.

Our bond pricing equations remain similar to (17), except that α and ω^e are now time-varying. This alters bond prices in two ways. First, time-variations in α will cause changes in idiosyncratic

23

unemployment risk, and thus in the precautionary demand for bonds by employed agents. Second, time-variations in ω^e will alter the number of agents who participate in bond markets and thus the quantity of bonds held by any single agent in equilibrium. Hence both the demand and the (per agent) supply will vary along the business cycle.

We conjecture that the price of bonds for each maturity is a function of both current and past aggregate states, so that the model now generates four yield curves instead of two (The previous aggregate state matters because the value of the liquidated portfolio depends on previous individual bond holdings). We call $p_k^{\kappa\zeta}$ the price of a bond of maturity k if the current aggregate state is ζ and the past state is κ. Pricing equations (17)–(18) now become:

$$\frac{p_k^{\kappa\zeta}}{z^\zeta} = \beta \sum_{s=h,l} \left[\alpha^\zeta \widetilde{\pi}^{\zeta s} \frac{p_{k-1}^{\zeta s}}{z^s} + (1-\alpha^\zeta)\widetilde{\pi}^{\zeta s} p_{k-1}^{\zeta s} u' \left(\delta + \sum_{j=1}^n p_{j-1}^{\zeta s} \frac{B_j}{\omega^{e,\kappa}} - \tau^{\zeta s} \right) \right], \qquad (25)$$

where the $\widetilde{\pi}^{\zeta s}$s summarize the transition probabilities across aggregate states (i.e., $\widetilde{\pi}^{hh} = \pi^h$, $\widetilde{\pi}^{hl} = 1 - \pi^h$, $\widetilde{\pi}^{ll} = \pi^l$ and $\widetilde{\pi}^{lh} = 1 - \pi^l$), and where from (3) the lump-sum tax is $\tau^{\zeta s} = \sum_{k=1}^n \left(p_{k-1}^{\zeta s} - p_k^{\zeta s} \right) B_k$. As in the economy with constant unemployment risk, we can check that prices are proportional to the current aggregate state: $p_k^{\kappa\zeta} = \widetilde{C}_k^{\kappa\zeta} z^\zeta$, where $\widetilde{C}_k^{\kappa\zeta}$ is a constant that only depends on the maturity of the bond, k, and on aggregate states ζ and κ. Using the usual continuity argument we can show that this equilibrium with four yield curves (and implied tax levels) exists provided that ω^e and α do not vary too much, and also that our previous results about the effect of volumes on the shape of the yield curve carry over to this more general case.

Before we turn to the cyclical pattern of bond premia implied by changes in idiosyncratic risk, let us discuss briefly their implications for the shape of the average yield curve. The following proposition summarizes how the latter is affected by the volatilities of unemployment risk and the unemployment rate.

Proposition 4 (Effect of time-varying idiosyncratic risk) *Suppose that α^h and α^l are close to 1, with $\alpha^h > \alpha^l$. Then a mean-preserving increase in the variance of α i) raises the yield curve, and ii) decreases the slope of the yield curve.*

To understand the first statement in Proposition 4, consider the joint effect of a rise in α^h and a fall in α^l. Employed agents in state h face limited unemployment risk and thus require less self-insurance; the implied lower demand for bonds lowers prices and raises yields. Conversely,

employed agents in state l face greater unemployment risk, leading to an increased demand for bonds and lower yields. However, employed agents have higher labour income and thus a lower marginal utility of consumption in state h than in state l, so the higher bond demand in the former aggregate state dominates the lower demand in the latter, leading to a higher average yield curve than under constant unemployment risk. The second result describes the effect of a change in the variance of idiosyncratic risk on the slope of the yield curve. Consider the same joint change in α^h and α^l. From our assumption that $\pi^h > \pi^l$, the economy is more often in the good aggregate state than in the bad aggregate state, and hence the slope of the mean yield curve is dominated by that in the good state. Since idiosyncratic risk is lower in that state, so are liquidation risk and the implied premium commanded by long bonds over shorter bonds.

Let us now analyze how changes in idiosyncratic risk lead to the rejection of the Expectations Hypothesis, which states that bond premia are not time-varying (Campbell and Shiller (1991)). For the sake of simplicity we focus on the time-pattern of term premia for two-period bonds and also assume that the aggregate state affects idiosyncratic probabilities but not technology (i.e., $z^h = z^l = 1$). More specifically, we define the premium on a two-period bond as the difference between the long yield and the average of future expected short yields, i.e.,

$$TP^s = r_2^s - \frac{1}{2} \left(r_1^s + E_s r_1 \right), \ s = h, l,$$

where $E_s r_1$ is the expected value of the future short yield, conditionally on the current state being s. We measure the degree of time-variations in the term premium by the difference $\Delta TP \equiv TP^h - TP^l$. We consider a small departure from the constant idiosyncratic risk case, i.e., $\alpha^h = \alpha + \eta$ and $\alpha^l = \alpha - \eta$, with $\eta > 0$ small. We show in Appendix E that ΔTP is then approximately given by:

$$\Delta TP = \frac{\left(\pi^h + \pi^l - 1 \right) \left(\pi^h - \pi^l \right)}{\left(\alpha + (1 - \alpha) u'(\delta) \right)^2} \left(u'(\delta) - 1 \right)^2 \eta^2 > 0. \tag{26}$$

Expression (26) implies that changes in liquidation risk generate time-varying risk premia along the business cycle, and thus contribute to the rejection of the Expectations Hypothesis. The reason for this is that when bad times are expected (i.e., next period's unemployment will be high) then agents' demand for liquidity increases and hence they are ready to accept a lower premium on long bonds. This implies that low premia are associated with low future output, as is consistent with the evidence (e.g., Hamilton and Kim (2002)).

6 Welfare

We now turn to the welfare impact of the supply of government bonds, both at the level of each agent type and in the aggregate.[7] For the sake of simplicity, we carry out this analysis in an economy without aggregate risk (i.e., $z^l = z^h = 1$) and where idiosyncratic uncertainty is not time-varying (i.e., $\alpha^h = \alpha^l$). Since all currently employed agents hold the same portfolio while all currently unemployed agents hold no assets, the type of an agent depends only on their current and previous employment states. We then have the following proposition:

Proposition 5 (Bond supplies and welfare) *i) A greater supply of bonds always increases the welfare of agents who stay employed or fall into unemployment, but increases the welfare of agents who leave unemployment or stay unemployed if and only if $\beta > [\alpha + (1 - \alpha)u'(\delta)]^{-1}$; and ii) a greater bond supply increases ex ante welfare (at date 0 and before agents know their type) if and only if $\beta > \frac{\alpha + \rho - 1}{\alpha + (1 - \alpha)u'(\delta)}$.*

Proposition 5 compares the intertemporal welfare of the four agent types in two economies that marginally differ in their supply of bonds. Agents who remain employed or fall into unemployment have accumulated assets in the previous period and thus currently enjoy greater self-insurance as the quantity of government bonds rises. In contrast, agents who stay unemployed or leave unemployment start the current period with no assets (since their held no assets at the end of the previous period), so that their current utility can only be negatively affected by the higher taxes associated with greater bond supply. For these individuals, the only source of higher intertemporal welfare is the prospect of benefiting from better self-insurance opportunities in the future, if they are sufficiently patient. When agents do not yet know their type (the second statement of the proposition), aggregate welfare is the average of each type's intertemporal utility weighted by their population sizes. Again, agents must be sufficiently patient for the welfare loss possibly suffered by some when they discover their type to be outweighed by the welfare gains enjoyed by others.

It is instructive to compare the welfare effects of greater liquidity to those generated by a direct unemployment-insurance scheme. For simplicity we consider the impact of a social-security system providing a constant benefit ν to the unemployed, which is funded by a social contribution $\iota = (\omega^u/\omega^e)\nu$ paid by the employed (this implies that the scheme is balanced regardless of taxes,

---

[7]Aiyagari and McGrattan (1998) and Floden (2001) have offered quantitative assessments of the aggregate welfare effect of changes in the stock of one-period government debt. Our analytical framework allows us to analyze this welfare impact on each agent type and hence to perform Pareto-comparisons of equilibria.

τ_t). We then have the following proposition:

Proposition 6 (Unemployment insurance and welfare) *i) Higher unemployment benefits always increase the welfare of the currently unemployed, but increase the welfare of the currently employed if and only if $\beta > [\rho + (1 - \rho)u'(\delta + \nu)]^{-1}$. ii) Higher unemployment benefits always increase ex ante welfare.*

The second statement is unsurprising: ex ante, social insurance makes up (at least partially) for the lack of private insurance through contingent securities and must thus be welfare-enhancing. Matters are different from an ex post point of view, however, since the currently employed bear the cost of higher social contributions; hence unless they are sufficiently patient to contemplate the possibility that they will benefit from better insurance in the future, their welfare will be negatively affected by more generous benefits.

Comparing the first statements in Propositions 5 and 6 shows that it is not the same types who benefit in either policy: those who stay unemployed in the current period may suffer from higher bond supply and taxes but would benefit from higher unemployment benefits, while the opposite is true of agents who stay employed in the current period. Moreover, some agents (i.e., those currently leaving unemployment) may suffer from *both* policies. Hence there is in general no Pareto-improving combination of these two policies unless agents are sufficiently patient.

7 Concluding remarks

This paper has analyzed the term-structure implications of an incomplete markets, general equilibrium model where agents hold bonds to self-insure against idiosyncratic shocks and face the risk of having to liquidate bonds in bad times. Our focus on the equilibrium with full asset liquidation has allowed us to derive analytical expressions for bond prices at any maturity and to study how changes in bond supplies or idiosyncratic volatility alter the shape of the yield curve as well as the welfare of (heterogeneous) agents. Our results are in contrast with the complete-markets model (e.g., the C-CAPM), where bond supplies do not affect the yield curve, and where full consumption insurance ensures that agents never have to sell assets before maturity to provide for current consumption.

It seems natural, when considering the impact of liquidation risk on asset prices, to start by focusing on real, zero-coupon bonds, which by construction bear no income risk and only differ by

their maturity. However, many long assets (e.g., equities) are likely to be affected by this risk, and hence to command a higher premium in equilibrium than that under complete markets. Similarly, the same properties should prevail in a monetary version of the model which would generate a nominal yield curve, regarding which a wealth of evidence is available. We leave both of these lines of investigation for future research.

A Proof of Proposition 1

We express the pricing equations in matrix form. Let first define, for $s = h, l$,

$$C_0^s = 1/z^s, \quad C \equiv \left[C_{n-k}^h \quad C_{n-k}^l \right]_{k=0,\ldots,n}^\top, \quad \text{and } X \equiv [z^h \quad z^l \quad B^\top], \text{ with } B = [B_{n-k}]_{k=0,\ldots,n-1}^\top.$$

Moreover, we use the following simplifying notation:

$$v^s \equiv v \left(\delta + z^s \sum_{j=1}^n \left(\frac{1 - \omega^e}{\omega^e} C_{j-1}^s + C_j^s \right) B_j \right), \text{ whether } v = u' \text{ or } u'' \text{ and } s = h, l. \tag{27}$$

For example, $u'^h \equiv u' \left(\delta + z^h \sum_{j=1}^n \left(\frac{1 - \omega^e}{\omega^e} C_{j-1}^h + C_j^h \right) B_j \right).$

$0_{m \times n}$ is the $m \times n$ null matrix, and we define $1_{cond.}$ as the function that takes value 1 when $cond.$ is true and to 0 otherwise. Finally, we define the 2×2 matrix M as:

$$M(C, X) \equiv \beta \begin{bmatrix} \pi^h(\alpha + (1 - \alpha) z^h u'^h) & (1 - \pi^h)(\alpha + (1 - \alpha) z^h u'^h) \\ (1 - \pi^l)(\alpha + (1 - \alpha) z^l u'^l) & \pi^l(\alpha + (1 - \alpha) z^l u'^l) \end{bmatrix}. \tag{28}$$

Then, the pricing equations (17)–(19) can be written as follows:

$$\left[C_k^h \quad C_k^l \right]^\top = M(C, X) \cdot \left[C_{k-1}^h \quad C_{k-1}^l \right]^\top \text{ for } k = 1, \ldots, n. \tag{29}$$

By stacking equalities, we rewrite (29) as $f(C, X) = 0_{(2n+2) \times 1}$, where f is the following $\mathcal{C}^1$ function:

$$f(C, X) \equiv C - \begin{bmatrix} 0_{2 \times 2} & M(C, X) & 0_{2 \times 2} & \ldots & 0_{2 \times 2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 0_{2 \times 2} & & & \ldots & M(C, X) \\ & & & & 0_{2 \times 2} \end{bmatrix} C - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/z^h \\ 1/z^l \end{bmatrix}.$$

To prove that C is a $\mathcal{C}^1$ function of B and Z, we show that the Jacobian of f w.r.t. C is invertible. The derivatives of f w.r.t. $\left(C_{n-i}^s \right)_{i=0,\ldots,n}$ can be written in a compact form as:

$$\frac{\partial f}{\partial C_{n-i}^s} = \Gamma_{n-i}^s + K_{n-i}^s \quad \text{for} \quad i = 0, \ldots, n.$$

with Γ^s_{n-i} defined, for $i = 0$, as $\Gamma^s_n \equiv [1_{s=h}, \ 1_{s=l}, 0_{2n}]^\top$ and, for $i = 1, \ldots, n$, as:

$$\Gamma^s_{n-i} \equiv \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\left(\alpha + (1-\alpha)z^s u'^s\right)\widetilde{\pi}^{hs} \\ -\beta\left(\alpha + (1-\alpha)z^s u'^s\right)\widetilde{\pi}^{ls} \\ 1_{s=h} \\ 1_{s=l} \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow \quad Rank\ 2i+1 \\ \longleftarrow \quad Rank\ 2i+2, \\ \\ \end{array}$$

where $\widetilde{\pi}^{hh} \equiv \pi^h, \widetilde{\pi}^{lh} \equiv 1 - \pi^l, \widetilde{\pi}^{ll} \equiv \pi^l, \widetilde{\pi}^{hl} \equiv 1 - \pi^h$ (cf. Section 5). K^s_{n-i} is defined as:

$$K^s_{n-i} \equiv -\beta\left(1-\alpha\right)\left[\frac{1-\omega^e}{\omega^e}B_{n+1-i}\,1_{i>0} + B_{n-i}\,1_{i<n}\right](z^s)^2\,u''^s$$

$$\times \left[\widetilde{\pi}^{hs}\,C^s_{n-1}, \ \widetilde{\pi}^{ls}\,C^s_{n-1}\,\ldots, \widetilde{\pi}^{hs}\,C^s_0, \ \widetilde{\pi}^{ls}\,C^s_0, \ 0, \ 0\right]^\top \quad \text{for } i = 0, \ldots, n.$$

The Jacobian $Df_Y = \left(\frac{\partial f}{\partial C^h_n}, \frac{\partial f}{\partial C^l_n}, \ldots, \frac{\partial f}{\partial C^h_{n-i}}, \frac{\partial f}{\partial C^l_{n-i}}, \ldots, \frac{\partial f}{\partial C^h_0}, \frac{\partial f}{\partial C^l_0}\right)$ of f w.r.t. to C can be expressed as the sum of an upper triangular matrix with only 1s on its diagonal and a matrix that is equal to 0 when $B = 0$ (because $K^s_{n-i} = 0$ if $B = 0$). The Jacobian is thus invertible for $B = 0$. Then, the implicit function theorem allows us to prove the first statement in the proposition. C is now a continuous (in fact $\mathcal{C}^1$) function of $\left[B^\top\ Z^\top\right]$ in a neighborhood V_1 of $\left[0_n^\top\ 1_2^\top\right]$. Moreover, if $\left[B^\top\ Z^\top\right] = \left[0_n^\top\ 1_2^\top\right]$, then C satisfies conditions (21). By continuity, there exists a neighborhood $V_2 \subset V_1$, such that condition (21) is satisfied if $\left[B^\top\ Z^\top\right] \in V_2$. QED.

B The two-maturity example

From (17)–(18) and our assumed shock process and utility function, one-period bond price is:

$$p^s_1 = C^s_1 z^s = \frac{\alpha\beta z^s}{2}\sum_{s'=l,h}\frac{1}{z^{s'}}$$

$$+ \frac{(1-\alpha)\,\beta z^s}{2}\sum_{s'=l,h}\left[u_1 - u_2\left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2\,C^{s'}_1\,z^{s'}) + (B_1\,C^{s'}_1 + B_2\,C^{s'}_2)z^{s'}\right)\right].$$

where $(s, s') \in \{l, h\}^2$ are the aggregate state in the current and the next period. Dividing both sides by z^s gives $C^h_1 = C^l_1 \equiv C_1$.

This in turn implies that the price of a two-period bond is:

$$p_2^s = C_2^s z^s = \alpha\beta z^s C_1$$
$$+ \frac{(1-\alpha)\beta z^s}{2} \sum_{s'=l,h} \left[u_1 - u_2 \left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2\,C_1\,z^{s'}) + (B_1 C_1 + B_2 C_2^{s'})z^{s'} \right) \right] C_1 z^{s'}.$$

Dividing both sides by z^s also gives $C_2^h = C_2^l \equiv C_2$. C_1 and C_2 then solve:

$$C_1 = \frac{\alpha\beta}{2} \sum_{s'=l,h} \frac{1}{z^{s'}} + \frac{(1-\alpha)\beta}{2} \sum_{s'=l,h} \left(u_1 - u_2 \left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2 C_1 z^{s'}) + (B_1 C_1 + B_2 C_2)z^{s'} \right) \right),$$

$$\frac{C_2}{C_1} = \alpha\beta + \frac{(1-\alpha)\beta}{2} \sum_{s'=l,h} \left(u_1 - u_2 \left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2 C_1 z^{s'}) + (B_1 C_1 + B_2 C_2)z^{s'} \right) \right) z^{s'}.$$

Finally, note that since the average of z is 1, C_1 and C_2 are also the average prices of one- and two-period bonds, respectively. For clarity we also denote these prices p_1 and p_2. Under the maintained assumptions that bond supplies and aggregate shocks are small, the previous equations give:

$$p_1 = \alpha\beta\left(1+\varepsilon^2\right) + (1-\alpha)\beta \left(u_1 - u_2 \left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2 p_1) + B_1 p_1 + B_2\,p_2 \right) \right) + o(\varepsilon^2),$$

$$\frac{p_2}{p_1} = \alpha\beta + (1-\alpha)\beta \left(u_1 - u_2 \left(\delta + \frac{1-\omega^e}{\omega^e}(B_1 + B_2\,p_1\left(1+\varepsilon^2\right)) + (B_1\,p_1 + B_2\,p_2)\left(1+\varepsilon^2\right) \right) \right) + o(\varepsilon^2),$$

where $o(\varepsilon^2)$ satisfies $\lim_{\varepsilon\to 0} o(\varepsilon^2)/\varepsilon^2 = 0$. From the first expression we get:

$$\left. \frac{\partial p_1}{\partial B_1} \right|_{B_1=B_2=0} = -(1-\alpha)\beta u_2 \left(\frac{1-\omega^e}{\omega^e} + p_1 \right) < 0,$$
$$\left. \frac{\partial p_1}{\partial B_2} \right|_{B_1=B_2=0} = -(1-\alpha)\beta u_2 \left(\frac{1-\omega^e}{\omega^e}\,p_1 + p_2 \right) < 0.$$

From the second expression we get:

$$\left. \frac{\partial p_2}{\partial B_1} \right|_{B_1=B_2=0} = (\alpha\beta + (1-\alpha)\beta\,(u_1 - u_2\delta)) \left. \frac{\partial p_1}{\partial B_1} \right|_{B_1=B_2=0} - p_1(1-\alpha)\beta u_2 \left(\frac{1-\omega^e}{\omega^e} + p_1\left(1+\varepsilon^2\right) \right) < 0,$$
$$\left. \frac{\partial p_2}{\partial B_2} \right|_{B_1=B_2=0} = (\alpha\beta + (1-\alpha)\beta\,(u_1 - u_2\delta)) \left. \frac{\partial p_1}{\partial B_2} \right|_{B_1=B_2=0} - p_1(1-\alpha)\beta u_2 \left(\frac{1-\omega^e}{\omega^e}\,p_1 + p_2 \right)\left(1+\varepsilon^2\right) < 0.$$

The mean yield-to-maturity of a k-period bond is $r_k = -\sum_{s=l,h} \ln p_k^s/2k$, $k=1,2$. Hence,

$$r_1 = -\frac{1}{2} \sum_{s=l,h} \ln C_1 z^s = -\ln C_1 - \frac{\ln z^h + \ln z^l}{2} = -\ln p_1 + \frac{1}{2}\varepsilon^2 + o(\varepsilon^2),$$

$$r_2 = -\frac{1}{4} \sum_{s=l,h} \ln C_2 z^s = -\frac{\ln C_2}{2} - \frac{\ln z^h + \ln z^l}{4} = -\frac{\ln p_2}{2} + \frac{1}{4}\varepsilon^2 + o(\varepsilon^2),$$

so that lower mean bond prices imply higher mean yields. The slope of the mean yield curve is:

$$S = r_2 - r_1 = -\frac{1}{2} \ln \frac{p_2}{p_1^2} - \frac{1}{4} \varepsilon^2 + o(\varepsilon^2).$$

From our expressions for p_1 and p_2/p_1 we get:

$$\frac{p_2}{p_1} = p_1 - \beta \left[\alpha + (1-\alpha) u_2 \left(\frac{1-\omega^e}{\omega^e} B_2 p_1 + B_1 p_1 + B_2 p_2 \right) \right] \varepsilon^2 + o(\varepsilon^2),$$

$$\frac{p_2}{p_1} \left(1 + \beta (1-\alpha) u_2 B_2 p_1 \varepsilon^2 \right) = p_1 - \beta \left[\alpha + (1-\alpha) u_2 \left(\frac{1-\omega^e}{\omega^e} B_2 p_1 + B_1 p_1 \right) \right] \varepsilon^2 + o(\varepsilon^2),$$

$$\frac{p_2}{p_1} = p_1 \left(1 - \beta (1-\alpha) u_2 B_2 p_1 \varepsilon^2 \right) - \beta \left[\alpha + (1-\alpha) u_2 \left(\frac{1-\omega^e}{\omega^e} B_2 p_1 + B_1 p_1 \right) \right] \varepsilon^2 + o(\varepsilon^2),$$

$$\frac{p_2}{p_1} = p_1 - \beta \left[\alpha + (1-\alpha) u_2 p_1 (B_1 + B_2) \right] \varepsilon^2 + o(\varepsilon^2). \tag{30}$$

Using (30) and rearranging, we find (24) in the body of the paper.

C   Proof of Proposition 2

C.1   Ranking of yield curves

We prove by inference that $C_k^h z^h > C_k^l z^l$ $(k \geq 1)$ for $B = 0$. By continuity this property will also hold when B is positive but small.

1. The result holds for $k = 1$: from (29), $C_1^h z^h > C_1^l z^l$ is equivalent to $\alpha (z^h - z^l) \left(\frac{1-\pi^l}{z^h} + \frac{1-\pi^h}{z^l} \right) > (1-\alpha) u'(\delta) (z^l - z^h)$, which is true since $z^h > z^l$.

2. Assume that $C_{k-1}^h z^h > C_{k-1}^l z^l$, for any $k \geq 2$. From (29), $C_k^h z^h > C_k^l z^l$ if and only if:

$$\alpha \left(\left(\pi^h + (\pi^l - 1) \frac{z^l}{z^h} \right) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l \right) >$$

$$(1-\alpha) u'(\delta) \left(\pi^l z^l - (1 - \pi^h) z^h - z^h (\pi^h + (\pi^l - 1) \frac{z^l}{z^h}) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} \right). \tag{31}$$

Since $\pi^h + (\pi^l - 1) \frac{z^l}{z^h} > 0$ and $z^h C_{k-1}^h > z^l C_{k-1}^l$, for (31) to hold it is sufficient to show that $\alpha (z^h - z^l) \left(\frac{1-\pi^l}{z^h} + \frac{1-\pi^h}{z^l} \right) > (1-\alpha) u'(\delta) (z^l - z^h)$ (as in the $k = 1$ case). QED.

C.2 Monotonicity of yield curves

We show that $r_k^h \leq r_{k+1}^h$ (the proof for r_k^l is similar). This inequality is equivalent to:

$$z^h \left(C_k^h \right)^{k+1} \geq \left(C_{k+1}^h \right)^k. \tag{32}$$

For α close to 1, $z^l = 1$ and $z^h > 1$ but close to 1, the coefficients Cs are approximately:

$$C_k^h = \beta^k \left(1 - \frac{z^h - 1}{2 - \pi^h - \pi^l} \left(1 - \pi^l + (1 - \pi^h)(\pi^h + \pi^l - 1)^k \right) \right) + o\left(z^h - 1 \right) + O\left(1 - \alpha \right), \tag{33}$$

$$C_k^l = \beta^k \left(1 - \frac{z^h - 1}{2 - \pi^h - \pi^l} \left(1 - \pi^l - (1 - \pi^l)(\pi^h + \pi^l - 1)^k \right) \right) + o\left(z^h - 1 \right) + O\left(1 - \alpha \right). \tag{34}$$

This can be shown recursively. First, equations (33)–(34) are true for $k = 0$ since $C_0^h = 1/(1 + z^h - 1) \approx 1 - (z^h - 1)$ and $C_0^l = 1$. Second, if (33)–(34) hold for $k \geq 0$, then by equality (29) they also hold for $k + 1$.

To conclude the proof, using (33)–(34) and z^h close to 1, inequality (32) can be written as:

$$1 - (\pi^h + \pi^l - 1)^k \left(k + 1 - k\left(\pi^h + \pi^l - 1 \right) \right) \geq 0.$$

To show that the last inequality holds, we define $P_k(t) = 1 - t^k(k + 1 - kt)$ for $0 \leq t \leq 1$. Then, for $k = 0$ we have $P_0(t) = 0$, whereas for $k \geq 1$ we have $P_k(0) = 1$, $P_k(1) = 0$ and $P_k'(t) = k(k+1)t^{k-1}(t-1) \leq 0$. This implies that $P_k(t) \geq 0$, for $0 \leq t \leq 1$. Since $0 \leq \pi^h + \pi^l - 1 \leq 1$, this establishes the result. QED.

C.3 Value of the long-run interest rate

We diagonalize the matrix $M(C, X)$ defined in (28): $M(C, X) = \beta Q D Q^{-1}$, where Q is a 2×2 invertible matrix and $D = Diag(d_{11}, d_{22})$ a diagonal matrix with:

$$d_{22} = H + d_{11} = \frac{1}{2} \left(\alpha \left(\pi^h + \pi^l \right) + (1 - \alpha) \left(z^h u'^h \pi^h + z^l u'^l \pi^l \right) + H \right),$$

$$H \equiv \left[\begin{array}{c} \left(\alpha(\pi^h + \pi^l) + (1-\alpha)(z^h \pi^h u'^h + z^l \pi^l u'^l) \right)^2 \\ -4(\pi^h + \pi^l - 1)(\alpha + (1-\alpha)z^h u'^h)(\alpha + (1-\alpha)z^l u'^l) \end{array} \right]^{1/2} > 0.$$

We check that H is well defined and that $-1 < d_{11}/d_{22} < 1$. The iteration of (29), after diagonal-

ization, now yields the following expressions for bond prices:

$$
\begin{bmatrix} p_k^h \\ p_k^l \end{bmatrix} = (\beta d_{22})^k \begin{bmatrix} z^h & 0 \\ 0 & z^l \end{bmatrix} Q \begin{bmatrix} (d_{11}/d_{22})^k & 0 \\ 0 & 1 \end{bmatrix} Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix}.
$$

As $|d_{11}/d_{22}| < 1$, $\lim_{k\to\infty} (d_{11}/d_{22})^k = 0$ and we get $\lim_{k\to\infty} \frac{1}{(\beta d_{22})^k} \begin{bmatrix} p_k^h & p_k^l \end{bmatrix}^\top = Const.$, where $Const.$ is a constant 2×1 vector that does not depend on k. This implies that $\lim_{k\to\infty} k^{-1} \ln \frac{p_k^h}{(\beta d_{22})^k} = \lim_{k\to\infty} k^{-1} \ln \frac{p_k^l}{(\beta d_{22})^k} = 0$. From the definition of interest rates, the common limit $\widetilde{r}^{\lim}$ in both states of the world is:

$$
\lim_{k\to\infty} r_k^h = \lim_{k\to\infty} r_k^l = \widetilde{r}^{\lim} = -\ln(\beta d_{22}). \tag{35}
$$

D Proof of Proposition 3

D.1 Impact of bond supplies on prices

We prove the result by inference for C_k^ζ, $\zeta = h, l$. Taking the derivative of (29) w.r.t. to B_i, $1 \le i \le n$, we get:

$$
\frac{\partial C_k^\zeta}{\partial B_i} = \beta \sum_{s=h,l} \widetilde{\pi}^{\zeta s} \left[(\alpha + (1-\alpha)z^s u'^s) \frac{\partial C_{k-1}^s}{\partial B_i} + (1-\alpha) C_{k-1}^s (z^s)^2 \left[\sum_{j=1}^n \left(\frac{1-\omega^e}{\omega^e} \frac{\partial C_{j-1}^s}{\partial B_i} + \frac{\partial C_j^s}{\partial B_i} \right) B_j + \frac{1-\omega^e}{\omega^e} C_{i-1}^s + C_i^s \right] u''^s \right]. \tag{36}
$$

where u'^s and u''^s are given by (27), and where, as before, $\widetilde{\pi}^{\zeta s}$ is defined as follows: $\widetilde{\pi}^{hh} \equiv \pi^h$, $\widetilde{\pi}^{lh} \equiv 1 - \pi^l$, $\widetilde{\pi}^{ll} \equiv \pi^l$, $\widetilde{\pi}^{hl} \equiv 1 - \pi^h$ (cf. Section 5).

1. The result stated in the proposition holds for $k = 1$, since (36) yields the following first-order approximation for small levels of bond supply (recall that $C_0^s = 1/z^s$):

$$
\frac{\partial C_1^\zeta}{\partial B_i} \approx \beta (1-\alpha) u''(\delta) \sum_{s=h,l} \widetilde{\pi}^{\zeta s} z^s \left[\frac{1-\omega^e}{\omega^e} C_{i-1}^s + C_i^s \right] < 0.
$$

2. Suppose that the result holds for $k - 1$: $\frac{\partial C_{k-1}^h}{\partial B_i}, \frac{\partial C_{k-1}^l}{\partial B_i} < 0$. Since C_{j-1}^s is a $\mathcal{C}^1$ function of B_i, $\frac{\partial C_{j-1}^s}{\partial B_i}$ is continuous in B_i and $B_j \frac{\partial C_{j-1}^s}{\partial B_i}$ is negligible relative to C_{i-1}^s for small bond supplies. Then, (36) implies that $\frac{\partial C_k^h}{\partial B_i} < 0$, so that greater bond supply decreases prices (i.e., raises yields). QED.

34

D.2   Impact of bond supplies on the slope of the yield curve

Using the expression for $\widehat{r}^{\lim}$ in (35) and that for r_1 computed from (29), we find that when α is close to 1 the derivative of Δ w.r.t. to B_i, $i \leq n$, is first-order approximated by the following expression:

$$\frac{\partial \Delta}{\partial B_i} \approx \frac{(1-\alpha)(1-\pi^l)(1-\pi^h)}{2 - \pi^h - \pi^l} \left[z^l \frac{\partial u''^l}{\partial B_i} - z^h \frac{\partial u''^h}{\partial B_i} \right] \left[\frac{z^h}{z^h \left(1-\pi^h\right) + \pi^h z^l} - \frac{z^l}{z^l \left(1-\pi^l\right) + \pi^l z^h} \right],$$

(37)

where $\frac{\partial u'^s}{\partial B_i} = z^s \left(\sum_{j=1}^n \left(\frac{1-\omega^e}{\omega^e} \frac{\partial C_{j-1}^s}{\partial B_i} + \frac{\partial C_j^s}{\partial B_i} \right) B_j + \frac{1-\omega^e}{\omega^e} C_{i-1}^s + C_i^s \right) u''^s$. When bond supplies are small we have that $\frac{\partial u'^s}{\partial B_i} = z^s(\frac{1-\omega^e}{\omega^e} C_{i-1}^s + C_i^s) u'' (\delta)$. The first term in the right-hand side of (37) is positive. The second term is positive since $C_k^h z^h > C_k^l z^l$ $(k \geq 1)$ and $z^h > z^l$. The third term is also positive since $z^h > z^l$. QED.

E   Proof of Proposition 4

We define the vector $\mathbf{C}_k = [C_k^{hh}, C_k^{lh}, C_k^{hl}, C_k^{ll}]^\top$ and, for $s_i = h, l$ and $i = 1, 2, 3$:

$$u'(s_1 s_2 s_3) = u' \left(\delta + \sum_{j=1}^n \left(\frac{1 - w^{e,s_1}}{\omega^{e,s_1}} C_{j-1}^{s_2 s_3} + C_j^{s_2 s_3} \right) z^{s_3} B_j \right),$$

$$\widetilde{M} = \begin{bmatrix} \pi^h \left[\alpha^h + (1-\alpha^h)z^h u'(hhh)\right] & 0 & (1-\pi^h)\left[\alpha^h + (1-\alpha^h)z^l u'(hhl)\right] & 0 \\ \pi^h \left[\alpha^h + (1-\alpha^h)z^h u'(lhh)\right] & 0 & (1-\pi^h)\left[\alpha^h + (1-\alpha^h)z^l u'(lhl)\right] & 0 \\ 0 & (1-\pi^l)\left[\alpha^l + (1-\alpha^l)z^h u'(hlh)\right] & 0 & \pi^l \left[\alpha^l + (1-\alpha^l)z^l u'(hll)\right] \\ 0 & (1-\pi^l)\left[\alpha^l + (1-\alpha^l)z^h u'(llh)\right] & 0 & \pi^l \left[\alpha^l + (1-\alpha^l)z^l u'(lll)\right] \end{bmatrix}.$$

We obtain the following recursion, which determines equilibrium bond prices (cf. (25)):

$$\mathbf{C}_k = \beta \widetilde{M} \mathbf{C}_{k-1}.$$

(38)

We first prove the following general result: for a function Φ of Y, where Y takes value y_1 with probability q and $y_2 < y_1$ with probability $1 - q$, the impact of a mean-preserving increase in the

variance of Y (i.e., an increase in $V[Y]$ holding $E[Y]$ constant) on Φ is:

$$\left.\frac{\partial \Phi}{\partial V[Y]}\right|_{E[Y] constant} = \frac{1}{2q(1-q)(y_1 - y_2)}\left((1-q)\frac{\partial \Phi}{\partial y_1} - q\frac{\partial \Phi}{\partial y_2}\right),$$

which can be shown by expressing y_1 and y_2 as functions of $E[Y]$ and $V[Y]$ and computing their derivatives w.r.t. $V[Y]$:

$$y_1 = E[Y] + (1-q)\sqrt{\frac{V[Y]}{q(1-q)}}, \qquad \frac{\partial y_1}{\partial V[Y]} = (1-q)\frac{1}{2\sqrt{q(1-q)V[Y]}},$$

$$y_2 = E[Y] - q\sqrt{\frac{V[Y]}{q(1-q)}}, \qquad \frac{\partial y_2}{\partial V[Y]} = -q\frac{1}{2\sqrt{q(1-q)V[Y]}}. \tag{39}$$

This establishes the result since $V[Y] = q(1-q)(y_1 - y_2)^2$.

E.1   Effect of the variance of α at zero volume

Suppose that $\omega^{e,s} = \omega$ $(s = h, l)$ and that bonds are in zero net supply. Then, (38) becomes:

$$\begin{bmatrix} C_k^h & C_k^l \end{bmatrix}^\top = \beta \widehat{M}^k \begin{bmatrix} 1/z^h & 1/z^l \end{bmatrix}^\top,$$

$$\text{with } \widehat{M} = \begin{bmatrix} \pi^h\left[\alpha^h + (1-\alpha^h)z^h u'(\delta)\right] & (1-\pi^h)\left[\alpha^h + (1-\alpha^h)z^l u'(\delta)\right] \\ (1-\pi^l)\left[\alpha^l + (1-\alpha^l)z^h u'(\delta)\right] & \pi^l\left[\alpha^l + (1-\alpha^l)z^l u'(\delta)\right] \end{bmatrix}.$$

The short yield.   The short yield is $r_1 = -\eta^h \ln C_1^h z^h - \eta^l \ln C_1^l z^l$ (cf. (3.2)). Using (39) with $q = \eta^h = 1 - \eta^l$, we find after some algebra that in the vicinity of $\alpha^h = \alpha^l = \alpha$ we have:

$$(\alpha^h - \alpha^l)\left.\frac{\partial r_1}{\partial V[\alpha]}\right|_{E[\alpha] constant} = \frac{(\pi^h + \pi^l - 1)\left(\frac{1}{z^l} - \frac{1}{z^h}\right)u'(\delta)}{2\left(\alpha(\frac{\pi^h}{z^h} + \frac{1-\pi^h}{z^l}) + (1-\alpha)u'(\delta)\right)\left(\alpha(\frac{1-\pi^l}{z^h} + \frac{\pi^l}{z^l}) + (1-\alpha)u'(\delta)\right)} > 0.$$

The long yield.   We use an argument similar to that in Section C.3. Since $\lim_{k\to\infty} k^{-1}[\ln \frac{P_k^h}{\beta^k \widehat{\lambda}^k}, \ \ln \frac{P_k^l}{\beta^k \widehat{\lambda}^k}] = 0_2$, where $\widehat{\lambda}$ is the largest eigenvalue of $\widehat{M}$, we have that $r_\infty = -\ln \beta\widehat{\lambda}$ and that

$$(\alpha^h - \alpha^l)\frac{\partial r_\infty}{\partial V[\alpha]} = -\frac{1}{2\widehat{\lambda}}\left[\frac{1}{\eta^h}\frac{\partial \widehat{\lambda}}{\partial \alpha^h} - \frac{1}{\eta^l}\frac{\partial \widehat{\lambda}}{\partial \alpha^l}\right], \qquad \text{where } \frac{\partial r_\infty}{\partial \alpha^s} = -\frac{1}{\widehat{\lambda}}\frac{\partial \widehat{\lambda}}{\partial \alpha^s}, \ s = h, l.$$

Some algebra shows that:

$$\widehat{\lambda} = \frac{1}{2}\left[\pi^h(\alpha^h + (1-\alpha^h)z^h\,u'(\delta)) + \pi^l(\alpha^l + (1-\alpha^l)z^l\,u'(\delta))\right.$$

$$\left. + \left((\pi^h(\alpha^h + (1-\alpha^h)z^h\,u'(\delta)) - \pi^l(\alpha^l + (1-\alpha^l)z^l\,u'(\delta)))^2 + 4(1-\pi^l)(1-\pi^h)(\alpha^l + (1-\alpha^l)z^h\,u'(\delta))(\alpha^h + (1-\alpha^h)z^l\,u'(\delta))\right)^{\frac{1}{2}}\right].$$

Assuming that α^h and α^l are both close to 1, so that $\widehat{\lambda} \approx 1$, we obtain:

$$(\alpha^h - \alpha^l)\frac{\partial r_\infty}{\partial V[\alpha]} = \frac{(\pi^h + \pi^l - 1)(z^h - z^l)\,u'(\delta)}{2} > 0,$$

$$(\alpha^h - \alpha^l)\frac{\partial \Delta}{\partial V[\alpha]} = \frac{(\pi^h + \pi^l - 1)(z^h - z^l)u'(\delta)}{2}\left(1 - \frac{1}{\left(\pi^h + (1-\pi^h)\frac{z^h}{z^l}\right)\left((1-\pi^l)\frac{z^l}{z^h} + \pi^l\right)}\right).$$

After some manipulations, we find that the curve flattens if and only if $\pi^l(1-\pi^h)z^h < \pi^h(1-\pi^l)z^l$, which is equivalent to $\pi^h > \pi^l$ when z is sufficiently close to 1.

F   Proof of Equation (26)

When $z^h = z^l = 1$, the coefficients determining the price of one- and two-period bonds are:

$$\begin{cases}
C_1^h &= \beta\pi^h\left(\alpha^h + (1-\alpha^h)u'(\delta)\right) + \beta(1-\pi^h)\left(\alpha^h + (1-\alpha^h)u'(\delta)\right), \\
C_1^l &= \beta\pi^l\left(\alpha^l + (1-\alpha^l)u'(\delta)\right) + \beta(1-\pi^l)\left(\alpha^l + (1-\alpha^l)u'(\delta)\right), \\
C_2^h &= \beta\pi^h\left(\alpha^h + (1-\alpha^h)u'(\delta)\right)C_1^h + \beta(1-\pi^h)\left(\alpha^h + (1-\alpha^h)u'(\delta)\right)C_1^l, \\
C_2^l &= \beta\pi^l\left(\alpha^l + (1-\alpha^l)u'(\delta)\right)C_1^l + \beta(1-\pi^l)\left(\alpha^l + (1-\alpha^l)u'(\delta)\right)C_1^h.
\end{cases}$$

The term premium in state $s = h, l$ is $TP^s = r_2^s - \frac{1}{2}(r_1^s + E_s r_1)$. Expressing interest rates as functions of prices, one deduces for $\Delta TP = TP^h - TP^l$ ($z^h = z^l = 1$):

$$e^{-2\Delta TP} = \frac{C_2^h}{C_2^l}\frac{(C_1^l)^{1+\pi^l}(C_1^h)^{1-\pi^l}}{(C_1^h)^{1+\pi^h}(C_1^l)^{1-\pi^h}} = \frac{C_2^h}{C_2^l}\left(\frac{C_1^l}{C_1^h}\right)^{\pi^h + \pi^l}.$$

Substituting the coefficients Cs by their values and taking a second order approximation with $\alpha^h = \alpha + \eta$ and $\alpha^l = \alpha - \eta$, one finds (26) in the body of the paper.

G Proof of Proposition 5

Agents can be of four different types only here, and we denote by ij, with $i, j = e, u$ the type of an agent who is in individual state j in the current period and was in individual state i in the previous period, where e stands for "employed" and u for "unemployed". Let U denote the vector of instantaneous utilities: $U = \left[u(c^k) - l^k\right]_{k=ee,ue,eu,uu}$ (no time index since $z_t = 1$). The c^ks that appear in U are given in Section 3.1, while labor supplies can be computed as residuals from the budget constraints of employed agents (see (5)) since the steady state consumption levels and bond holdings of the different types of agents are known. We simplify these expression using the fact that in the no-trade equilibrium $C_k = \beta\left(\alpha + (1-\alpha)u'(\delta)\right)C_{k-1} \equiv \theta C_{k-1}$ (cf. (28)), and we evaluate the derivatives of U w.r.t. the B_ks at the no-trade equilibrium:

$$U = \begin{bmatrix} u(u'^{-1}(1)) - u'^{-1}(1) - \frac{1-\omega^e}{\omega^e}\sum_{k=1}^{n}(C_k - C_{k-1})B_k \\ u(u'^{-1}(1)) - u'^{-1}(1) - \sum_{k=1}^{n}(C_{k-1} + C_k\frac{1-\omega^e}{\omega^e})B_k \\ u(\sum_{k=1}^{n}(C_{k-1}\frac{1-\omega^e}{\omega^e} + C_k)B_k + \delta) \\ u(\delta - \sum_{k=1}^{n}(C_{k-1} - C_k)B_k) \end{bmatrix}, \quad \frac{\partial U}{\partial B_k} = C_{k-1}\begin{bmatrix} -\frac{1-\omega^e}{\omega^e}(\theta - 1) \\ -(1 + \frac{1-\omega^e}{\omega^e}\theta) \\ (\frac{1-\omega^e}{\omega^e} + \theta)u'(\delta) \\ -(1-\theta)u'(\delta) \end{bmatrix}.$$

Let $\mathcal{U}$ denote the vector of intertemporal utilities: $\mathcal{U} = \sum_{k=0}^{\infty}\beta^k\,\Omega^k\,U = \sum_{k=0}^{\infty}\beta^k\,Q\,D^k\,Q^{-1}\,U$, where Ω (the transition matrix for the four types $\{ee\ ue\ eu\ uu\}$), Q and D are given by:

$$\Omega = \begin{bmatrix} \alpha & 0 & 1-\alpha & 0 \\ \alpha & 0 & 1-\alpha & 0 \\ 0 & 1-\rho & 0 & \rho \\ 0 & 1-\rho & 0 & \rho \end{bmatrix} = Q.D.Q^{-1}, \text{ with } Q = \begin{bmatrix} 1 & 1-\alpha & 0 & 1-\alpha \\ 1 & 0 & \rho & 1-\alpha \\ 1 & -\alpha & 0 & -(1-\rho) \\ 1 & 0 & -(1-\rho) & -(1-\rho) \end{bmatrix},$$

$$\text{and } D = \text{Diag}(1\ 0\ 0\ \alpha + \rho - 1).$$

The impact of bond volumes on ex post utilities is then given by:

$$\frac{\partial \mathcal{U}}{\partial B_k} = C_{k-1}\begin{bmatrix} \frac{1-\omega^e}{\omega^e} + \frac{\beta^2(1-\alpha)(u'(\delta)-1)(1-\rho+(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\ \quad - \frac{(1-\beta(\alpha+u'(\delta)(1-\alpha)))(1-\beta\rho+\beta(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\ \frac{1-\omega^e}{\omega^e}u'(\delta) + \frac{\beta(1-\alpha\beta)(u'(\delta)-1)(1-\rho+(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\ \quad - \frac{(1-\beta(\alpha+(1-\alpha)u'(\delta)))(\beta(1-\rho)+(1-\beta\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \end{bmatrix}.$$

From $\frac{\partial \mathcal{U}}{\partial B_k}$ we find that $\frac{\partial \mathcal{U}^{ee}}{\partial B_k}, \frac{\partial \mathcal{U}^{eu}}{\partial B_k} > 0$, but $\frac{\partial \mathcal{U}^{ue}}{\partial B_k}, \frac{\partial \mathcal{U}^{uu}}{\partial B_k} < 0$ if and only if $\beta < \beta^{\text{ex post}} = \left[(\alpha + (1-\alpha)u'(\delta)\right]^{-1}$.

To derive the impact of changes in bond supplies on ex ante welfare, we premultiply the ex post utility vector by the vector of population weights $W = \frac{1}{2-\alpha-\rho}[\alpha(1-\rho),\ (1-\alpha)(1-\rho),\ (1-\alpha)(1-\rho),\ \rho(1-\alpha)]$:

$$W\frac{\partial \mathcal{U}}{\partial B_k} = C_{k-1}\,(1-\alpha)\frac{(1-\beta+\beta u'(\delta))(\alpha+(1-\alpha)u'(\delta))-(1-\rho+\rho\,u'(\delta))}{(1-\beta)(2-\alpha-\rho)}.$$

This expression is negative if and only if $\beta < \beta^{\text{ex ante}} = \frac{\alpha+\rho-1}{\alpha+(1-\alpha)u'(\delta)}(<\beta^{\text{ex post}})$. QED.

H Proof of Proposition 6

Similarly, computing the instant and intertemporal utilities of the four agent types when the employed pay $(\omega^u/\omega^e)\,\nu$ and the unemployed receive ν, we find that expressions of instantaneous $\widehat{U}$ and intertemporal $\widehat{\mathcal{U}}$ utility vectors are:

$$\frac{\partial \widehat{U}}{\partial \nu} = \begin{bmatrix} -\frac{\omega^u}{\omega^e}\,1_2 & u'(\delta+\nu)\,1_2 \end{bmatrix}^{\top},$$

$$W\frac{\partial \widehat{\mathcal{U}}}{\partial \nu} = \frac{(1-\alpha)(u'(\delta+\nu)-1)}{(1-\beta)(2-\alpha-\rho)} > 0,$$

$$\frac{\partial \widehat{\mathcal{U}}}{\partial \nu} = \frac{1}{(1-\beta)(1-\beta(\alpha+\rho-1))}\begin{bmatrix} (\beta\,(\rho+(1-\rho)\,u'(\delta+\nu))-1)\frac{1-\alpha}{1-\rho} \\ (\beta\,(\rho+(1-\rho)\,u'(\delta+\nu))-1)\frac{1-\alpha}{1-\rho} \\ 1-\beta+(u'(\delta+\nu)-1)(1-\beta\,\alpha) \\ 1-\beta+(u'(\delta+\nu)-1)(1-\beta\,\alpha) \end{bmatrix}.$$

We have that $\frac{\partial \widehat{\mathcal{U}}^{eu}}{\partial \nu}, \frac{\partial \widehat{\mathcal{U}}^{uu}}{\partial \nu} > 0$ but $\frac{\partial \widehat{\mathcal{U}}^{ee}}{\partial \nu}, \frac{\partial \widehat{\mathcal{U}}^{ue}}{\partial \nu} < 0$ if and only if $\beta < \beta_\nu^{\text{ex post}} = (\rho+u'\,(\delta+\nu)\,(1-\rho))^{-1}$. In contrast, ex ante welfare always increases with ν. QED.

References

AIYAGARI, R. S. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109(3), 659–684.

AIYAGARI, R. S., AND E. R. MCGRATTAN (1998): "The Optimum Quantity of Debt," *Journal of Monetary Economics*, 42(3), 447–469.

ANG, A., AND M. PIAZZESI (2003): "A No–Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50(4), 745–787.

BARRO, R. J. (1974): "Are Government Bonds Net Wealth?," *Journal of Political Economy*, 82(6), 1095–1117.

BEWLEY, T. F. (1980): "The Optimum Quantity of Money," in *Models of Monetary Economies*, ed. by J. Kareken, and N. Wallace, pp. 169–210. Federal Reserve Bank of Minneapolis.

——— (1983): "A Difficulty with the Optimum Quantity of Money," *Econometrica*, 54(5), 1485–1504.

BRUNNERMEIER, M. K., AND L. H. PEDERSEN (2009): "Market Liquidity and Funding Liquidity," *The Review of Financial Studies*, 22(6), 2201–2238.

CAMPBELL, J. Y., AND R. J. SCHILLER (1991): "Yield Spreads and Interest Rate Movements: A Bird's Eye View," *The Review of Economic Studies*, 58(3), 495–514.

CARROLL, C. D., K. E. DYNAN, AND S. D. KRANE (2003): "Unemployment Risk and Precautionary Wealth: Evidence from Households' Balance Sheets," *The Review of Economics and Statistics*, 85(3), 586–604.

CHOWDHRY, B., AND V. NANDA (1998): "Leverage and market stability: the role of margin rules and price limits," *Journal of Business*, 71(2), 179–210.

CONSTANTINIDES, G. M., AND D. DUFFIE (1996): "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104(2), 219–240.

DAI, Q., AND T. PHILIPPON (2006): "Fiscal Policy and the Term Structure of Interest Rates," *Working Paper*.

DAI, Q., AND K. J. SINGLETON (2006): "Term Structure Dynamics in Theory and Reality," *The Review of Financial Studies*, 16(3), 631–678.

DEN HAAN, W. J. (1996): "Heterogeneity, Aggregate Uncertainty, and the Short-Term Interest Rate," *Journal of Business and Economic Statistics*, 14(4), 399–411.

DONALDSON, J. B., T. JOHNSEN, AND R. MEHRA (1990): "On the Term Structure of Interest Rates," *Journal of Economic Dynamics and Control*, 14(3-4), 571–596.

DUFFIE, D., N. GARLEANU, AND L. H. PEDERSEN (2005): "Over-the-Counter Markets," *Econometrica*, 73(6), 1815–1847.

ELMENDORF, D. W., AND G. N. MANKIW (1999): "Government Debt," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, vol. 1, chap. 25, pp. 1615–1669. Elsevier, 1st edn.

ENGEN, E. M., AND J. GRUBER (2001): "Unemployment Insurance and Precautionary Saving," *Journal of Monetary Economics*, 47(3), 545–579.

FAHRI, E., AND J. TIROLE (2008): "Competing Liquidities: Corporate Securities, Real Bonds and Bubbles," *NBER Working Paper*, No. 13955.

FLODEN, M. (2001): "The Effectiveness of Government Debt and Transfers as Insurance," *Journal of Monetary Economics*, 48(1), 81–108.

GOMES, F. J., AND A. MICHAELIDES (2008): "Asset Pricing with Limited Risk Sharing and Heterogeneous Agents," *The Review of Financial Studies*, 21(1), 415–448.

HAMILTON, J. D. (1994): *Time Series Analysis*. Princeton University Press.

HAMILTON, J. D., AND D. H. KIM (2002): "A Reexamination of the Predictability of Economic Activity Using the Yield Spread," *Journal of Money, Credit and Banking*, 34(2), 340–360.

HEATHCOTE, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *The Review of Economic Studies*, 72(1), 161–188.

HEATON, J., AND D. J. LUCAS (1992): "The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model," *Journal of Economic Dynamics and Control*, 16(3-4), 601–620.

——— (1996): "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy*, 104(3), 443–487.

HOLMSTRÖM, B., AND J. TIROLE (1998): "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106(1), 1–40.

——— (2001): "LAPM: A Liquidity-Based Asset Pricing Model," *The Journal of Finance*, 56(5), 1837–1867.

HUGGETT, M. (1993): "The Risk Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

KEHOE, T. J., AND D. K. LEVINE (2001): "Liquidity Constrained Markets versus Debt Constrained Markets," *Econometrica*, 69(3), 575–578.

KRISHNAMURTHY, A., AND A. VISSING-JORGENSEN (2008): "The Aggregate Demand for Treasury Debt," *mimeo*.

KRUSELL, P., AND A. A. SMITH (1997): "Income and Wealth Heterogeneity, Portfolio Choice and Equilibrium Asset Returns," *Macroeconomic Dynamics*, 1(2), 387–422.

KRUSSEL, P., T. MUKOYAMA, AND A. A. J. SMITH (2008): "Asset Prices in a Huggett Economy," *mimeo*.

LAGOS, R., AND G. ROCHETEAU (2007): "Search in Asset Markets: Market Structure, Liquidity, and Welfare," *The American Economic Review*, 97(2), 198–202.

——— (2009): "Liquidity in Asset Markets with Search Frictions," *Econometrica*, 77(2), 403–426.

LAUBACH, T. (2009): "New Evidence on the Interest Rate Effects of Budget Deficits and Debt," *Journal of the European Economic Association*, 7(4), 858–885.

LONGSTAFF, F. A. (2004): "The Flight-to-Liquidity Premium in U.S. Treasury Bond Prices," *Journal of Business*, 77(3), 511–526.

MANKIW, G. (1986): "The Equity Premium and the Concentration of Aggregate Shocks," *Journal of Financial Economics*, 17(1), 211–219.

MEHRA, R., AND E. C. PRESCOTT (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15(2), 145–161.

PAGANO, M. (1989): "Endogenous Market Thinness and Stock Price Volatility," *The Review of Economic Studies*, 56(2), 269–287.

REINHART, V., AND B. SACK (2000): "The Economic Consequences of Disappearing Government Debt," *Brookings Papers on Economic Activity*, 31(2000-2), 163–220.

SCHEINKMAN, J., AND L. WEISS (1986): "Borrowing Constraints and Aggregate Economic Activity," *Econometrica*, 54(1), 23–45.

STORESLETTEN, K., C. TELMER, AND A. YARON (2007): "Income and Wealth Heterogeneity, Portfolio Choice and Equilibrium Asset Returns," *Review of Economic Dynamics*, 10(4), 519–548.

VAYANOS, D., AND J.-L. VILA (2007): "A Preferred-Habitat Model of the Term Structure of Interest Rates ," *mimeo*.

VAYANOS, D., AND P.-O. WEILL (2008): "A Search-Based Theory of the On-the-Run Phenomenon," *The Journal of Finance*, 63(3), 1361–1398.

WEIL, P. (1992): "Equilibrium Asset Prices with Undiversifiable Labor Income Risk," *Journal of Economic Dynamics and Control*, 16(3-4), 769–790.

WOODFORD, M. (1990): "Public Debt as Private Liquidity," *The American Economic Review*, 80(2), 382–388.

Documents de Travail

280. D. Durant et L. Frey, "Une Première comparaison des droits à pension des ménages français et américains," Avril 2010

281. G. Bertola, A. Dabusinskas, M. Hoeberichts, M. Izquierdo, C. Kwapil, J. Montornès and D. Radowski, "Price, Wage and Employment Response to Shocks: Evidence from the WDN Survey," May 2010

282. J. Montornès and J-B. Sauner-Leroy, "Wage-setting Behavior in France: Additional Evidence from an Ad-hoc Survey," May 2010

283. R. Bourlès, G. Cette, J. Lopez, J. Mairesse and G. Nicoletti, "Do product market regulations in upstream sectors curb productivity growth? Panel data evidence for OECD countries," June 2010

284. Ph. Askenazy, Th. Breda and D. Irac, "Innovation and Advertising: Theory and Evidence," May 2010

285. M. Lemoine and C. Mougin, "The Growth-Volatility Relationship: New Evidence Based on Stochastic Volatility in Mean Models," July 2010

286. C. Bouthevillain and G. Dufrénot, "Are the effects of fiscal changes different in times of crisis and non-crisis? The French Case," July 2010

287. S. Avouyi-Dovi, D. Fougère and E. Gautier, "Wage rigidity, collective bargaining and the minimum wage: evidence from french agreement data," July 2010

288. P. Askenazy, C. Célérier et D. Irac, "Vente à distance, internet et dynamiques des prix," juillet 2010

289. G. Dufrénot, P. Frouté and C. Schalck, "The French Regions' Borrowing Behaviours," july 2010

290. C. Bordes and L. Clerc, "The ECB art of central banking and the separation principle," August 2010

291. R. Jimborean and J-S. Mésonnier, "Banks' financial conditions and the transmission of monetary policy: a FAVAR approach," September 2010

292. G. Dufrénot and L. Paul, "Fiscal development in the euro aera beyond the crisis: some lessons drawn from fiscal reaction functions," October 2010

293. R. Cooper, H. Kempf and D. Peled, "Insulation impossible: monetary policy and regional fiscal spillovers in a federation," October 2010

294. C. Célérier, "Compensation in the financial sector: are all bankers superstars?," October 2010

295. O. de Bandt and S. Malik, "Is there evidence of shift-contagion in international housing markets?," October 2010

296. F. Ferroni, "Did Tax Policies mitigate US Business Cycles?," October 2010

297. E. Challe and X. Ragot, "Fiscal policy in a tractable liquidity-constrained economy," October 2010

298. P. Cahuc and E. Challe, "Produce or speculate? Asset bubbles, occupational choice and efficiency," October 2010

299. H. Kempf and G. Rota Graziosi, "Endogenizing leadership in tax competition: a timing game perspective," October 2010

300. X. Ragot, "The Case for a Financial Approach to Money Demand," October 2010

301. E. Challe, F. Le Grand and X. Ragot, "Incomplete markets, liquidation risk, and the term structure of interest rates," October 2010

Pour accéder à la liste complète des Documents de Travail publiés par la Banque de France veuillez consulter le site :
http://www.banque-france.fr/fr/publications/documents_de_travail/documents_de_travail_10.htm

For a complete list of Working Papers published by the Banque de France, please visit the website:
http://www.banque-france.fr/fr/publications/documents_de_travail/documents_de_travail_10.htm

Pour tous commentaires ou demandes sur les Documents de Travail, contacter la bibliothèque de la Direction Générale des Études et des Relations Internationales à l'adresse suivante :

For any comment or enquiries on the Working Papers, contact the library of the Directorate General Economics and International Relations at the following address :

BANQUE DE FRANCE
49- 1404 Labolog
75049 Paris Cedex 01
tél : 0033 (0)1 42 97 77 24 ou 01 42 92 62 65 ou 48 90 ou 69 81
email :   marie-christine.petit-djemad@banque-france.fr
          jeannine.agoutin@banque-france.fr
          michael.brassart@banque-france.fr
          veronique.jan-antuoro@banque-france.fr
          nathalie.bataille-salle@banque-france.f