Asset returns, idiosyncratic and aggregate risk exposure*

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Abstract

We present a class of tractable incomplete-market models, where agents face both aggregate risk and financial market participation costs. Tractability relies on the assumption of a periodic utility function which is linear beyond a threshold, following a contribution of Fishburn (1977) in decision theory. We prove equilibrium existence and derive theoretical results about asset prices and consumption choices. The model is able to match US data and to quantitatively reproduce a low safe return and a high equity premium, together with a realistic exposure of households to both idiosyncratic and aggregate risks.

Keywords: Incomplete markets, risk sharing, consumption inequalities.

JEL codes: E21, E44, D91, D31.

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1 Introduction

Infinite-horizon incomplete-insurance market models with credit constraints are known to be difficult to solve in the presence of aggregate shocks. These models generate a large amount of heterogeneity, typically a time-varying distribution of agents with continuous support, such that numerical methods are needed to approximate the equilibrium (Krusell and Smith, 1998). The existence of simple recursive equilibria in such environments is still an open question (Miao, 2006). These economies are nevertheless appealing. First, they can account for a significant heterogeneity across households in consumption, wealth or income as is observed in the data. Second, incomplete markets can contribute to the explanation of certain asset price properties, which are otherwise hard to rationalize in complete market environments.

In this paper, we present a class of incomplete market models allowing for theoretical investigations of equilibrium allocations and asset prices. More precisely, we are able to prove the existence of an equilibrium with aggregate shocks, heterogeneous levels of idiosyncratic risks and stock market participation costs, where we can analytically analyze the main determinants of the risk allocation and of asset prices. The model is based on two assumptions. First, we assume that the periodic utility function is linear beyond a certain threshold, while strictly concave before, though being globally smooth. This utility function was first introduced in decision theory by Fishburn (1977) to analyze risk for "below-target returns".[1] We show that this utility function provides tractability in incomplete-market models in an interesting way when compared to alternatives. In particular, incomplete market models have often relied on quasi-linearity in the labor supply to reduce the dimension of the state space (Scheinkman and Weiss, 1986; Lagos and Wright, 2005; Challe, LeGrand and Ragot, 2013, among others). This assumption has nevertheless the drawback that the consumption of agents with infinite labor elasticity is constant and pinned down by model parameters. In particular, consumption is independent of wealth and income. The infinite Frisch elasticity of labor supply is also far too high at household level (see Hall, 2010, for a recent survey). Linearity in the periodic utility function may thus be an attractive alternative choice to study consumption dynamics. Our second assumption is that the supply of securities is not too large. This implies that credit constraints bind after a small number of periods, generating an equilibrium with a small number of heterogeneous agents. Our

[1] This captures the idea that investors are averse to risk for low returns (below a given target), while they care much less about risk for high returns.

economy therefore features a "small-trade" equilibrium, where prices can be analytically studied, as in the no-trade equilibrium of Constantinides and Duffie (1996) or Krusell, Mukoyama and Smith (2011), but where we can also investigate consumption allocations as well as the role of security volumes and liquidity. Further financial frictions, such as participation costs, can also be introduced.

We show that our model is able to reproduce realistic household risk exposure and asset prices. It is known that models with only incomplete insurance markets fail to reproduce the equity premium, for a realistic calibration (Krusell and Smith, 1998 and Krusell, Mukoyama and Smith, 2011), while limited participation and preference heterogeneity can help in reproducing relevant aspects of asset prices (Guvenen, 2009). In this paper, we prove that incomplete markets together with heterogeneous idiosyncratic risk exposure and participation costs can generate a low return for the safe asset, a high equity premium and a realistic household risk exposure, in a model where agents have identical preferences. Indeed, the stock market participation cost for high idiosyncratic risk agents generates market segmentation. These high-risk agents do not hold stocks, but trade safe bonds to self-insure against their idiosyncratic risk, which generates a low bond return. Low-idiosyncratic-risk agents participate in the stock market, and thus face a higher exposure to aggregate risk, which implies a high return for holding stocks. The combination of heterogeneous individual risk exposure with participation costs is therefore a key ingredient of our model. In this setup, we show that a higher volume of securities decreases asset prices and improves consumption smoothing, whereas a higher level of idiosyncratic risk generates both a decrease in the bond interest rate and an increase in stock prices. Interestingly, these features are consistent with the trends observed in the aftermath of the 2008 crisis. We finally take advantage of the tractability of our framework to estimate key parameters and show that the model is able to match the data on consumption allocations and asset prices. As in the data, the model generates a higher volatility of the consumption growth rate for low-income households than for high-income households (Gomes and Michaelides, 2008; Guvenen, 2009; De Giorgi and Gambetti, 2012; Gârleanu, Kogan and Panageas, 2012; Meyer and Sullivan, 2013, among others), and a consumption growth rate for high-income households, which is more correlated with the aggregate consumption growth rate than the one of low-income agents. In addition, high-income households are found to bear a larger fraction of the aggregate risk in our model than low income households, while the latter face a larger total risk than the former (Parker and Vissing-Jorgensen (2009)). Consistently with Krusell and Smith (1998) and Krusell, Mukoyama and Smith (2011), we verify that the model, when calibrated in absence of participation costs, fails to jointly reproduce risk exposure and asset prices.

The paper contributes to the theoretical literature on incomplete-market models. In this literature, analytical tractability can be obtained in a no-trade equilibrium, as in Constantinides and Duffie (1996), where idiosyncratic risk is assumed to be persistent. Alternatively, Krusell, Mukoyama and Smith (2011) study transitory shocks with credit constraints, assuming that the volume of assets in the economy is zero. In these economies, no trade takes place by construction but assets can be priced. In our model, trades do occur at the equilibrium and shocks are transitory. Our assumption of linearity in the utility function is reminiscent of several papers that consider linearity in consumption or leisure utility or in the production function, so as to reduce ex-post heterogeneity, such as Scheinkman and Weiss (1986). While Lagos and Wright (2005) consider a model with linear dis-utility in labor in order to reduce heterogeneity in the time-dimension (every agent has the same marginal utility of consumption at the end of each period), we use linearity to reduce heterogeneity in the state-dimension: all the agents in the same individual "state" will have the same marginal utility. In a similar vein, Kiyotaki and Moore (2005, 2008), and Miao and Wang (2015) consider that entrepreneurs face idiosyncratic investment opportunities with a constant return-to-scale. This constant marginal productivity of idiosyncratic investments reduces the heterogeneity among entrepreneurs. Dang, Holmstrom, Gorton, and Ordoñez (2014) introduce a piecewise linear utility function to model the urgency to consume a certain amount of goods.[2] Finally, this paper generalizes some previous works (Challe, Le Grand and Ragot 2013; Challe and Ragot 2014 or Le Grand and Ragot 2015). We develop here a new framework, which allows us to theoretically analyze properties of both prices and consumption allocations.

The paper is also related to the vast quantitative literature on asset prices with heterogeneous agents. In this literature, our contribution consists in explaining asset prices and household risk exposure with just two assumptions, i.e., limited insurance markets and participation costs. Guvenen (2009) shows that asset price properties can be rationalized in an economy with exogenous limited participation and household heterogeneity in intertemporal elasticities of substitution.

[2] Portfolio choices would be indeterminate in our setup with such a utility function, because marginal utility would not depend on households' portfolio composition. The (strict) concavity below a threshold of Fishburn's utility function ensures that portfolio choices are determinate in our model.

Constantinides and Ghosh (2014) build on Constantinides and Duffie (1996) to construct a no-trade equilibrium with Epstein-Zin preferences. Chien, Cole and Lustig (2011, 2012) consider an incomplete market model featuring exogenous trading restrictions, which can easily be simulated. Gomes and Michaelides (2008), following the seminal contribution of Krusell and Smith (1998), show that the equity premium can be reproduced in a model with several ingredients, such as preference heterogeneity, incomplete markets and limited participation.

The remainder of the paper is organized as follows. In Section 2, we present the model and derive our existence result. In Section 3, we present the intuition underlying our model in simplified versions of our framework. In Section 4, we perform a quantitative exercise to show that the model can reproduce household risk exposures and asset returns. Section 5 discusses the key assumptions of the model. Section 6 concludes.

2 The model

The model has three assumptions. First, insurance markets are incomplete. Second, agents face stock market participation costs. Third, utility function is linear beyond a certain threshold.

2.1 Risks and securities

The economy is populated by a large number of infinitely-lived agents. Time is discrete and indexed by $t = 0, 1, \ldots$. There are two populations of ex-ante different agents (the ex-ante heterogeneity will be made clearer later on). Each population $i = 1, 2$ is distributed on a segment J_i according to an non-atomic measure, denoted ℓ_i. Each segment is of length 1: $\ell_i(J_i) = 1$.[3] We call these populations "type-1" and "type-2" agents.

2.1.1 Aggregate risk

There is a single aggregate shock z_t in each period t, which can take n different values in the state space $Z = \{z_1, ..., z_n\}$. Values of z_k are assumed to be pairwise distinct. The index $k = 1, \ldots, n$ characterizes the aggregate state. Moreover, we assume that the aggregate risk

[3] Among others, Feldman and Gilles (1985) have identified issues when applying the law of large number to a continuum of random variables. Green (1994) describes a construction of the sets J_i and of the measures ℓ_i to ensure that our statements hold. Feldman and Gilles (1985), Judd (1985), and Uhlig (1996) also propose other solutions to this issue. From now on, we assume that the law of large numbers applies.

process $\tilde{Z} = (z_t)_{t=0,1,\ldots}$ is a time-homogeneous first-order Markov chain whose transition matrix is denoted $\Pi = (\pi_{kj})_{k,j=1,\ldots,n}$. The probability of moving from state $k = 1,\ldots,n$ to state $j = 1,\ldots,n$ is thus constant and denoted π_{kj}. For every date $t \geq 0$, $z^t \in Z^{t+1}$ denotes a possible history of aggregate shocks up to date t, which is defined as a possible realization of the $(t+1)-$tuple $(z_0,\ldots,z_t)$.

2.1.2 Asset markets

Agents can hold two types of assets –a risky stock and a riskless bond. Although the introduction of other assets would not be difficult in our setting, we confine our attention to the simplest market structure with limited participation.

The risky asset. Agents can trade shares of a Lucas tree, whose mass V_X remains constant over time. The tree dividend payoff is stochastic and the dividend in state k be y_k $(k = 1,\ldots,n)$. At any date t, the (endogenous) price of one unit of the tree is P_t. Shares of the tree will be called "stocks" or "risky assets" in the remainder of the paper.

The bond. Agents can also purchase riskless bonds of maturity one. Purchased at any date t at price Q_t, these short-term bonds pay off one unit of the consumption good at the next date in all states of the world. The total supply of bonds is constant and equal to V_B. These bonds are issued by the State and funded by taxes, as explained below. We will also refer to bonds as "safe assets" in the remainder of the paper.

As we will see later on, participation in the bond market is free, while participation in the stock market may require the payment of a periodic participation cost.

2.1.3 Idiosyncratic risk

Agents face an idiosyncratic risk in addition to the aforementioned aggregate risk. This individual risk can neither be avoided nor insured. We call this a productivity risk even though it may cover many different individual risks (such as the risks of unemployment, income, health, etc.) that are likely to affect their productivity (see Chatterjee, Corbae, Nakajima and Rios-Rull, 2007 for a quantitative discussion). At any point in time, type-i agents can either be *productive* (denoted herein by p) earning income $\omega^i(z_t)$ or *unproductive* (denoted by u), earning income

δ^i.[4] Both incomes may depend on the agent type i. To simplify the exposition, we assume that δ^i does not depend on z_t, but all our results can be easily extended to stochastic incomes δ^i. It is assumed that, regardless of the aggregate state, $\omega^i(z_t)$ is greater than δ^i for both agent types. Moreover, type-1 agents when productive have a higher income than type-2 agents. We refer to type-1 agents as "high-income" agents and to type-2 as "low-income" agents. These assumptions will be summarized in Assumption B below.

For each type-i agent $j \in J_i$ at any date t, the function $\xi_t^{i,j}(z^t)$ is a random variable that characterizes the current status of the agent's productivity, taking the value 1 when the agent is productive and 0 when unproductive. Therefore, agent j earns a total income of $(1-\xi_t^{i,j}(z^t))\delta^i + \xi_t^{i,j}(z^t)\omega^i(z_t)$. We assume that for each agent, the productivity risk process $(\xi_t^{i,j}(z^t))_{t=0,1,\ldots}$ is a two-state Markov-chain. When productive in period $t-1$, the probability of remaining productive in the next period t for any type-i agent is $\alpha_t^i(z^t) \in (0,1)$, while the probability of becoming unproductive is thus $1-\alpha^i(z^t)$. Similarly, the probability of remaining unproductive in date t is $\rho_t^i(z^t) \in (0,1)$. The transition matrix is thus $T_t^i = \begin{pmatrix} \alpha_t^i(z^t) & 1-\alpha_t^i(z^t) \\ 1-\rho_t^i(z^t) & \rho_t^i(z^t) \end{pmatrix}$. Time-varying transition rates are introduced for the sake of generality as idiosyncratic risks might depend on the state of the world.[5] For every date t, $\xi^{i,j,t} \in \{0,1\}^t = E^t$ denotes a possible history of individual shocks for agent $j \in J_i$ up to date t, which is defined as a possible realization of the tuple $(\xi_0^{i,j}, \ldots, \xi_t^{i,j})$.

We call $\eta_t^i \in (0,1)$ the share of productive agents among type-i population. Initial values η_0^1 and η_0^2 being given, the laws of motion of productive shares are

$$\eta_t^i(z^t) = \alpha_t^i(z^t)\eta_{t-1}^i(z^{t-1}) + (1-\rho_t^i(z^t))(1-\eta_{t-1}^i(z^{t-1})), \text{ for } i = 1,2 \text{ and } t \geq 1. \tag{1}$$

To obtain a tractable framework, we impose the following constraint:

Assumption A (Population shares) *The probability of remaining productive in the next period depends solely on the current aggregate state: $\alpha_t^i(z^t) = \alpha^i(z_{t-1})$.*

The shares of unproductive and productive agents depend only on the current state of the world z_t: $\eta_t^i(z^t) = \eta^i(z_t)$ for $t \geq 0$ and $i = 1,2$.

[4] Our idiosyncratic productivity risk is reminiscent of Kiyotaki and Moore (2005, 2008), Kocherlakota (2009) and Miao and Wang (2015), although the model and the paper scope are very different.

[5] Krusell and Smith (1998) provide estimations of time-varying transition rates for employment risk.

This assumption simplifies the dynamics of the population structure, and thus the algebra, but is not sufficient to obtain analytical tractability. In the general case, even if this condition is fulfilled, the wealth distribution of agents is continuous, preventing us from obtaining an analytical characterization. Assumption A includes the standard case where the transition probabilities for each type-i agent are constant and equal to α^i and ρ^i. In this case, if the initial share of productive type-i agents is $\eta_0^i = \frac{1-\rho^i}{2-\alpha^i-\rho^i} \in (0,1)$, then the share remains constant over time (i.e., $\eta_t^i = \eta_0^i$ at all dates t). Assumption A also includes the case where the shares, $\eta^i(z_t)$, and the probability of remaining a productive agent, $\alpha^i(z_{t-1})$, are stochastic. In this case, the transition rate ρ_t^i adjusts so that the law of motion in equation (1) holds. It notably implies that the probability of remaining unproductive in the next period depends only on the current and previous aggregate states: $\rho_t^i(z^t) = \rho_t^i(z_{t-1}, z_t)$. Note that Assumption A implies that the primitives of our model are the probabilities α_t^i and the shares η_t^i, while the probabilities ρ_t^i adjust. The severity of the idiosyncratic risk and the initial endowments are the sole source of ex-ante heterogeneity among agents.

2.2 Agents' preferences

Agents' preferences are a crucial feature of the model for tractability, enabling us to derive our small-trade equilibrium with a finite number of states.

Description of preferences. The periodic utility function is strictly increasing and globally concave. It is strictly concave for low values of consumption and has possibly 2 linear parts. This assumption can formally be written through conditions imposed on marginal utilities

$$\tilde{u}'(c) = \begin{cases} u'(c) & \text{if } c \leq c_1^*, \\ \lambda^2 & \text{if } c_2^* \leq c \leq c_3^*, \\ \lambda^1 < \lambda^2 & \text{if } c_4^* \leq c \leq c_5^*. \end{cases} \tag{2}$$

Figure 1 plots the shape of such a periodic utility function.

When agents consume a low amount, they value their consumption with the marginal utility $u'(\cdot)$, which is the derivative of a function $u : \mathbb{R}^+ \to \mathbb{R}$ assumed to be twice derivable, strictly increasing, and strictly concave. When agents consume a higher amount, their marginal utility is constant for two consumption intervals, on which it is equal to λ^i for $i = 1, 2$.

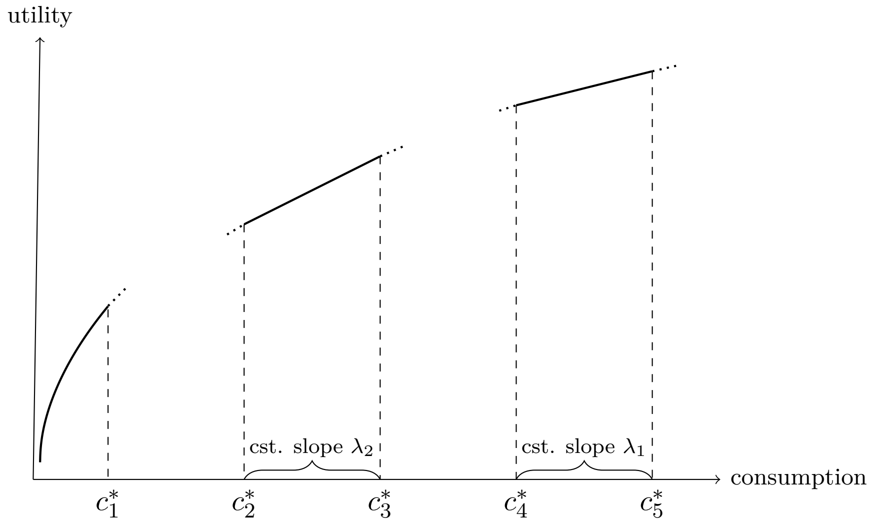

Figure 1: Shape of the periodic utility function

Assumption B (Income processes) *We assume that in any state* $k = 1, \ldots, n$*, we have* $c_2^* < \omega^2(z_k) < c_3^*$*,* $c_4^* < \omega^1(z_k) < c_5^*$ *and* $\delta^i < c_1^*$ *for* $i = 1, 2$*. This notably implies that in any state* $k = 1, \ldots, n$*, we have* $\delta^i < \omega^i(z_k)$ *for both types* $i = 1, 2$ *and that* $\omega^1(z_k) > \omega^2(z_k)$*.*

Assumption B states that the income of productive agents lies in the set where the utility function is linear, and that the income of unproductive agents lies in the set where the utility is strictly concave. A straightforward corollary of this assumption is that in the absence of trade, unproductive agents are worse off than productive ones (for both types). In other words, $u'(\delta^i) > \lambda^i$ for $i = 1, 2$.

Consequence on the equilibrium. To push the interpretation further, we need to slightly anticipate the equilibrium construction below. We construct an equilibrium where unproductive agents of both types consume a low amount that they value at the strictly concave part of the utility function. Productive agents of type i consume a higher amount that they value with a linear utility of slope λ^i. This assumption of constant marginal utility for productive agents helps generate a limited heterogeneity equilibrium. Indeed, this assumption implies that the individual history of productive agents does not matter for the pricing of securities since their marginal utility depends only on their type. These marginal utilities are independent of their wealth or past saving choices. Assumption B guarantees that when security supply is null and when agents do not trade, consumption levels are consistent with marginal utilities:

unproductive agents are endowed with a strictly decreasing marginal utility, while productive ones are endowed with constant marginal utilities.

Interpretation. The utility function introduced in equation (2) can be seen as a generalization of Fishburn (1977), who considers a concave-linear utility function. Fishburn's utility function is linear above a given threshold and strictly concave below it. In a portfolio choice problem, the agent endowed with such a utility, is risk-neutral for large payoffs and risk-averse for low ones. Loosely speaking, this functional form reflects the asymmetry in risk perception. Payoff realizations that are lower than a given threshold are perceived as actual risks, while payoffs greater than the threshold are perceived as being "nice surprises". The concave-linear utility function attributes therefore different statuses to under- and over-performances. As explained by Fishburn (1977, p. 123), this concave-linear functional form is "motivated by the observation that decision makers in investment contexts frequently associate risk with failure to attain a target return". In this paper, we introduce this utility function in an incomplete-market framework, especially to gain tractability. Furthermore, we generalize this assumption and assume that there are two consecutive thresholds and two linear parts. Note that it would be strictly equivalent to assume that each agent type $i = 1, 2$ is endowed with a Fishburn concave-linear utility, where utilities of both types differ in the threshold and the slope of the linear part (but have identical concave parts). In this case, we would have two utilities $\tilde{u}_i$ $(i = 1, 2)$ with a concave part u –the same for both types– and a linear part λ_i. We discuss this assumption further in Section 5, after the quantitative exercise.

2.3 Agent's program

Timing. At the beginning of every period, each agent observes their current productivity status and the current risky dividend payoff for the period. Thus the agent knows the entire history of both aggregate and idiosyncratic shocks up to that date.

Allocations. Due to the timing of the agent's program, agents' choices –consumption levels and demands for stocks and bonds– at date t are mappings defined over the state space of possible shock histories $Z^t \times E^t$. We call $(c_t^{i,j} : Z^t \times E^t \to \mathbb{R}^+)_{t \geq 0}$ the consumption plan for a type-i agent $j \in J_i$, whose consumption levels are assumed to be positive. The stock demand plan is $(x_t^{i,j} : Z^t \times E^t \to \mathbb{R})_{t \geq 0}$ while $(b_t^{i,j} : Z^t \times E^t \to \mathbb{R})_{t \geq 0}$ is the demand plan for bonds.

Participation costs. Participation in the stock market is costly for agents, even though trading riskless bonds is free. Trading stocks requires type-i agents to pay in every period a lump-sum participation cost χ_i, $i = 1, 2$. This periodic participation cost will generate endogenous stock market limited participation. Note that we have made the choice that the participation cost is paid in every period when the agent purchases stocks. However, no cost has to be paid for the agent to liquidate his stock portfolio. Participation costs are a frequent device to understand limited participation in the stock market. Vissing-Jorgensen (2002) provides a discussion and lower bounds for participation costs.[6]

We state an assumption regarding stock market participation cost. Without further constraints on χ_1 and χ_2, a number of different market structures are possible. Consistently with the empirical data presented in Section 4, we set participation costs, such that type-1 agents trade stocks, while type-2 agents do not. More precisely:

Assumption C (Participation costs) *We assume that* $\chi_2 = \overline{\chi}_2$ *is large enough for type-2 agents not to participate to the stock market. Type-1 agents do not pay participation costs:* $\chi_1 = 0$.

The intuition for the existence of a value $\overline{\chi}_2$, which guarantees that type-2 agents do not wish to buy stocks, is quite straightforward. The return of savings in stocks for type-2 agents, net of participation cost, must be lower than the return of savings in bonds, in all states of the world. The formula for the lower bound of $\overline{\chi}_2$ is given in Section 2.7, after the description of the equilibrium properties. It would have been possible to introduce a non-zero participation cost for type-1 agents and to discuss more general market structures.[7] However, for the sake of simplicity, this paper focuses on the minimal setup that generates endogenous participation for type-2 agents, which allows our model to be consistent with empirical facts. We discuss this in Section 4.

Budget and borrowing constraints. At each date, the choices of a type-i agent $j \in J_i$ are limited by a budget constraint in which total resources made up of income, stock dividends, and security-sale values are used to consume, pay taxes, and purchase securities. The budget

[6] Removing the participation cost would not impair equilibrium existence. However, it would change market participation structure and limit the ability of the model to jointly reproduce asset prices and consumption inequalities, as shown in the quantitative exercise.

[7] The finite-state equilibrium indeed simplifies to some extent the analysis of endogenous limited market participation.

constraint can be expressed at any date t as follows:[8]

$$c_t^{i,j} + P_t\, x_t^{i,j} + Q_t b_t^{i,j} + \chi_i 1_{x_t^{i,j}>0} = (1-\xi_t^{i,j})\delta^i + \xi_t^{i,j}(\omega_t^i - \tau_t^i) + (P_t + y_t)x_{t-1}^{i,j} + b_{t-1}^{i,j}, \tag{3}$$

where $\xi_t^{i,j} = 1$ if agents are productive, $\xi_t^{i,j} = 0$ if agents are unproductive and where $1_{x_t^{i,j}>0}$ is the indicator function.

In addition, agents face borrowing constraints. They can neither produce any share of stocks nor short-sell the bond. This implies, for any agent $j \in J_i$ at date $t \geq 0$, that:[9]

$$x_t^{i,j}, b_t^{i,j} \geq 0. \tag{4}$$

A feasible allocation is a collection of plans $(c_t^{i,j}, x_t^{i,j}, b_t^{i,j})_{t\geq 0}^{j\in J_i, i=1,2}$ such that equations (3) and (4) hold at any date t. We call $\mathcal{A}_i$ the set of feasible allocations for a type-i agent.

Agent's program. The program of a type-i agent $j \in J_i$ consists in finding the feasible allocation that maximizes his intertemporal utility subject to the budget constraint (3), the borrowing limits (4) and a transversality condition ruling out exploding paths. Instantaneous utilities are discounted by a common factor $\beta \in (0,1)$ representing the common exogenous time preference. The operator $E_0[\cdot]$ is the unconditional expectation over the aggregate and uninsurable idiosyncratic shocks. The initial financial asset endowments are denoted by $x_{-1}^{i,j}$ and $b_{-1}^{i,j}$. The agent's program can be expressed as ($j \in J_i$ and $i = 1, 2$):

$$\max_{(c_t^{i,j}, x_t^{i,j}, b_t^{i,j})_{t\geq 0} \in \mathcal{A}_i} E_0\left[\sum_{t=0}^{\infty} \beta^t \tilde{u}(c_t^{i,j})\right] \tag{5}$$

$$\text{s.t.} \forall t \geq 0,\ c_t^{i,j} + P_t\, x_t^{i,j} + Q_t b_t^{i,j} + \chi_i 1_{x_t^{i,j}>0} = (1-\xi_t^{i,j})\delta^i + \xi_t^{i,j}(\omega_t^i - \tau_t^i) + (P_t + y_t)x_{t-1}^{i,j} + b_{t-1}^{i,j},$$

$$\forall t \geq 0,\ x_t^{i,j}, b_t^{i,j} \geq 0,$$

$$\lim_{t\to\infty} \beta^t E_0\left[\tilde{u}'(c_t^{i,j}) x_t^{i,j}\right] = \lim_{t\to\infty} \beta^t E_0\left[\tilde{u}'(c_t^{i,j}) b_t^{i,j}\right] = 0,$$

$$\{x_{-1}^{i,j}, b_{-1}^{i,j}, \xi_0^{i,j}, z_0\} \text{ are given.}$$

Agents' risk-sharing is limited along three dimensions. First, as in the Bewley-Huggett-

[8] We drop the dependence on z^t and $\xi^{i,j,t}$ to lighten notations.

[9] It would be possible to have strictly negative (but not too loose) borrowing constraints on bonds and stocks, while preserving the equilibrium existence. However, the set $\mathcal{V}$ of admissible security volumes defined below in Proposition 1 would be different.

Aiyagari literature, individual risk is uninsurable because no asset is contingent on the productivity status. Second, agents face participation and borrowing constraints and are prevented from short-selling assets. Finally, the set of securities may not complete the insurance market for aggregate risk as the number of aggregate states, n, may be (strictly) larger than the number of tradable securities (which is 2).

2.4 The government

The government issues short-term bonds that pay off one unit of the consumption good in the next period. The aggregate supply of short-term bonds is assumed to remain constant over time and equal to V_B. The government levies taxes on productive agents to finance public debt. In the absence of government consumption, taxes and the issuing of new bonds exactly cover the payoffs of maturing bonds. Moreover, the tax τ_t on productive agents of both types is assumed to be proportional to the productive agent's income.[10] Hence, a balanced government budget constraint at any date t implies that the tax rate is

$$\tau_T = \frac{(1-Q_t)V_B}{\omega_t^1\eta_t^1 + \omega_t^2\eta_t^2}. \tag{6}$$

2.5 Equilibrium definition

Before turning to the definition of the competitive equilibrium, we express the security market clearing conditions. Each security market clears whenever aggregate demand equals total supply, which is equal to V_X for stocks and V_B for bonds. We define the probability measure $\Lambda_t^i : \mathcal{B}(\mathbb{R})^2 \times \mathcal{B}(E^t) \to [0,1]$ describing the distribution of type-i agents as a function of their security holdings and the history of their individual status.[11] As an example, $\Lambda_t^i(X, B^S, I)$ (with $(X, B^S, I) \in \mathcal{B}(\mathbb{R})^2 \times \mathcal{B}(E^t)$) is the measure of agents of type i, with holdings in risky assets $x \in X$, in bonds $b \in B^S$, and with an individual history $\xi \in I$: $\Lambda_t^i(X, B^S, I) = \ell_i(\{j \in J_i : (x_t^{i,j}, b_t^{i,j}, \xi_t^{i,j}) \in$

[10] A time-varying bond supply would not change the conclusions of the paper. Moreover, a lump-sum tax would not quantitatively change the results, as will be clear after the presentation of the equilibrium structure.

[11] For any metric space X, $\mathcal{B}(X)$ denotes the borel sets of X.

$(X, B^S, I)\}$). The market-clearing conditions can therefore be written as

$$\sum_{i=1,2} \int_{\mathbb{R}^2 \times E^t} x \Lambda_t^i(dx, db, d\xi) = V_X, \tag{7}$$

$$\sum_{i=1,2} \int_{\mathbb{R}^2 \times E^t} b \Lambda_t^i(dx, db, d\xi) = V_B. \tag{8}$$

Finally, by Walras' law, the good market clears when the asset markets clear. We can now define a sequential competitive equilibrium.

Definition 1 (Sequential competitive equilibrium) *A sequential competitive equilibrium is a collection of allocations* $(c_t^{i,j}, x_t^{i,j}, b_t^{i,j})_{t\geq 0}^{j\in J_i}$ *for* $i = 1, 2$ *and of price processes* $(P_t, Q_t)_{t\geq 0}$ *such that, for an initial distribution of stock and bond holdings and of idiosyncratic and aggregate shocks* $\{(x_{-1}^{i,j}, b_{-1}^{i,j}, \xi_0^{i,j})_{i=1,2}^{j\in J_i}, z_0\}$, *we have:*

1. *given prices, individual strategies solve the agents' optimization program in equation (5);*
2. *the security markets clear at all dates: for any* $t \geq 0$, *equations (7) and (8) hold;*
3. *the probability measures* Λ_t^i *evolves consistently with individual strategies in each period.*

2.6 Equilibrium existence

In standard economies featuring uninsurable idiosyncratic shocks, credit constraints and aggregate shocks, the equilibrium cannot be explicitly derived since it involves an infinite distribution of agents (typically of agents' wealth) with different individual histories. The usual strategy follows Krusell and Smith (1998) by computing approximate equilibria assuming a recursive structure. But, as pointed out by Heathcote, Stroresletten, and Violante (2009), the existence of such an equilibrium is still an open question.

In this paper, we prove the existence of an equilibrium and derive its theoretical properties under the assumption that the supply of both risky and riskless assets is not too large. In this case, unproductive agents (i.e., low-income agents) of both types remain credit-constrained even after selling off their entire portfolio. They will not participate in the financial markets while productive agents are trading securities.

More precisely, we construct an equilibrium where the portfolio chosen by each agent depends only on its type, its current productive status, and the aggregate state. In other words, at each date, all type-1 productive agents have the same (time-varying) portfolio and all type-2 productive agents have the same portfolio. In this economy, there are thus only four different portfolios at each point in time. This enables us to construct an equilibrium in which the consumption of productive type-2 agents lies in the set $[c_2^*, c_3^*]$ while the consumption of productive type-1 agents lies in the set $[c_4^*, c_5^*]$. Productive type-1 agents have marginal utility λ^1 and productive type-2 agents have a marginal utility λ^2. Assumption B guarantees that this is the case when securities are in zero supply. Proposition 1 below extends it to positive supplies and proves the existence of a small-trade equilibrium.

Proposition 1 (Equilibrium existence) *We assume that:*

$$\forall k \in \{1, \dots, n\},\ \beta \left(\alpha^1(z_k) + (1 - \alpha^1(z_k)) \frac{u'(\delta^1)}{\lambda^1} \right) < 1. \tag{9}$$

If security volumes (V_B, V_X) *belong to a set* $\mathcal{V} \subset \mathbb{R}_+ \times \mathbb{R}_+$ *defined in (55) –and containing* $(0, 0)$*–, then there exists an equilibrium with the following features:*

1. *the end-of-period security holdings of unproductive type-1 and type-2 agents is* 0 *for both the risky and the riskless assets;*
2. *the end-of-period security holdings of productive agents depend only on their type (*1 *or* 2*) and the current aggregate state;*
3. *the end-of-period holdings in stocks of type-2 agents is always* 0*;*
4. *the security prices depend only on the current aggregate state.*

The proof can be found in the Appendix. In the heterogeneous agent literature, several existence results can already be found. Huggett (1993) proves the equilibrium existence when agents trade short-lived riskless bonds in the absence of aggregate shocks. Recently, Kuhn (2013) extended Huggett's result to economies in which agents face permanent idiosyncratic shocks. In contrast, Krebs (2004) has proven that in this Aiyagari-type setup, if credit constraints never bind, no equilibrium can exist.[12] To our knowledge, Miao (2006) proves the sole existence result

[12] In a seminal paper, Duffie et al. (1994) consider endowment economies in which a finite number of ex-ante heterogeneous agents face aggregate risks and trade long-lived assets with borrowing constraints. They then prove the existence of ergodic equilibria, whose recursive characterization state space includes all endogenous variables

in an economy featuring asset trades, credit constraints, and idiosyncratic and aggregate risks with a continuum of agents. He considers an economy with general preferences in which agents can trade one short-lived asset, which are claims on capital, and where expected discounted utilities are introduced as additional state variables. Our existence result concerns a setup where there is endogenous limited participation and where agents can trade both a short- and long-lived asset.[13]

To prove existence, we do not start from a recursive formulation when state variables only include the security distribution but no endogenous variables as we do not know the conditions for the existence of such a formulation. We prove the equilibrium from first-order conditions, as in Coleman (1991). An additional difficulty is that we cannot directly apply the Kuhn-Tucker theorem to derive first order conditions since it notably requires a Hermitian space of allocations, which is not the case for the set of bounded real sequences (which (c_t^i), (e_t^i), (x_t^i), and (b_t^i) belong to). As a consequence, we follow the steps of the proof of Theorem 4.15 in Stockey and Lucas (1989) to derive the first order conditions.

The equilibrium exists under three conditions. The first one, $\beta(\alpha^1(z_k)+(1-\alpha^1(z_k))\frac{u'(\delta^1)}{\lambda^1}) < 1$ for all k, ensures that stock prices are well defined. If this condition does not hold, the stock price can possibly be infinite because agents are too patient or their desire to self-insure is too high. This existence condition is less restrictive when the discount factor β is low or the idiosyncratic shock is not too severe.[14]

The second condition for ensuring the equilibrium existence is that the volumes of both securities is constrained to belong to a given set $\mathcal{V}$, including the zero volume case $V_X = V_B = 0$. In other words, this assumption implies that security volumes should not be too high. Agents cannot therefore hold too large an amount of assets to self-insure against the idiosyncratic

(such as prices). In a similar vein, Becker and Zilcha (1997) prove the existence of a stationary equilibrium in a production economy with ex-ante heterogeneous agents facing aggregate risk. Krebs (2006) proves the existence of a no-trade equilibrium in a Krusell-Smith economy. Kubler and Schmedders (2002) prove existence of recursive equilibrium with a finite number of agents. These papers consider a finite number of households. This assumption helps in proving existence but makes the analysis of the properties of the equilibrium more difficult as all shocks are "aggregate". It may explain the wide use of Bewley-type model with a continuum of agents.

[13]In our setup it would also be possible to prove that the sequential competitive equilibrium is also a recursive competitive equilibrium in which the state variables are: current aggregate and idiosyncratic shocks and beginning-of-period security holdings for both agent types.

[14]Technically, this condition ensures that the mapping $P \mapsto \beta E_k\left[(\alpha_k^1 + (1-\alpha_k^1)u'(\delta_{k'}^1))(P_{k'} + y_{k'})\right]$, derived from the Euler equation, is a contraction with modulus strictly smaller than 1, where $E_k\left[X_{k'}\right] = \sum_{k'=1}^{n} \pi_{k,k'} X_{k'}$. The Banach fixed-point theorem then allows us to deduce the properties of the price. This condition does not appear in economies with only short-lived assets, which are simpler in this respect.

risk, which guarantees that agents, when becoming unproductive, are credit-constrained. The quantity of available securities is not sufficient for them to unwind their borrowing constraints. A second consequence of this condition on security volumes is that it guarantees that consumption levels of agents belong to the proper sets of the utility functions. More precisely, we already know from Assumption B that in the zero supply equilibrium, consumption levels of unproductive agents are valued with a strictly concave utility, while consumption levels of productive agents are valued with an affine utility. The set $\mathcal{V}$ ensures that consumption levels variations do not remain too large in the positive supply equilibrium, so that unproductive agents still value consumption with a strictly concave utility and productive agents with an affine one.

The equilibrium we consider presents four particular features. First, all unproductive agents are credit constrained. The intuition is as follows. Productive agents want to buy the securities –both the bond and the stock– so as to hedge the risk of becoming unproductive in the next period. If the quantity of available securities is not too large, productive agents agree to pay a high price for these securities. But at that price, unproductive agents would like to short-sell them (but are prevented from doing so by the positive wealth constraint), because their current income is low and they expect to be wealthier in the future. The second feature of our equilibrium is that the saving choices of productive agents only depend on the current aggregate state and on the agent's type. This property critically relies on the quasi-linearity of the utility function, which implies that all productive agents of each type have the same marginal utility and therefore the same demand for assets. The third aspect of our equilibrium is that only type-1 agents trade stocks, while type-2 do not. As already discussed, this result stems from stock market participation costs and in particular from Assumption C. The quasi-linearity of utility function also explains the fourth feature of our equilibrium, according to which security prices depend only on the current aggregate state.

We simplify our notations using the results of Proposition 1. Since security prices depend only on the current aggregate state, we call P_k the price of the risky asset and Q_k the price of the bond in state z_k $(k = 1, \dots, n)$. Bond holdings only depend on the aggregate states and unproductive agents do not hold any assets. We therefore call b_k^i the holdings in bonds of any productive type-i agent in state z_k $(k = 1, \dots, n)$. Since type-2 agents do not trade stocks $(x^2 = 0)$, productive agents hold all stocks and $x_k^1 = \frac{V_X}{\eta_k^1}$ The equilibrium is therefore characterized by a finite sequence of $4 \times n$ variables $(b_k^1, b_k^2, P_k, Q_k)_{k=1 \dots n}$ instead of continuous distributions, as is standard in incomplete market models.

2.7 Equilibrium structure

Due to the finite characterization of our equilibrium, we have a deeper understanding of the structure of the model. In the risky asset market, productive type-1 agents will always be the sole participants. In the bond market, both productive type-1 and productive type-2 agents can participate, even though an agent type may choose not to hold bonds in some states of the world. The next proposition summarizes this market structure.

Proposition 2 (Equilibrium properties) *There exist two distinct subsets $I_i \subset \{1,\dots,n\}$ ($i = 1, 2$), characterizing the states of the world in which only type-i agents trade bond, such that the $4 \times n$ variables $(b^1_k, b^2_k, P_k, Q_k)_{k=1\dots n}$ defining the equilibrium are given by the following $4 \times n$ equations:*

$$P_k = \beta \sum_{j=1}^{n} \pi_{kj} (\alpha^1_k + (1-\alpha^1_k) \frac{1}{\lambda^1} u'(\delta^1 + (P_j + y_j)\frac{V_X}{\eta^1_k} + b^1_k))(P_j + y_j),\ k \in \{1,\dots,n\}, \quad (10)$$

$$Q_k = \beta \sum_{j=1}^{n} \pi_{kj} (\alpha^1_k + (1-\alpha^1_k) \frac{1}{\lambda^1} u'(\delta^1 + (P_j + y_j)\frac{V_X}{\eta^1_k} + b^1_k)),\ k \in \{1,\dots,n\} - I_2, \quad (11)$$

$$Q_k = \beta \sum_{j=1}^{n} \pi_{kj} (\alpha^2_k + (1-\alpha^2_k) \frac{1}{\lambda^2} u'(\delta^2 + b^2_k)),\ k \in \{1,\dots,n\} - I_1, \quad (12)$$

$$V_B = \eta^1_k b^1_k \text{ and } 0 = b^2_k,\ k \in I_1, \quad (13)$$

$$V_B = \eta^2_k b^2_k \text{ and } 0 = b^1_k,\ k \in I_2, \quad (14)$$

$$V_B = \eta^1_k b^1_k + \eta^2_k b^2_k,\ k \in \{1,\dots,n\} - I_1 - I_2. \quad (15)$$

Our equilibrium is characterized by equalities (10)–(15). The first three sets of Euler equations provide security prices. Due to stock market limited participation, the risky asset price is only defined by a Euler equation (10) of productive type-1 agents, who hold all the stocks. Bonds may be traded by productive type-1 or type-2 agents possibly depending on the state of the world. Since our equilibrium features security prices that only depend on the current state of the world, there is one subset of states of the world, characterized by the index subset I_1, in which only type-1 agents bonds, while type-2 agents are excluded. In states of the world I_1: (i) the Euler equation (12) of type-2 agents does not hold, (ii) the bond supply equals the demand of type-1 agents in equation (13). By the same token, there are states of the world, characterized by the subset index I_2, in which only type-2 agents hold bonds, while type-1 agents only hold stocks. This corresponds to the Euler equation (11) and the resource equality (14). Subsets I_1 and I_2 are possibly empty. For instance, if both are empty, it means that type-1 and type-2 agents always trade bonds.

We provide in Appendix the inequality conditions determining the financial market participation of both types of productive agents –see equations (56) and (57). Equations (58)–(60) in Appendix too, are the conditions ensuring that unproductive agents do not trade any assets. The latter equations obviously matter for the equilibrium existence and in fact partly implicitly define the set $\mathcal{V}$ of admissible security supplies V_X and V_B.

This system of equations can easily be simulated and estimated, as we do in Section 4. More importantly, this equilibrium enables us to derive theoretical insights about the household risk exposure and the equity premium throughout the business cycle.

Consumption. From the budget constraint given in (3), credit constraints (4), and the equilibrium properties, we can deduce the consumption levels of our 8 different agent classes. For each agent type $i = 1, 2$, there are 4 different agent classes, each of which depends on the current and past productive status of the agent. For instance, $c_{hk}^{i,pu}$ is the consumption at date t of type-i agents that are currently unproductive in state k but were productive in the previous period in state h. Letting $k \in \{1, \ldots, n\}$ denote the current aggregate state and $h \in \{1, \ldots, n\}$ the previous state, we have:

$$
\begin{aligned}
c_{hk}^{i,pp} &= \omega_k^i(1-\tau_k) + \left((P_k + y_k)\frac{V_X}{\eta_h^1} - P_k\frac{V_X}{\eta_k^1}\right) 1_{i=1} + b_h^i - Q_k b_k^i, \\
c_k^{i,up} &= \omega_k^i(1-\tau_k) - P_k\frac{V_X}{\eta_k^1}\, 1_{i=1} - Q_k b_k^i, \\
c_{hk}^{i,pu} &= \delta^i + (P_k + y_k)\frac{V_X}{\eta_h^1}\, 1_{i=1} + b_h^i, \\
c_k^{i,uu} &= \delta^i,
\end{aligned}
$$

where τ_k is given in Equation (6) and $1_{i=1} = 1$ if $i = 1$ and 0 otherwise. Note that participation costs do not explicitly affect consumption level expressions at the equilibrium.[15] Indeed, type-1

[15]Obviously, they do affect equilibrium market structure and indirectly consumption levels.

agents, who trade stocks, have a zero participation cost, while type-2 agents with a non-zero cost, do not participate in the stock market and never pay that cost.

Participation costs. We now derive explicitly the condition on the participation cost $\overline{\chi}_2$ of Assumption C for type-2 agents never to participate in the stock market. More precisely, we compute the value of $\overline{\chi}_2$, for which this decision to trade stocks is a dominated strategy. To do so, we determine the (fictive) portfolios of type-2 agents (given actual equilibrium prices), if these agents had chosen to participate in the stock market in state k.

If type-2 agents participate in stock markets, their portfolio choice is denoted $\{\tilde{x}_k^2, \tilde{b}_k^2\}_{k=G,B}$ given equilibrium prices $(P_j, Q_j)_{j=1,\ldots,n}$. Purchasing the quantity of stock $\tilde{x}_k^2$ is a dominated strategy in any state of the world k, if investing the same amount in bonds offers in every state a greater payoff. Due to participation cost, purchasing $\tilde{x}_k^2$ costs $P_k\tilde{x}_k^2 + \chi_2$ and pays off $\tilde{x}_k^2(P_j + y_j)$ in the next period when the state of the world is $j = 1, \ldots, n$. Investing the same amount $P_k\tilde{x}_k^2 + \overline{\chi}_2$ in bonds pays off $\frac{P_k\tilde{x}_k^2+\overline{\chi}_2}{Q_k}$ units of consumption in all states of the next period. In consequence, if $\frac{P_k\tilde{x}_k^2+\chi_2}{Q_k} > \tilde{x}_k^2(P_j + y_j)$ for any k, j, type-2 agents never wish to trade stocks. We deduce the following expression for $\overline{\chi}_2$ that ensures Assumption C to hold:

$$\overline{\chi}_2 = \max_{k,j=1,\ldots,n} (Q_k(P_j + y_j) - P_k)\tilde{x}_k^2. \tag{16}$$

Note that to compute the portfolio choice $\{\tilde{x}_k^2, \tilde{b}_k^2\}_{k=G,B}$, we can follow the same steps as in Proposition 2 and obtain:[16]

$$P_k \geq \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^2 + (1-\alpha_k^2)\frac{1}{\lambda^2}u'(\delta^2 + (P_j + y_j)\tilde{x}_k^2 + \tilde{b}_k^2))(P_j + y_j),\ k \in \{1, \ldots, n\}, \tag{17}$$

$$\text{with equality if } \tilde{x}_k^2 > 0, \tag{18}$$

$$Q_k \geq \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^2 + (1-\alpha_k^2)\frac{1}{\lambda^2}u'(\delta^2 + (P_j + y_j)\tilde{x}_k^2 + \tilde{b}_k^2)),\ k \in \{1, \ldots, n\} - I_2, \tag{19}$$

$$\text{with equality if } \tilde{x}_k^2 > 0. \tag{20}$$

[16]We have used the fact that credit constraints bind for unproductive type-2 agents after this deviation, which is true in the equilibrium under consideration.

3 Intuitions through simpler setups

Our model features heterogeneous uninsurable individual risk, aggregate risk and positive security volumes. We now examine, in turn, the role of the different model features in explaining asset returns (the equity premium, in particular), heterogeneity in consumption levels, and consumption growth.

Throughout this section –and only in this section–, we further simplify our setup so as to make mechanisms as transparent as possible. In particular, we make the following two assumptions:

1. aggregate risk follows an IID process;
2. productivity transition probabilities α^i and ρ^i are constant.

No idiosyncratic risk. As a first benchmark, we study the case where agents do not face idiosyncratic risk (i.e., $\alpha^i = 1$ for $i = 1, 2$). Due to limited participation, only type-1 agents trade the risky asset, whose constant price P^{NIR} (NIR stands for "No Idiosyncratic Risk") verifies:

$$P^{NIR} = \beta E^{\tilde{z}}[P^{NIR} + y(\tilde{z})],$$

where $\tilde{z}$ denotes the IID aggregate risk in the next period. $E^{\tilde{z}}[\cdot]$ is the expectation over $\tilde{z}$ while the dividend, $y(\tilde{z})$, depends only on $\tilde{z}$. The gross average stock return R_s^{NIR} is therefore constant and equal to β^{-1}.

The riskless bond is traded by both agents and its price Q^{NIR} can be expressed as $Q^{NIR} = \beta$. The riskless gross interest rate R_f^{NIR} is identical to the stock return. The equity premium in this environment is null: $R_s^{NIR} - R_f^{NIR} = 0$. In our setup, limited participation alone does not imply a non-zero risk premium. Actual idiosyncratic risk is mandatory.

Zero volumes. We now assume that agents face heterogeneous but constant transition probabilities across idiosyncratic states. Aggregate risk effects dividends while the income of productive (ω^i) and unproductive (δ^i) agents are constant. Moreover, both riskless and risky securities are in zero supply. In what follows, ZV stands for "Zero Volume."

Proposition 3 (Zero volumes) *In this economy, the equilibrium features a complete asset market segmentation where productive type-1 agents trade the stock and productive type-2 agents* *trade the bond if:*

$$(1-\alpha^2)(\frac{u'(\delta^2)}{\lambda^2}-1) > (1-\alpha^1)(\frac{u'(\delta^1)}{\lambda^1}-1). \tag{21}$$

The equity premium can then be expressed as:

$$R_s^{ZV} - R_f^{ZV} = \frac{(1-\alpha^2)(\frac{u'(\delta^2)}{\lambda^2}-1) - (1-\alpha^1)(\frac{u'(\delta^1)}{\lambda^1}-1)}{(1-\alpha^1)(1-\alpha^2)(\frac{u'(\delta^1)}{\lambda^1}-1)(\frac{u'(\delta^2)}{\lambda^2}-1)}, \tag{22}$$

while average consumption of a type-i agent, denoted $\bar{c}_i^{ZV}$*, is:*

$$\bar{c}_i^{ZV} = \frac{1-\rho^i}{2-\alpha^i-\rho^i}\omega^i + \frac{1-\alpha^i}{2-\alpha^i-\rho^i}\delta^i, \text{ for } i = 1,2. \tag{23}$$

If $\tilde{\gamma}_c^{i,ZV}$ *is consumption growth of a type-i agent, its variance can be expressed as:*

$$V[\tilde{\gamma}_c^{i,ZV}] = \frac{(1-\rho^i)(1-\alpha^i)}{4(2-\alpha^i-\rho^i)}\left(\frac{\alpha^i(1-\rho^i)+\rho^i(1-\alpha^i)}{2-\alpha^i-\rho^i}\left(\frac{\omega^i}{\delta^i}+\frac{\delta^i}{\omega^i}-2\right)^2 + \left(\frac{\omega^i}{\delta^i}-\frac{\delta^i}{\omega^i}\right)^2\right). \tag{24}$$

In Proposition 3, beyond stating the existence condition for an equilibrium with complete market segmentation, we obtain a simple expression for the risk premium in Equation (22). We also derive expressions for average consumption and the volatility of consumption growth for each agent type. Because of market segmentation and inequality (21), the risk premium is strictly positive, even in the absence of correlation between dividend payouts and marginal utility (which is independent of aggregate uncertainty). In our setup, the risk premium stems from the heterogeneity in the expected magnitude of productivity risk. Indeed, for a type-i agent, the expression $(1-\alpha^i)(\frac{u'(\delta^i)}{\lambda^i}-1)$ can be interpreted as the expected magnitude of productivity risk since (i) $(1-\alpha^i)$ is the probability that a type-i agent faces a bad outcome due to the productivity risk, and (ii) $\frac{u'(\delta^i)}{\lambda^i}-1$ is the relative fall in marginal utility experienced by a type-i agent due to the productivity risk. This expected magnitude of productivity risk drives the demand for self-insurance against the uninsurable individual risk. On the one hand, the stronger the self-insurance need of type-2 agents, the more they demand riskless bonds to hedge against the risk, causing the return on riskless bonds to decrease. On the other hand, the lower the self-insurance need of type-1 agents, the less they demand stocks and the greater the risky return they require to purchase stocks. As a result, heterogeneous demands for self-insurance –in combination with limited stock market participation– are sufficient to generate a strictly positive risk premium, even though both asset payouts are not correlated with marginal utilities.

Expression (23) is very simple and states that consumption only depends on incomes since securities are in zero supply. The volatility of consumption growth in Equation (24) also depends solely on the volatility of wage growth. This volatility increases when the difference between wages in both possible individual states rises.

Positive volumes. We now relax the assumption of zero volumes. For the sake of simplicity, we assume that both bond and stock volumes are small, so as to allow us to derive closed-form expressions –as first-order expressions– for the equity premium and consumption levels and growth rates. In addition, and to simplify expressions, it is assumed that the income of productive and unproductive agents of each type are not time-varying : δ^i and ω^i are constant for $i = 1, 2$. Only stock dividends are time-varying.

Proposition 4 (Small positive volumes) *If the condition (21) of Proposition 3 holds, the economy exhibits the following features:*

- *the equity premium can be expressed as:*

$$R_s^{PV} - R_f^{PV} \approx R_s^{ZV} - R_f^{ZV} + \beta(1-\alpha^1)\frac{-\frac{u''(\delta^1)}{\lambda^1}}{\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1}} \tag{25}$$

$$\times \left(\frac{E\left[P^{ZV} + y(\tilde{z})\right]}{P^{ZV}} (E_t\left[P^{ZV} + y(\tilde{z})\right]\frac{V}{\eta^1} + b^1) + \frac{V\left[P^{ZV} + y(\tilde{z})\right]}{P^{ZV}}\frac{V}{\eta^1} \right),$$

$$\textit{with: } P^{ZV} = \frac{\beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})}{1 - \beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})} E[y(\tilde{z})]. \tag{26}$$

- *the bond holdings of productive agents are such that*
  - *either $b^1 = 0$ and $\eta^2 b^2 = V_B$ in case of (endogenous) complete market separation;*

– *or $b_1 \geq 0$ and $b_2 \geq 0$ verify the two following equations:*

$$\eta^2 \mu b^2 \approx (1-\alpha^2)(\frac{u'(\delta^2)}{\lambda^2} - 1) - (1-\alpha^1)(\frac{u'(\delta^1)}{\lambda^1} - 1) \tag{27}$$
$$- (1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1 \eta^1}(E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]V_X + V_B),$$

$$\eta^1 \mu b^1 \approx (1-\alpha^1)(\frac{u'(\delta^1)}{\lambda^1} - 1) - (1-\alpha^2)(\frac{u'(\delta^2)}{\lambda^2} - 1) \tag{28}$$
$$+ (1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1 \eta^1}E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]V_X - (1-\alpha^2)\frac{u''(\delta^2)}{\eta^2 \lambda^2}V_B,$$

$$\textit{with: } \mu = -(1-\alpha^1)\frac{u''(\delta^1)}{\eta^1 \lambda^1} - (1-\alpha^2)\frac{u''(\delta^2)}{\eta^2 \lambda^2} > 0. \tag{29}$$

- *The average consumptions*

$$\bar{c}_1^{PV} \approx \bar{c}_1^{ZV} + E[y(\tilde{z})]V_X + \eta^1(1 - Q^{ZV})b^1 - \eta^1 \omega^1 \tau, \tag{30}$$

$$\bar{c}_2^{PV} \approx \bar{c}_2^{ZV} + \eta^2(1 - Q^{ZV})b^2 - \eta^2 \omega^2 \tau, \tag{31}$$

$$\textit{with: } \quad Q^{ZV} = \beta(\alpha^2 + (1-\alpha^2)\frac{u'(\delta^2)}{\lambda^2}), \tag{32}$$

$$\tau = \frac{(1 - Q^{ZV})V_B}{\omega^1 \eta^1 + \omega^2 \eta^2}. \tag{33}$$

- *The volatility consumption growth rates $V[\tilde{\gamma}_c^{i,PV}]$ depends on the volatility of financial payoffs:*

$$V[\tilde{\gamma}_c^{1,PV}] \approx V[\tilde{\gamma}_c^{1,ZV}] - \nu_{10}\tau - \nu_{11}b^1 - \nu_{12}\frac{V}{\eta^1} + \nu_{13}(\frac{V}{\eta^1})^2 V[y(\tilde{z})], \tag{34}$$

$$V[\tilde{\gamma}_c^{2,PV}] \approx V[\tilde{\gamma}_c^{2,ZV}] - \nu_{20}\tau - \nu_{21}b^2, \tag{35}$$

where ν_{1i} $(i = 0, \ldots, 3)$ and ν_{2i} $(i = 0, 1)$ are positive terms whose expressions can be found in (70)–(75).

This proposition illustrates the role of positive asset volumes along three dimensions: the equity premium, average consumption levels, and consumption growth rates. Equation (25) describes the role of positive volumes on the equity premium. P^{ZV} in Equation (26) refers to the price of the risky stock in the zero-volume economy. Positive volumes increase the equity premium through two channels. First, positive volumes increase the ability to self-insure for productive type-1 agents who can trade both risky and riskless securities. Type-1 agents therefore require a greater return to hold the risky asset, increasing the equity premium. Second, once they become unproductive, type-1 agents now sell a positive volume of risky assets whose liquidation value is uncertain. Type-1 agents want to be compensated for the uncertainty related to the liquidation value. This corresponds to the variance term in Equation (25). Since the riskless bond pays off a certain outcome in every state of the world, there is not any liquidation premium related to short-term riskless bonds. Challe, LeGrand, and Ragot (2013) use a similar mechanism for long-term bonds and the term structure of interest rates.

Equations (27) and (28) determine the bond demands b^1 and b^2, provided that there is no full market segmentation. In the event of full market segmentation, all bonds are held by productive type-2 agents –under condition (21), type-1 agents cannot hold all bonds. We deduce that the bond demand b^2 of type-2 agents is mainly driven by two factors. The first determinant is the heterogeneity in the magnitude of expected productivity risk between both agents. The greater this risk for type-2 agents –with respect to type-1 agents–, the more type-2 agents need to self-insure themselves and the more they demand bonds. The second determinant is the total quantity of securities available, of both bonds and stocks. The greater the security supply, the smaller the bond price and the more type-2 agents can purchase bonds. For type-1 agents, the intuition is similar except for the role of stock volumes. Indeed, type-1 agents can purchase either stocks or bonds, which are therefore partly substitutes. An increase in stock volumes makes stock cheaper and therefore crowds out bonds in favor of stocks for type-1 agents. These agents purchase more stocks, but need fewer bonds to achieve the same degree of self-insurance.

Equations (30) and (31) explain how positive volumes effect the average consumption of both agent types. The quantity Q^{ZV} in (32) is the price of the riskless bond in the zero-volume economy, while τ is the (first-order expansion of the) tax rate that productive agents of both types have to pay on their income ω^i. Average consumption is augmented by the payoffs of securities, net of tax effects (due to the funding of bond issuance). The heterogeneity in average consumption levels therefore reflects (i) the heterogeneity in average incomes (i.e., the zero net supply case), and (ii) the heterogeneity in net returns of financial wealth. In consequence, the effect of an increase in risky asset volumes on consumption inequality is twofold. First, it raises the equity premium and reinforces the fact that risky stocks pay off more than riskless bonds. Second, it increases stockholder consumption while leaving that of non-stockholders unchanged.

The impact of a larger public debt will imply higher taxes (see equation (33)) for both agent types and will therefore mainly benefit agents with a greater need of bonds for self-insurance.

Finally, equations (34) and (35) illustrate the role of positive security volumes on the volatility of consumption growth, that comes in addition to the zero volume volatility due to heterogeneity in productive and unproductive wages. A first effect (related to coefficient ν_{10} or ν_{20} depending on the agent type) corresponds to the effect of taxes. A greater bond volume increases tax burden and diminishes the variance of consumption growth rate. This effect is related to our taxation scheme: productive agents are taxed, while unproductive are not. A second effect (related to coefficients ν_{11} and ν_{12} for type-1 agents and ν_{21} for type-2 agents) also contributes to a reduction in the consumption growth volatility. Indeed, positive security volumes allow agents to self-insure themselves better against productivity risk. In consequence, agents are more able to smooth out their consumptions across the different idiosyncratic states, which diminishes the volatility of consumption growth. A third effect (related to coefficient ν_{13} for type-1 agents) of positive security volumes raises consumption growth volatility for stock-holders. Indeed, the consumption growth is affected by the uncertain payoff of future dividends, whose volatility is directly transmitted to consumption growth volatility. Therefore, the consumption growth of type-1 agents, who hold stocks, can potentially increase because of positive stock volumes.

4 Quantitative exercise

We now assess the ability of our model to match both asset prices and the allocation of risk across households. In the description of Section 2, Assumption C implies that only type-1 agents hold stocks, while type-2 do not. Since the population of both types is identical, the proportions of agents who participate and those who do not participate in the stock market both equals 50%. This is consistent with the empirical observation that the top 50% of households in the income distribution hold stocks, either directly or indirectly (and are thereby exposed to aggregate risk through this channel), whereas the bottom 50% do not (Bricker et al., 2014). We identify type-1 participating households to the top 50% of US households in the income distribution and type-2 nonparticipating agents to the bottom 50%. We henceforth refer to these two groups respectively as the *top 50%* and *bottom 50%*.

We now calibrate the model to assess its ability to jointly reproduce consumption allocations and asset prices. The model parameters can be divided into three groups: parameters driving the aggregate risk, preference parameters and parameters governing income risks. We start with calibrating aggregate risk and preference parameters, which are standard in the literature. We use the tractability of our framework to estimate the idiosyncratic risk parameters, by simulating the model to match several targets (described below). This empirical exercise enables us to assess to what extent our small-trade equilibrium reproduces observed risk allocations and asset prices.

4.1 Aggregate risk and asset volumes

The period is a quarter. The aggregate state can be either G (for good) or B (for bad). Following Krusell and Smith (1998), the payoff of the risky asset is $y_G = 1.01$ in the good state of the world and $y_B = 0.99$ in the bad state. Moreover, the good state is more persistent than the bad one. For transition probabilities, we use the estimation of Hamilton (1994) and choose the values $\pi_{GG} = 0.75$ and $\pi_{BB} = 0.5$.

The stock volume V_X is assumed to be small. As the model is able to reproduce the volatility of the consumption growth rate for type-1 agents (who are the only agents holding stocks), this small value is not crucial. The income process will indeed reproduce the overall risk exposure of type-1 agents. We assume the stock volume is $V_X = 0.002$ and that the volume of bonds is $V_B = 0.01$. These low values ensure that type-2 agents liquidate their portfolio within one period after becoming unproductive and that consumption levels of unproductive and productive agents lie in the proper sets of the concave-linear utility function. We discuss in Section 5 the realism of this assumption.

4.2 Preference parameters

The shape of the periodic utility function $\tilde{u}$ is defined by three parameters –see equation (2). First, σ determines the constant concavity of the utility function in the non-linear part. We set $\sigma = 1$, implying that we have $\tilde{u}(c) = \log(c)$ in the non-linear part. The two other parameters are the slopes λ^1 and λ^2 (with $\lambda^1 < \lambda^2$) of the two linear parts of the utility function. To avoid arbitrariness in the choice of these slopes, we require them to be equal to the slopes –computed at the relevant point– of a periodic utility function with constant intertemporal elasticity of substitution equal to $1/\sigma = 1$. More precisely, we impose:

$$\lambda^i = \frac{1}{c_{GG}^{i,pp}},\ i = 1, 2. \tag{36}$$

Removing the arbitrariness in the selection of λ^1 and λ^2 is consistent with our interpretation of the linear parts of the utility function as first-order approximations, where variations in marginal utilities due to productivity risk are taken into account while those due to marginal variations in consumption are neglected when consumption is high. Obviously, the values of $c_{GG}^{i,pp}$, $i = 1, 2$ depend on the choice of λ^1 and λ^2 and on all other parameters. Note that equation (36) defining λ^i involves solving a fixed-point problem. Finally, the coefficient β must be low for asset prices to be well defined (see condition (9)). We set $\beta = 0.9$ as a benchmark and we check that the value of β is not key to match the data.

4.3 Income process

To bring discipline to the calibration strategy and to avoid having too many free parameters, we impose several constraints on the model parameters, which are consistent with the mechanisms identified in Section 3. First, we set $\eta_k^1 = \eta_k^2 = \eta = 0.95$ ($k = G, B$) such that 5% of the population is credit-constrained in every period, which is in line with the literature (Krusell and Smith, 1998). The income process is now defined by 10 parameters: ω_k^i, δ^i, and α_k^i for $i = 1, 2$ and $k = G, B$. Using equation (1), the values of $\rho_{k_1 k_2}^i$ for $i = 1, 2$ and $k_1, k_2 = G, B$ are indeed uniquely determined by the values of η^i and α_k^i for $i = 1, 2$ and $k = G, B$. Second, we assume that the transition probabilities for productive agents of both types are not time-varying: $\alpha^1 \equiv \alpha_G^1 = \alpha_B^1$ and $\alpha^2 \equiv \alpha_G^2 = \alpha_B^2$. The values of α^1 and α^2 are sufficient to pin down the average riskless interest rate and the average stock return. Third, we impose that the income risk faced by type-2 agents is not time-varying. As a consequence, $\omega^2 \equiv \omega_G^2 = \omega_B^2$. Finally, we set $\omega_B^1 = 1.0$ to scale the income process of type-1 agents.

As a result, we are left with 6 parameters to estimate: α^1, α^2, ω^2, δ^1, δ^2 and ω_G^1. To do so, we match 6 empirical targets, specified below. The vector of targets is denoted $T = [T_1, \ldots, T_6]$. For any value of the six model parameters α^1, α^2, ω^2, δ^1, δ^2, ω_G^1, we compute the vector of the six corresponding moments $\tilde{T}$ generated by the model. We then find the values of the six model parameters that minimize the distance between the 6 moments generated by the model and their empirical counterparts. Our estimation strategy can therefore be expressed as:

$$\min_{\alpha^1,\alpha^2,\omega^2,\delta^2,\omega_G^1,H} \quad (\tilde{T}-T)\Omega(\tilde{T}-T)',$$

where Ω is the weighting matrix. As we have 6 moments for 6 parameters, we can simply use the identity matrix $\Omega = I_{6\times 6}$ to obtain unbiased estimation results.[17] We now describe the six targets, characterizing both the risk exposure of households and asset prices.

4.3.1 Consumption and the risk exposure of households

Following the literature (Parker and Vissing-Jorgensen, 2009 among others), the risk faced by each category of households is proxied by the volatility of the consumption growth rate for non-durable goods and services. Consumption is measured by quarterly expenditures on non-durable goods and on a subset of services deflated with the relevant price index. We use data of the Consumer Expenditure Survey (CEX) from 1980 to 2007.[18] A detailed discussion can be found in Appendix D. The variance of the consumption growth rate is found to be 0.19 for the bottom 50% and is 0.14 for the top 50%. The bottom 50% face higher total risk than the top 50%, which is a standard result in the literature. Our first two targets are $T_1 = 0.19$ and $T_2 = 0.14$.

We also compute the exposure of both groups to aggregate shocks. Following Parker and Vissing-Jorgensen (2009), we decompose the fraction of total CEX consumption fluctuations borne by each group. More precisely, we compute, for each group, the coefficient equal to: (Change in real group consumption per household)∗(Group share of population)/(Lagged aggregate real consumption per household). The coefficients for both groups sum to one. We can interpret them as the fraction of aggregate risk born by each group. According to this metric, the top 50% bear 84% of aggregate risk, whereas the bottom 50% bears the remaining 16%. Our third target is then $T_3 = 84\%$.

Our fourth target is the consumption share of the top 50%, which drives consumption in-

[17] We solve the minimization problem using a "hill-climbing" algorithm. At each step of the algorithm, for any value of α^1, α^2, ω^2, δ^1, δ^2, ω_G^1, we solve the two fixed-point problems (36) determining the values of λ^i, $i = 1, 2$. We provide in Section E of the Appendix a detailed description of the algorithm.

[18] In what follows, we apply the methodology of Parker and Vissing-Jorgensen (2009) to a different subset of households, so as to be consistent with our model. Moreover, it is known that consumption data are not as accurate as data on household income (see Aguiar and Bils, 2011, among others, for a discussion). Nevertheless, as our results are consistent with those derived using different datasets, we are confident that the facts presented here are robust.

equalities in our economy. This share averages to 72.1% in the CEX, over the period 1980 to 2007. Our fourth target is thus $T_4 = 72.1\%$.

4.3.2 Asset returns

We focus on the unconditional average returns of stocks and bonds. We rely on Guvenen (2009)'s estimates computed using Campbell (1999)'s dataset. The stock returns are computed from the S&P 500 Index while the bond returns are computed from the six-month commercial paper rate. These are real returns and correspond to historical US data from 1890–1991. The average stock return equals 6.2%, while the average bond interest rate is equal to 1.9%. This gives our last two targets $T_5 = 6.2\%$ and $T_6 = 1.9\%$. The households' risk allocation must therefore be consistent with an equity premium of around 4%.

4.4 Results

Table 1 summarizes the parameter values. The first set of parameters are the calibrated parameters. The second set of parameters are the six parameters α^1, α^2, ω^2, δ^1, δ^2, ω^1_G obtained by our estimation strategy described above. The third set of parameters are two values λ^i $(i = 1, 2)$ such that equalities (36) hold.

We recall that the income of productive type-1 agents is normalized to 1 in the bad state of the world. Their income reaches 1.2600 in the good state. For unproductive type-1 agents, their income is constant and equal to 0.2411. For type-2 agents, their income is constant through states and amounts to 0.4491 when productive and to 0.1337 when unproductive. These values imply that the replacement ratio is around 0.2 for type-1 agents and around 0.3 for type-2.

The values of α^1 and α^2 are equal to 0.9579 and 0.9499, respectively. Interestingly, the estimation of our model on household risk exposure and asset prices delivers a higher idiosyncratic severity for type-2 agents than for type-1 agents. Consistently with the data, our model quantitatively features richer agents bearing a larger share of aggregate risk and a smaller probability of leaving their productive status than poorer agents. Even though we did not have any prior about the interpretation of this "productivity" risk, it is interesting to observe that the values of α^1 and α^2 are not inconsistent with an interpretation of the idiosyncratic risk as an employment risk. On US data from 1948Q1-2007Q4, the quarterly probability of remaining employed equals 0.953 (Challe and Ragot, 2014).

Calibrated parameters									
y_G	y_B	π_{GG}	π_{BB}	β	σ	η	ω_B^1	V_X	V_B
1.01	0.99	0.75	0.50	0.90	1.00	0.95	1.00	0.002	0.100

Estimated parameters							
α^1	α^2	ω_1^G	ω^2	δ^1	δ^2	λ^1	λ^2
0.9579	0.9499	1.2151	0.4491	0.2407	0.1337	0.8217	2.2241

Note: See the text for parameter description and calibration

Table 1: Parameter values

Variable	Target	Model	Description
C_1/C^{tot}	72.1%	71.8%	Consumption share of the top 50% of households
$\sigma(g_1)$	14%	14%	Volatility of type-1 household consumption growth rate
$\sigma(g_2)$	19%	19%	Volatility of type-2 household consumption growth rate
R_f	1.9%	1.8%	Annualized average interest rate of the safe bond
R_s	6.2%	6.2%	Annualized average return of stocks
S_1	84%	84%	Share of aggregate risk born by type-1 households

Table 2: Model outcome

Table 2 presents the outcome of the model. Overall, the model matches data pretty well. Regarding the consumption allocation, the share of consumption of type-1 agents in the total consumption, C_1/C^{tot}, is 71.8%, which is very close to its empirical counterpart of 72.1%. The volatilities of consumption growth for both types of agents, $\sigma(g_1)$ and $\sigma(g_2)$, are almost equal to their empirical targets. Type-1 agents bear 84% of the aggregate risk as in the data, while, type-2 agents bear the remaining 16% of the aggregate risk. Second, the model reproduces a high average stock return R_s and a low average riskless interest rate R_f. Again, model outcomes are very close to their empirical counterparts. The equity premium generated by the model amounts to 4.3%. In terms of allocations, the model hits the targets for an equilibrium where type-1 agents do not buy the safe bonds: type-1 agents only use stocks for consumption smoothing and self-insurance. The model outcome therefore yields endogenous market segmentation.

Implicit valuation of the risky asset by type-2 agents. In Assumption C, we set a high participation cost of type-2 agent to ensure that these agents will not trade any stock. We provide in equation (16) a value of $\overline{\chi}_2$ which ensures that holding stocks is dominated. Using our calibration, we obtain a participation cost that amounts to 0.052% of quarterly income. As the median annual income of US households was $52,250 in 2014, this represents an annual participation cost of $ 7. This is relatively small compared to other estimates of the participation cost (Vissing-Jorgensen, 2002, among others).[19]

Effect of public debt on the real interest rate. As an additional investigation of the performance of the model, we increase the level of public debt by 1%. We find that the annual real interest rate increases by 5 basis points from 1.80% to 1.85%. When the volume of securities to self-insure is higher, an additional unit of security is less valuable as households are more insured. This effect is qualitatively consistent with the results of Laubach (2009).

Effect of an increase in idiosyncratic risk on asset prices. We study the effect of an increase in idiosyncratic risk as a final investigation of the properties of the model. We consider an economy with the same calibration as before except that α^1 and α^2 decrease by the same small amount, from 0.9579 to 0.9549 and from 0.9499 to 0.9469, respectively. For the sake of consistency, we also change accordingly the values of λ^i $(i = 1, 2)$ for conditions (36) to hold. Those are the only differences with the benchmark economy. Consistently with intuition, this increase in idiosyncratic risk decreases both the riskless interest rate and the average stock return. The riskless interest rate decreases from 2.0% to 0.0% and the stock return from 6.2% to 5.5%. As agents want more insurance due to the higher idiosyncratic risk, their valuation of assets increases (and thus returns decrease). The price of the risky asset increases by around 10% whereas the riskless interest rate decreases to 0%. It is interesting to observe that these movements in asset prices are qualitatively consistent with evolutions observed in the US bond and stock markets after the 2008 crisis, which can thus be partly rationalized by an increase in perceived idiosyncratic risk.

4.5 The model without participation costs

The ability of the previous model to reproduce asset prices crucially relies on the participation cost which generates limited stock market participation. To see this, we perform the same quan-

[19] As this estimation may depend on the small volume of debt, we also compute the implicit valuation of the risky asset by type-2 agents (i.e., their valuation with their own pricing kernel). We find that type-2 agents will never participate in the stock market, if they face a proportional participation cost as low as 1.1%.

titative analysis as in the previous section but we relax Assumption C and set all participation costs to zero: $\chi_1 = \chi_2 = 0$. As explained in Proposition 2, both agents may trade stocks and bonds. We choose the same calibration as in Table 1, and we estimate the same 6 parameters as in the model with participation costs to match the same 6 targets. Table 3 presents the parameter values after the estimation procedure, and the model outcome lies in Table 4 .

Calibrated parameters									
y_G	y_B	π_{GG}	π_{BB}	β	σ	η	ω_B^1	V_X	V_B
1.01	0.99	0.75	0.50	0.90	1.00	0.95	1.00	0.002	0.100

Estimated parameters							
α^1	α^2	ω_1^G	ω^2	δ^1	δ^2	λ^1	λ^2
0.9820	0.9600	1.020	0.9920	0.1856	0.1820	0.9806	1.0059

Note: See the text for parameter description and calibration

Table 3: Parameter values, no participation cost

Variable	**Target**	**Model**	**Description**
C_1/C^{tot}	72.1%	50.5%	Consumption share of the top 50% of households
$\sigma(g_1)$	14%	11%	Volatility of type-1 household consumption growth rate
$\sigma(g_2)$	19%	16%	Volatility of type-2 household consumption growth rate
R_f	1.9%	7.86%	Annualized average interest rate of the safe bond
R_s	6.2%	7.86%	Annualized average return of stocks
S_1	84%	29%	Share of aggregate risk born by type-1 households

Table 4: Model outcome, no participation cost

Although the model can roughly replicate individual consumption volatilities, it fails to reproduce realistic asset prices. The reason is the following. The equilibrium allocation without participation cost is such that type-2 agents hold all stocks and bonds. Indeed, a higher idiosyncratic risk for type-2 agents is necessary to match the difference in the consumption growth rate volatilities. Such a difference generates a higher desire to self-insure for type-2 agents and thus a higher valuation of all assets. In consequence, only one type of agents prices all the assets at the equilibrium. Krusell, Mukoyama and Smith (2011) have shown that in such an economy, it is not possible to reproduce empirical asset prices with realistic idiosyncratic risks. Participation cost is thus a key ingredient for the ability of our model to match empirical data.

5 Discussion of our assumptions

As we explained in the discussion of Proposition 1, our equilibrium crucially relies on two assumptions: (i) the linear parts in the utility function and (ii) the upper bound on security volumes (i.e., the set $\mathcal{V}$). The concave-linear utility function has been introduced as a generalization of Fishburn (1977). In the context of our incomplete-market model, a more precise interpretation can be provided. The shape of the periodic utility function implies that agents who have a low level of consumption are sensitive to small variations in consumption levels. Agents consuming a higher amount (i.e., those in the linear part of the utility function) have a marginal utility which is invariant to small variations in consumption. However, these agents can experience a sensible increase in marginal utility if they are hit by a negative idiosyncratic shock that would force them to consume a low amount (that would be valued by the strictly concave part of the utility function). The concave-linear utility function accounts for extensive variations in consumption due to individual shocks but neglects the impact of small intensive variations in consumption for productive agents. We believe it to be a relevant representation of consumption smoothing and of the behavior with respect to risk. Consumption variations matter much more when consumption levels are low than when they are high. For productive agents, who consume significantly more than unproductive ones, the marginal utility variations implied by their saving choices is small as long as they remain productive. The quantitative exercise has shown that the linear part can be chosen for the marginal utility to be consistent with a globally concave utility function. Finally, productive agents are *not* risk neutral with respect to aggregate risk as they always have a positive probability of valuing next period consumption with a strictly concave utility function.

The upper bound on security volumes is the second crucial assumption. Indeed, for our limited-heterogeneity equilibrium, we have to limit the amount of self-insurance, for unproductive agents to be credit-constrained. Considering the bottom 50% of US households (in the consumption distribution), these households hold a small amount of liquid wealth –using the

SCF, this amounts to less than one thousand dollars. As a consequence, the assumption of small asset volumes is not unrealistic for this fraction of the population. The top 50% of US households obviously hold a much higher amount of assets. For them, we justify our assumption by the result of our estimation exercise. Indeed, the estimation shows that this simple model reproduces quite well the aggregate risk exposure together with the equity premium and the risk-free rate. The availability of assets for participating agents to self-insure can thus be captured by parameter values in our small-trade equilibrium.

Other assumptions are much less critical with respect to our equilibrium existence. Assumption A about the timing of the idiosyncratic risk could replaced by a less strict assumption but this would imply a greater number of agent classes in the equilibrium. As discussed in Footnote 9, the credit constraints preventing agents from having negative positions could also be partially relaxed. We could easily allow agents to have negative wealth, as long as the borrowing limit is not too loose. Our equilibrium would still exist, provided that the set $\mathcal{V}$ of admissible volumes in Proposition 1 is changed accordingly.[20]

6 Conclusion

We have constructed an analytically tractable incomplete insurance market model with participation cost, heterogeneity in risk exposure, and aggregate shocks. Our small-trade equilibrium relies on not-too-large security volumes and a concave-linear utility function introduced by Fishburn (1977). Although simple, this model can reproduce asset prices and household risk exposure quite well. This parsimonious setup could be used to study other forms of heterogeneity with aggregate shocks. In particular, it could be used to model the heterogeneity of agents according to both sides of their balance sheet (i.e., for the asset side and the liability side). This would allow us to study financial intermediation in an incomplete market setting with aggregate shocks. Such an environment could improve our understanding of the functioning of financial markets.

[20]There is a kind of substitution between the negative wealth constraint and the maximal bounds (in $\mathcal{V}$) allowing equilibrium existence.

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Appendix

A Proof of Proposition 1

We prove that the market arrangement implied by Assumption C, in which type-1 agents trade stocks, while type-2 do not, is an equilibrium. We proceed in two steps: (i) we prove that we can find prices and quantities such that equations (10)–(15) hold and (ii) we check that unproductive agents do not participate in security markets and that equilibrium consumption levels lie in proper definition sets of the utility function.

First step: the existence of prices and quantities. We define a correspondence on a compact set to invoke the Kakutani's fixed-point theorem. First, we define the compact convex sets $D_b = \{b \in \mathbb{R} : \ V_B \geq b \geq 0\}$ and $D_p = \{(P, Q) \in \mathbb{R}^2 : \underline{P} \leq P \leq \overline{P} \text{ and } 0 \leq Q \leq \overline{Q}\}$ with $\underline{P} = \frac{\beta \min_{z\in Z} \alpha^1(z)y(z)}{1-\beta \min_{z\in Z} \beta\alpha^1(z)} > 0$, $\overline{P} = \frac{\beta \max_{z\in Z}(\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))y(z)}{1-\beta \max_{z\in Z} \alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1)} \geq \underline{P}$ and $\overline{Q} = \max_{z\in Z} \beta(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))$.[21]

We define the mapping T^x from $(b\mathcal{C}(Z \times D_b), \|\cdot\|_\infty)$ onto itself as follows:[22]

$$T^x : X \mapsto y(z) + \beta\left[(\alpha^1(z) + (1-\alpha^1(z))f(X, z, z', b))X\right],$$
$$f(X, b, z, z') = \frac{1}{\lambda^1}u'\left(\delta^1 + X\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}\right),$$

where f decreasing in its first argument and where we can find $\underline{\beta}$ and $\overline{\beta}$ such that for all $z \in Z$ and for all $X \geq 0$, $0 < \underline{\beta} \leq \beta(\alpha^1(z) + (1-\alpha^1(z))f(X, z, z', b)) \leq \overline{\beta} < 1$ (condition (9)). We wish to prove that T^x is a contraction. We define $R : (X, X') \mapsto \frac{T^x X - T^x X'}{X - X'}$ for $0 \leq X' < X$. First, we can notice that $R(X, X') \leq R(X, 0)$. Indeed, it is equivalent to: $\frac{T^x X - T^x 0}{X} \leq \frac{T^x X' - T^x 0}{X'}$, which holds since f is decreasing in the first argument. Second, we have for $X' < X$ (since f is decreasing in the first argument):

$$R(X, X') = \beta\alpha^1(z) + \beta(1-\alpha^1(z))\frac{f(X,\cdot) - f(X',\cdot)}{X - X'} \geq \beta\alpha^1(z).$$

We deduce that for all $X \geq 0$ and $X' \geq 0$, we have $|E^z[T^x X] - E^z[T^x X']| \leq \overline{\beta}|E^z[X] - E^z[X']|$, where $E^z[\zeta] = \sum_{z'\in Z} \pi_{zz'}\zeta_{z'}$ is the conditional expectation of ζ. The Banach fixed-point theorem implies that there exists a unique $X \in b\mathcal{C}(Z \times D_b)$ such that:

$$X(z, b) = y + \beta E^z\left[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + X(z',\cdot)\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))X(z',\cdot)\right].$$

[21] It will be straightforward to check that equilibrium prices and quantities respectively belong to D_p and D_b.
[22] $b\mathcal{C}(\star)$ is the set of continuous bounded functions over the metric space $\star$, endowed with the sup. norm.

X is by construction a continuous function in b. We have just proven that the stock price $P(\cdot) = X(\cdot) - y$ is well-defined and is a continuous function of bond demand.

We now define the following correspondence $\psi^P : \mathcal{F}(Z, D_b) \rightrightarrows \mathcal{P}(\mathcal{F}(Z, D_p))$, as:[23]

$$
\begin{aligned}
\psi^P(b) = \{(P, Q) \in \mathcal{F}(Z, D_p)| \\
& P = \beta E^z[(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + (P(z') + y(z'))\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)}))(P(z') + y(z'))], \\
& Q = \beta E^z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + \frac{b}{\eta^2(z)})]\Big\}.
\end{aligned}
$$

If security demands solely depend on the current aggregate and idiosyncratic states, we deduce from Assumption A that the bond market clearing implies that $\forall z \in Z,\ b^2(z) = \frac{b(z)}{\eta^2(z)}$ and $b^1(z) = \frac{V_B - b(z)}{\eta^1(z)}$ where b^i denotes the bond demand of a type-i agent ($i = 1, 2$).

We introduce the correspondence $\psi^x : \mathcal{F}(Z, D_p) \rightrightarrows \mathcal{P}(\mathcal{F}(Z, D_b))$, as follows:

$$
\psi^x(P, Q) = \Big\{b \in \mathcal{F}(Z, D_b)| T^p_{P,Q}(b) = 0, V_B \geq b(z) \geq 0\Big\} \tag{37}
$$

$$
\begin{aligned}
&\text{where } \forall (P, Q) \in \mathcal{F}(Z, D_p),\ T^p_{P,Q} : b \in \mathcal{F}(Z, D_b) \mapsto \forall z \in Z, \\
&\quad (V_B - b(z)) \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2+\frac{V_B}{\eta^2(z)})] > E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)})]} \\
&+ b(z) \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] < E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)}+\frac{V_B}{\eta^1(z)})]} \\
&+ (E^z[\alpha^2(z) + (1-\alpha^2(z))\frac{u'(\delta^2 + \frac{b(z)}{\eta^2(z)})}{\lambda^2}] \\
&\qquad\qquad - E^z[\alpha^1(z) + (1-\alpha^1(z))\frac{u'(\delta^1 + (P + y(z'))\frac{V_X}{\eta^1(z)} + \frac{V_B - b}{\eta^1(z)})}{\lambda^1}]) \\
&\quad \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2+\frac{V_B}{\eta^2(z)})] \leq E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)})]} \\
&\quad \times 1_{E^z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] \geq E^z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+(P+y(z'))\frac{V_X}{\eta^1(z)}+\frac{V_B}{\eta^1(z)})]},
\end{aligned}
$$

where $1_A = 1$ if A is true and 0 otherwise. The mapping $T^p_{P,Q}$ considers the three possible cases of bond market participation. Bonds are traded by: (i) only type-2 agents, (ii) only type-1 agents and (iii) both agents. These three cases correspond to three mutually exclusive conditions. We can therefore check that ψ^x is compact- and convex-valued and upper semi-continuous (since it is compact-valued and its graph is closed).[24] ψ^x is also non-empty: either there is complete market separation (with only type-1 or type-2 agents holding bonds), or both types of agents

[23] Correspondences are set-valued functions (see Mas-Collel, Whinston and Green(1995), Section M.H). $\mathcal{P}(\star)$ is the set of all subsets of $\star$. For any compact K, $\mathcal{F}(Z, K)$ is the set of functions from Z to K and is isomorphic to K^n (and thus compact) since Z is of a cardinal n.

[24] Considering $\phi : p \mapsto \big\{x \in [\underline{x}, \overline{x}], (\overline{x} - x)1_{p>k_2} + (x - \frac{k_2-p}{k_2-k_1}\underline{x} - \frac{p-k_1}{k_2-k_1}\overline{x})1_{k_2 \geq p \geq k_1} + (x - \underline{x})\ 1_{p<k_1} = 0\big\}$ ($k_2 > k_1$) may clarify this point. $\phi(p) = \{\overline{x}\}$ for $p > k_2$; $\phi(p) = \{\frac{k_2-p}{k_2-k_1}\underline{x} + \frac{p-k_1}{k_2-k_1}\overline{x}\}$ for $k_2 \geq p \geq k_1$ and $\phi(p) = \{\underline{x}\}$ for $p < k_2$. The set $\{(p, \phi(p))\ ,\ p \in \mathbb{R}\}$ is closed.

trade bonds.

We finally define the correspondence $\psi : ((P,Q),b) \in \mathcal{F}(Y,D_p)\times\mathcal{F}(Y,D_b) \rightrightarrows (\psi^p(b),\psi^x(P,Q)) \in \mathcal{P}(\mathcal{F}(Y,D_p)\times\mathcal{F}(Y,D_b))$. Since ψ^p and ψ^x are non-empty, compact- and convex-valued and upper semi-continuous, ψ also is. The Kakutani's theorem then ensures the existence of a fixed point $((P^*,Q^*),b^*) \in (\psi^p(b^*),\psi^x(P^*,Q^*))$. It is then straightforward to check that this fixed-point defines a competitive equilibrium. Moreover, for this equilibrium sets D_p and D_b are well-defined.

We now check that unproductive agents are kept out of the financial market.

Second step: unproductive agents do not participate in security markets. First note that the fixed-point generates an equilibrium with endogenous bond market participation of productive type-1 and type-2 agents. However, we need to determine, under which conditions unproductive agents of both types choose not to trade any security.

Security zero-supplies. We first assume $V_X = V_B = 0$. No security is traded and security prices are given by:

$$P(z) = \beta(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))E^z\left[(P(z') + y(z'))\right], \tag{38}$$

$$Q(z) = \beta(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)). \tag{39}$$

The equilibrium existence conditions are as follows (here $\hat{z}$ is the former state, z the current one and z' the next one):

$$P(z)\frac{1}{\lambda^1}u'(\delta^1) > \beta E^z\left[(1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1))(P(z')+y(z'))\right], \tag{40}$$

$$Q(z)\frac{1}{\lambda^1}u'(\delta^1) > \beta(1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1)), \tag{41}$$

$$Q(z)\frac{1}{\lambda^2}u'(\delta^2) > \beta(1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2)). \tag{42}$$

First notice that condition (40) can be expressed using (38) as:

$$E^z\left[\left(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))\frac{1}{\lambda^1}u'(\delta^1) - (1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1))\right)(P(z')+y(z'))\right] > 0.$$

For conditions (40)–(42) to hold, it is sufficient that using (38)–(39), we have:

$$(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1))\frac{1}{\lambda^1}u'(\delta^1) > 1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1), \tag{43}$$

$$(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))\frac{1}{\lambda^1}u'(\delta^1) > 1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1), \tag{44}$$

$$(\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2))\frac{1}{\lambda^2}u'(\delta^2) > 1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2). \tag{45}$$

We can check that equations (43) and (45) can be seen as positivity inequalities of polynomial functions in $\frac{1}{\lambda^1}u'(\delta^1)$ and $\frac{1}{\lambda^2}u'(\delta^2)$ respectively. Each polynomial function admits one negative root and another root equal to 1. Both polynomials are thus always positive since $\frac{1}{\lambda^1}u'(\delta^1) > 1$ and $\frac{1}{\lambda^2}u'(\delta^2) > 1$ (see Assumption B). Conditions (43) and (45) therefore always hold. The condition (44) can similarly be written as a positivity inequality of a polynomial function in $\frac{1}{\lambda^1}u'(\delta^1)$ and $\frac{1}{\lambda^2}u'(\delta^2)$, which is increasing in both arguments. We therefore deduce that: (i) when $\frac{1}{\lambda^1}u'(\delta^1) \geq \frac{1}{\lambda^2}u'(\delta^2)$, condition (44) holds whenever condition (45) does and (ii) when $\frac{1}{\lambda^1}u'(\delta^1) \leq \frac{1}{\lambda^2}u'(\delta^2)$, condition (44) holds whenever condition (43) does. In consequence, condition (44) always holds.

We finally check that consumptions of productive (resp. unproductive) agents lie in the linear (resp. concave) part of the utility function. Since our equilibrium features limited-heterogeneity, there are only 4 different agents classes per type, each of which depends on the current and past productive status. For instance, $c^{i,pu}_{\hat{z},z}$ is the consumption of type-i agents, who are currently unproductive (in state z) but were productive in the previous period (in state $\hat{z}$) . The consumption levels of the different classes ($i = 1, 2$) are:

$$c^{i,pp}_{\hat{z},z} = c^{i,up}_{\hat{z},z} = \omega^i(z),$$
$$c^{i,pu}_{\hat{z},z} = c^{i,uu}_{\hat{z},z} = \delta^i.$$

Assumption B readily implies that consumptions lie in the proper regions of the utility function.

The equilibrium always exists in zero volume.

Positive supply economy. We assume that $V_B > 0$, and $V_X > 0$. Security prices are:

$$P(z) = \beta E_z\left[\left(\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + b^1(z) + \frac{V_X}{\eta^1(z)}(P(z') + y(z')))\right)(P(z') + y(z'))\right], \tag{46}$$

$$Q(z) = \beta E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + b^2(z))], \tag{47}$$

where the quantities b^1 and b^2 are determined by three cases (see definition (37) of ψ^x):

- $b^1(z) = 0$ and $b^2(z) = \frac{V_B}{\eta^2(z)}$ if $E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + \frac{V_B}{\eta^2(z)})] \geq E_z[\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + \frac{V_X}{\eta^1(z)}(P(z') + y(z')))]$: the equilibrium features complete market segmentation;

- $b^1(z) = \frac{V_B}{\eta^1(z)}$ and $b^2(z) = 0$ if $E_z[\alpha^2(z)+(1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2)] \leq E_z[\alpha^1(z)+(1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1+\frac{V_X}{\eta^1(z)}(P(z') + y(z')) + \frac{V_B}{\eta^1(z)})]$: the equilibrium also features complete market segmentation;

- $b^1(z) = \frac{V_B - \eta^2 b^2(z)}{\eta^1(z)}$ and $b^2(z)$ solves $E_z[\alpha^2(z) + (1-\alpha^2(z))\frac{1}{\lambda^2}u'(\delta^2 + b^2(z))] = E_z[\alpha^1(z) + (1-\alpha^1(z))\frac{1}{\lambda^1}u'(\delta^1 + \frac{V_X}{\eta^1(z)}(P(z') + y(z')) + \frac{V_B - \eta^2(z)b^2(z)}{\eta^1(z)})]$: both agents types trade bonds.

Since prices and bond quantities depend on the security supplies V_X and V_B (in addition to other model parameters), equilibrium existence conditions can be expressed as $\Theta(V_X, V_B) > 0$, where:

$$\Theta(V_X,V_B) = \left(\begin{array}{c} P(z)\frac{1}{\lambda^1}u'(\delta^1 + b^1(\hat{z}) + \frac{V_X}{\eta^1(\hat{z})}(P(z)+y(z)))\dots \\ \dots - \beta E_z\left[(1-\rho^1(z,z') + \rho^1(z,z')\frac{1}{\lambda^1}u'(\delta^1))(P(z')+y(z'))\right] \\ Q(z)\frac{1}{\lambda^1}u'(\delta^1 + b^1(\hat{z}) + \frac{V_X}{\eta^1(\hat{z})}(P(z)+y(z)))\dots \\ \dots - \beta(1 - E^z\left[\rho^1(z,z')\right] + E^z\left[\rho^1(z,z')\right]\frac{1}{\lambda^1}u'(\delta^1)) \\ Q(z)\frac{1}{\lambda^2}u'(\delta^2 + b^2(\hat{z}))\dots \\ \dots - \beta(1 - E^z\left[\rho^2(z,z')\right] + E^z\left[\rho^2(z,z')\right]\frac{1}{\lambda^2}u'(\delta^2)) \end{array} \right)_{(\hat{z},z)\in Z^2} > 0. \tag{48}$$

Since the set Z is of cardinal n, $\Theta(V_X, V_B) \in \mathbb{R}^{3n^2}$. Note that $\Theta(V_X, V_B) > 0$ means that every component of $\Theta(V_X, V_B)$ is strictly positive.

We define

$$\mathcal{V}_\Lambda = \left\{(V_X, V_B) \in (\mathbb{R}^+)^2 | \Theta(V_X, V_B) > 0\right\}. \tag{49}$$

The zero supply part implies that $\mathcal{V}_\Lambda$ is not empty and, by continuity, includes an open set (of $(\mathbb{R}^+)^2$ endowed with the Euclidean norm) containing $(0,0)$. In other words, there exist $\overline{V}_X^\Lambda > 0$ and $\overline{V}_B^\Lambda > 0$, such that for all $0 \le V_X \le \overline{V}_X^\Lambda$ and $0 \le V_B \le \overline{V}_B^\Lambda$, $(V_X, V_B) \in \mathcal{V}_\Lambda$.

We now turn to the consumption expression. The consumption levels of the different classes $(i = 1, 2)$ can be expressed as follows

$$c_{\hat{z},z}^{i,pp} = \omega^i(z)(1-\tau(z)) + \left(P(z)(\frac{V_X}{\eta^1(\hat{z})} - \frac{V_X}{\eta^1(z)}) + y(z)\frac{V_X}{\eta^1(\hat{z})} - \chi^1\right)1_{i=1} + b^i(\hat{z}) - Q(z)b^i(z), \tag{50}$$

$$c_{\hat{z},z}^{i,up} = \omega^i(z)(1-\tau(z)) - \left(P(z)\frac{V_X}{\eta^1(z)} + \chi^1\right)1_{i=1} - Q(z)b^i(z), \tag{51}$$

$$c_{\hat{z},z}^{i,pu} = \delta^i + (P(z)+y(z))\frac{V_X}{\eta^1(\hat{z})}1_{i=1} + b^i(\hat{z}), \tag{52}$$

$$c_{\hat{z},z}^{i,uu} = \delta^i. \tag{53}$$

where taxes are given by $\tau(z) = \frac{(1-Q(z))V_B}{\omega^1(z)\eta^1(z)+\omega^2(z)\eta^2(z)}$. The vector of consumptions is denoted $C(V_X, V_B) = [c_{\hat{z},z}^{1,pp}, c_{\hat{z},z}^{1,up}, c_{\hat{z},z}^{2,pp}, c_{\hat{z},z}^{2,up}, c_{\hat{z},z}^{1,pu}, c_{\hat{z},z}^{1,uu}, c_{\hat{z},z}^{2,pu}, c_{\hat{z},z}^{2,uu}]_{(\hat{z},z)\in Z^2}$ and depends on V_B and V_X. The space of admissible consumptions is $\Gamma = \left([c_4^*, c_5^*]^2 \times [c_2^*, c_3^*]^2 \times [0, c_1^*]^4\right)^{n^2}$. We now define

$$\mathcal{V}_\Gamma = \left\{(V_X, V_B) \in (\mathbb{R}^+)^2 | C(V_X, V_B) \in \Gamma\right\}. \tag{54}$$

As for $\mathcal{V}_\Lambda$, we know, from the zero supply part, that $\mathcal{V}_\Gamma$ is not empty and by continuity that an open set containing $(0,0)$ is included in $\mathcal{V}_\Lambda$.

We conclude with defining the set $\mathcal{V}_1$ containing all volumes for which the equilibrium, where only type-1 agents trade stocks, exists:

$$\mathcal{V}_1 = \mathcal{V}_\Gamma \cap \mathcal{V}_\Gamma. \tag{55}$$

Previous remarks imply that $\mathcal{V}_1$ is non-empty and includes an open set containing $(0, 0)$.

B Proof of Proposition 2

From Proposition 1 and in particular from its proof above, it is straightforward to deduce that there exist two distinct subsets $I_i \subset \{1, \ldots, n\}$ $(i = 1, 2)$, characterizing the states of the world in which only type-i agents trade bond, such that the $4 \times n$ variables $(b_k^1, b_k^2, P_k, Q_k)_{k=1\ldots n}$ characterizing the equilibrium are given by the $4 \times n$ equations (10)–(15).

For the equilibrium to exist, we need to check two sets of conditions. The first one concerns the states of world, in which productive agents of a given type are excluded from bond markets. The second one concerns all states of the world, since unproductive agents of both types are permanently excluded from both financial markets.

In the states of the world I_2 where only type-2 agents trade bonds, type-1 agents are excluded due to too high bond prices and the following inequality has to hold for all $k = 1, \ldots, n$:

$$Q_k > \beta \sum_{j=1}^{n} \pi_{kj} (\alpha_k^1 \lambda^1 + (1 - \alpha_k^1) \frac{1}{\lambda^1} u'(\delta^1 + (P_j + y_j) \frac{V_X}{\eta^1})), \text{ for } k \in I_2. \tag{56}$$

By the same token, in the states of the world I_1, when only type-1 agents trade bonds, type-2 agents are excluded and the following inequality has to hold for all $k = 1, \ldots, n$:

$$Q_k > \beta(\alpha_k^2 \lambda^2 + (1 - \alpha_k^2) \frac{1}{\lambda^2} u'(\delta^2)), \ k \in I_1. \tag{57}$$

Regarding unproductive agents, type-1 (unproductive) agents are excluded from both stock and bond markets. The two following inequalities therefore need to hold for all $k, h = 1, \ldots, n$:

$$P_k \frac{1}{\lambda^1} u'(\delta^1 + b_h^1 + \frac{V_X}{\eta_h^1}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^1 + \rho_{kj}^1 \frac{1}{\lambda^1} u'(\delta^1))(P_j + y_j), \tag{58}$$

$$Q_k \frac{1}{\lambda^1} u'(\delta^1 + b_h^1 + \frac{V_X}{\eta_h^1}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^1 + \rho_{kj}^1 \frac{1}{\lambda^1} u'(\delta^1)). \tag{59}$$

Unproductive type-2 agents cannot participate to stock markets. For them to be excluded from bond markets, the following inequality needs to hold for all $k, h = 1, \ldots, n$:

$$Q_k \frac{1}{\lambda^2} u'(\delta^2 + b_h^2 + \frac{V_X}{\eta_h^2}(P_k + y_k)) > \beta \sum_{j=1}^{n} \pi_{kj} (1 - \rho_{kj}^2 + \rho_{kj}^2 \frac{1}{\lambda^2} u'(\delta^2)). \tag{60}$$

C Proof of propositions in Section 3

C.1 Proof of Proposition 3

Equity premium. Since dividends are IID, stock prices are constant. Provided that condition (9) holds, the Euler equation for the stock implies:

$$P^{ZV} = \frac{\beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})}{1-\beta(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1})} E^{\tilde{z}}[y(\tilde{z})], \tag{61}$$

where $E^{\tilde{z}}[\cdot]$ is the expectation with respect to $\tilde{z}$. Type-2 agents are trading riskless bonds, while the bond price is too expensive for type-1 agents, i.e.:

$$Q^{ZV} = \beta\left(\alpha^2 + (1-\alpha^2)\frac{u'(\delta^2)}{\lambda^2}\right), \tag{62}$$

$$Q^{ZV} > \beta\left(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1)}{\lambda^1}\right), \tag{63}$$

where condition (63) holds thanks to condition (21). The zero supply economy therefore features full market segmentation, where type-1 agents hold stocks, while type-2 agents hold bonds. This equilibrium always exists from Proposition 1.

From price expressions (61) and (62), we deduce the equity premium of equation (22).

Average consumptions. Agents of type i have the unconditional probability η^i to be productive and consume ω^i and the probability $1-\eta^i$ to be unproductive and consume δ^i. The expression (23) is then immediate to obtain.

Consumption growth variances. We start with computing the consumption growth $\tilde{\gamma}_c^{i,ZV}$ of a type-i agent. Remark that the consumption growth is: (i) 1 with probability $\alpha^i\eta^i+\rho^i(1-\eta^i)$ (the agent remains productive or unproductive); (ii) $\frac{\omega^i}{\delta^i}$ with probability $(1-\alpha^i)\eta^i$; and (iii) $\frac{\delta^i}{\omega^i}$ with the same probability $(1-\alpha^i)\eta^i$. We can then easily derive the expression (24) for the variance using Lemma 1 below.

Lemma 1 (A particular variance expression) *Let consider a real random variable X taking the values z_1 with probability p_1, z_2 with probability p_2, x with probability $\frac{1-p_1-p_2}{2}$ and y with probability $\frac{1-p_1-p_2}{2}$, where $1-p_1-p_2 \geq 0$. Then, the variance $V[X]$ of X can be expressed as:*

$$V[X] = (1-p_1-p_2)\left(p_1(z_1 - \frac{x}{2} - \frac{y}{2})^2 + p_2(z_2 - \frac{x}{2} - \frac{y}{2})^2 + (\frac{x}{2} - \frac{y}{2})^2\right) + p_1p_2(z_1-z_2)^2. \tag{64}$$

Proof.

After some algebra, the variance $V[X] = E[(X - E[X])^2]$ can be expressed as follows:

$$\begin{aligned}
V[X] &= p_1 \left((1-p_1) \left(\frac{x}{2} + \frac{y}{2} - z_1 \right) + p_2 \left(z_2 - \frac{x}{2} - \frac{y}{2} \right) \right)^2 \\
&+ p_2 \left(p_1 \left(z_1 - \frac{x}{2} - \frac{y}{2} \right) + (1-p_2) \left(\frac{x}{2} + \frac{y}{2} - z_2 \right) \right)^2 \\
&+ (1-p_1-p_2) \left(p_1 \left(z_1 - \frac{x}{2} - \frac{y}{2} \right) + p_2 \left(z_2 - \frac{x}{2} - \frac{y}{2} \right) \right)^2 \\
&+ (1-p_1-p_2) \left(\frac{x}{2} - \frac{y}{2} \right)^2 .
\end{aligned}$$

Developing and rearranging the squares in $z_1 - \frac{x}{2} - \frac{y}{2}$ and $z_2 - \frac{x}{2} - \frac{y}{2}$ yield the expression (64). ∎

C.2 Proof of Proposition 4

Equity premium. Because the dividend process is IID, stock and bond prices, as well as bond holdings, are constant. The Euler equations for both securities become:

$$\begin{aligned}
P^{PV} &= \beta E^{\tilde{z}} \left[\left(\alpha^1 + (1-\alpha^1) \frac{u'(\delta^1 + (P_{t+1} + y_{t+1})\frac{V_X}{\eta^1} + b^1)}{\lambda^1} \right) (P^{PV} + y(\tilde{z})) \right], \\
Q^{PV} &= \beta(\alpha^2 + (1-\alpha^2) \frac{u'(\delta^2 + b^2)}{\lambda^2}).
\end{aligned}$$

We solve for the price expression in the neighborhood of zero volumes. We assume that $0 < V_X \ll 1$ and $0 < V_B \ll 1$. Since bonds cannot be short-sold, we also have $0 \leq b^i \ll 1$. We obtain $P^{PV} \approx P^{ZV} + \pi_x V_X + \pi_b b^1$,[25] where P^{ZV} defined in equation (61) is the stock price in zero volume and where:

$$\pi_x(1 - \beta\kappa^1) = \beta(1-\alpha^1) \frac{u''(\delta^1)}{\lambda^1} E^{\tilde{z}} \left[(P^{ZV} + y(\tilde{z}))^2 \right], \tag{65}$$

$$\pi_b(1 - \beta\kappa^1) = \beta(1-\alpha^1) \frac{u''(\delta^1)}{\lambda^1} E^{\tilde{z}} \left[P^{ZV} + y(\tilde{z}) \right], \tag{66}$$

$$\text{with: } \kappa^i = \alpha^i + (1-\alpha^i) \frac{1}{\lambda^i} u'(\delta^i), \ i = 1, 2. \tag{67}$$

For the bond, we obtain $Q^{PV} \approx Q^{ZV} + \beta(1-\alpha^2)\frac{u''(\delta^2)}{\lambda^2} b^2$ for type-2 agents, where Q^{ZV} defined in equation (62) is the bond price in zero volume. For type-1 agents, we have $Q^{PV} \gtrsim \beta\kappa^1 + \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}(b^1 + E^{\tilde{z}}[P^{ZV} + y(\tilde{z})])$. If type-1 agents do not participate to the bond market, the previous inequality is strict and we have $b^1 = 0$ and $b^2 = \frac{V_B}{\eta^2}$. If type-1 agents trade bonds, the previous inequality is an equality and noticing that $b^1 = \frac{V_B}{\eta^1} - \frac{\eta^2}{\eta^1} b^2$, we deduce the bond expressions (27) and (28). Because of condition (21), type-2 agents cannot be credit-

[25] The approximation sign $\approx$ refers to a first order development with respect to security volumes. It should be understood as $\ldots = \ldots + o(V_X, V_B)$. We assume that both volumes have the same

constrained. Otherwise, we would have $(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}(\frac{V_B}{\eta^1}+\frac{V_X}{\eta^1}E^{\tilde{z}}[P^{ZV}+y(\tilde{z})])>\kappa^2-\kappa^1>0$, contradicting positive volumes. We derive then from bond and stock prices the equity premium in (22).

Average consumptions. Since idiosyncratic and aggregate shocks are independent, $\overline{c}_i^{PV} = E[c_i^{PV}] = E^{\xi}[E^{\tilde{z}}[c_i^{PV}]]$, where $E[\cdot]$ is the total expectation, $E^{\xi}[\cdot]$ the expectation with respect to idiosyncratic risk and $E^{\tilde{z}}[\cdot]$ the expectation with respect to aggregate risk. Let us start with computing the (approximative) different realizations of $E^{\tilde{z}}[c_i^{PV}]$. Agents of type 1 consume:

- $\omega^1-\omega^1\tau+E[y(\tilde{z})]\frac{V_X}{\eta^1}+(1-Q^{ZV})b^1$ with (unconditional) probability $\alpha^1\eta^1$ (i.e., *pp* agents);
- $\omega^1-\omega^1\tau-P^{ZV}\frac{V_X}{\eta^1}-Q^{ZV}b^1$ with probability $(1-\alpha^1)\eta^1$ (i.e., *pu* agents);
- $\delta^1+(P^{ZV}+E[y(\tilde{z})])\frac{V_X}{\eta^1}+b^1$ with probability $(1-\rho^1)(1-\eta^1)$ (i.e., *up* agents);
- δ^1 with probability $\rho^1(1-\eta^1)$ (i.e., *uu* agents);

Since $(1-\alpha^1)\eta^1=(1-\rho^1)(1-\eta^1)$ and $\tau=\frac{1-Q^{ZV}}{\omega^1\eta^1+\omega^2\eta^2}V_B$, the expression (30) is straightforward to derive. By the same token, we can easily obtain the expression (31) for type-2 agents.

Consumption growth variances. To compute these variances, we use the law of total variance.

We first define for $i=1,2$:

$$K^i(p,\pi)=-2\frac{(1-\rho^i)(1-\alpha^i)}{2-\alpha^i-\rho^i}\frac{1}{\omega^i}\times\Bigg((1-E[\tilde{\gamma}_c^{i,ZV}])(\alpha^i-\rho^i\frac{\omega^i}{\delta^i})\pi \tag{68}$$
$$+\frac{\delta^i}{\omega^i}(\frac{\delta^i}{\omega^i}-E[\tilde{\gamma}_c^{i,ZV}])\left(\pi(\frac{\omega^i}{\delta^i}-\alpha^i)+p\right)-\frac{\omega^i}{\delta^i}(\frac{\omega^i}{\delta^i}-E[\tilde{\gamma}_c^{i,ZV}])\left(p+\pi(1-\rho^i)\frac{\omega^i}{\delta^i}\right)\Bigg),$$
$$E[\tilde{\gamma}_c^{i,ZV}]=1+\frac{(1-\rho^i)(1-\alpha^i)}{2-\alpha^i-\rho^i}(\frac{\delta^i}{\omega^i}+\frac{\omega^i}{\delta^i}-2). \tag{69}$$

For type-1 agents, we obtain that $V[\tilde{\gamma}_c^{1,PV}]\approx V[\tilde{\gamma}_c^{1,ZV}]-\nu_{10}\tau-\nu_{11}b^1-\nu_{12}\frac{V_X}{\eta^1}+\nu_{13}\frac{V_X}{\eta^1}V^{\tilde{z}}[y(\tilde{z})]$, where $V[\tilde{\gamma}_c^{1,ZV}]$ is the total variance of consumption growth in zero volumes in equation (24) and where using (68) we have:

$$\nu_{10}=2\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\left(\frac{\omega^1}{\delta^1}-\frac{\delta^1}{\omega^1}\right)\left(1+\frac{1-\alpha^1\rho^1}{2-\alpha^1-\rho^1}(\frac{\delta^1}{\omega^1}+\frac{\omega^1}{\delta^1}-2)\right), \tag{70}$$
$$\nu_{11}=K^1(Q^{ZV},1), \tag{71}$$
$$\nu_{12}=K^1(P^{ZV},P^{ZV}+E^{\tilde{z}}[y(\tilde{z})]), \tag{72}$$
$$\nu_{13}=\eta^1(\frac{1}{\omega^1})^2+\eta^1\alpha^1(\frac{1}{\omega^1})^2(\alpha^1+(1-\alpha^1)(\frac{\delta^1}{\omega^1})^2)+\eta^1(1-\alpha^1)(\frac{1}{\delta^1})^2(\rho^1+(1-\rho^1)(\frac{\omega^1}{\delta^1})^2). \tag{73}$$

Using Lemma A in the electronic supplementary material, we obtain that $\nu_{10}, \nu_{11}, \nu_{12}, \nu_{13} > 0$.

We obtain a very similar expression for the consumption growth variance of type-2 agents: $V[\tilde{\gamma}_c^{2,PV}] \approx V[\tilde{\gamma}_c^{2,ZV}] - \nu_{20}\tau - \nu_{21}b^2$, where the expressions of $\nu_{20} > 0$ and $\nu_{21} > 0$ (using (68)) are symmetric to the ones of ν_{10} and ν_{11} in (70) and (71) respectively:

$$\nu_{20} = 2\frac{(1-\rho^2)(1-\alpha^2)}{2-\alpha^2-\rho^2}\left(\frac{\omega^2}{\delta^2} - \frac{\delta^2}{\omega^2}\right)\left(1 + \frac{1-\alpha^2\rho^2}{2-\alpha^2-\rho^2}(\frac{\delta^2}{\omega^2} + \frac{\omega^2}{\delta^2} - 2)\right), \tag{74}$$

$$\nu_{21} = K^2(Q^{ZV}, 1). \tag{75}$$

D Data Appendix

We consider the dataset used by Heathcote, Perri and Violante (2010). To measure the consumption of non-durable and services, we use the sum of expenditures on non-durable goods, including: the vehicle services and other vehicle expenses (insurance, maintenance, etc.), the housing services, the rent paid, other lodging expenses, household equipment and entertainment. These items are deflated using the CPI. This measure corresponds to the variable *ndpnd*0 in Heathcote et al. (2010). We use the weights given in the CEX to define in each quarter the bottom 50% and the top 50% of households in the consumption distribution.

To compute the volatility of consumption growth for a given group in each quarter, we use the variance of the consumption growth rate between quarter t and quarter $t+1$ among all households belonging to said group at date t (regardless the household's group in $t+1$). We then compute the average variance per group over the time period.

E Description of the estimation algorithm

We describe here the algorithm of Section 4 that we use to minimize the distance between the 6 moments generated by the model and their empirical counterparts and that allows us to estimate our model through the simulated method of moments. We denote $\chi_v = [\alpha^1, \alpha^2, \omega^2, \delta^1, \delta^2, \omega_G^1] \in \mathbb{R}_+^6$ the vector of model parameters we have to estimate. We start from an initial guess vector χ_v^0.

1. We compute the six moments $\widetilde{T}^0$ generated by the model when parameters are equal to χ_v^0. We compute the score $S^0 = (\widetilde{T}^0 - T)\Omega(\widetilde{T}^0 - T)'$, where $\Omega = I_{6\times 6}$ is the weight matrix and T is the vector of empirical moments we match.

2. We construct the hyper-cube of the $2^6 = 64$ neighbors of χ_v^0 by considering marginal increase or decrease in each parameter: $\chi_v^i = [\chi_{v,1} \pm \epsilon, \dots, \chi_{v,6} \pm \epsilon]$ $(i = 1, \dots, 64)$ where we set $\epsilon = 10^{-3}$.

3. For every vector χ_v^i, we compute the moments generated by the model $\widetilde{T}^i$ $(i = 1, \ldots, 64)$. We then also compute the related score $S^i = (\widetilde{T}^i - T)\Omega(\widetilde{T}^i - T)'$ for $i = 1, \ldots, 64$.

4. If $S^0 \leq S^i$ for all $i = 1, \ldots, 64$, we stop the algorithm and we have just found a minimum. We then set our model parameters equal to χ_v^0. If not, we start the algorithm in step 1 with the new initial value $\chi_v^0 = \chi_v^{i_{\min}}$, where $i_{\min} = \arg\min_i \; S_i$.

This algorithm generates a path in $\mathbb{R}_+^6$ converging towards a (local) minimum. We try different starting points χ_v^0 to find a global minimum.

F The model without participation costs

Following the same steps as in the paper, we deduce the structure of the model without participation costs. At the equilibrium, both agents types may trade or not bonds and stocks. In particular, stock holdings x_k^1 and x_k^2 in state k are determined by Euler equations. There exist sets $I_i^B, I_i^X \subset \{1, \ldots, n\}$ $(i = 1, 2)$, such that the $6 \times n$ variables $(b_k^1, b_k^2, x_k^1, x_k^2, P_k, Q_k)_{k=1\ldots n}$ defining the equilibrium are given by the following $6 \times n$ equations:

$$P_k = \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^i + (1 - \alpha_k^i)\frac{1}{\lambda^i}u'(\delta^i + (P_j + y_j)x_k^i + b_k^i))(P_j + y_j), \; k \in \{1, \ldots, n\} - I_i^X, \; i = 1, 2, \tag{76}$$

$$Q_k = \beta \sum_{j=1}^{n} \pi_{kj}(\alpha_k^i + (1 - \alpha_k^i)\frac{1}{\lambda^i}u'(\delta^i + (P_j + y_j)x_k^i + b_k^i)), \; k \in \{1, \ldots, n\} - I_i^B, \; i = 1, 2, \tag{77}$$

and

$$V_X = \eta_k^i \, x_k^i \text{ and } 0 = x_k^j, \; k \in I_i^X, \; i \neq j = 1, 2, \tag{78}$$

$$V_X = \eta_k^1 \, x_k^1 + \eta_k^2 \, x_k^2, \; k \in \{1, \ldots, n\} - I_1^X - I_2^X, \tag{79}$$

$$V_B = \eta_k^i \, b_k^i \text{ and } 0 = b_k^j, \; k \in I_i^B, \; i \neq j = 1, 2, \tag{80}$$

$$V_B = \eta_k^1 \, b_k^1 + \eta_k^2 \, b_k^2, \; k \in \{1, \ldots, n\} - I_1^B - I_2^B. \tag{81}$$

The set I_i^X $i = 1, 2$ gathers states of the world, in which type-i agents do not trade stocks. The set I_i^B has the same meaning for bond market. The sets I_1^B, I_2^B on one side and I_1^X, I_2^X on the other side must be disjoint. This means that there should not exist a state of the world, in which no one is trading bond or stocks. Note that since our equilibrium features security prices that only depend on the current state of the world, the sets I_i^B, I_i^X are not time-dependent.

As in the core of the paper, several inequalities have to hold for the above equations to define a small-trade equilibrium. These inequalities guarantee that: (i) productive agents who do not trade a given security do not want to do so (i.e., this implies inequalities similar to (56)–(56)), and (ii) that unemployed agents do not want to trade (i.e., this implies inequalities similar to (58)–(60)). For the sake of conciseness, we do not report here these inequalities, which are rather straightforward to deduce from the previous equilibrium but rather lengthy to write down.