OPTIMAL MONETARY POLICY IN A LIQUIDITY TRAP WITH HETEROGENEOUS AGENTS

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ABSTRACT. This paper derives the optimal money injection at the Zero Lower Bound (ZLB), in a tractable model where households hold heterogeneous money holdings due to explicit financial frictions, such as limited participation or temporary binding credit constraints. This framework is motivated by recent empirical findings. A deleveraging shock generates deflationary pressure and a fall in the real interest rate, pushing the economy to the ZLB. The main result is that open-market operations can stabilize the economy at the ZLB whereas lump-sum money transfers cannot. Moreover, an optimal money injection does not avoid the economy being at the ZLB.

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Keywords : liquidity trap, zero lower bound, heterogeneous agents, optimal policy.

1. INTRODUCTION

The role of monetary policy in a liquidity trap, i.e when the nominal interest rate reaches the zero lower bound (ZLB), is now acknowledged to be a major issue. After Japan, the Euro area and the US central banks have decided to inject massive quantities of money, a policy labeled quantitative easing, at zero interest rates. The channels through which such money creation could have real effects are still not well understood, and the empirical assessment of the real effects of such policies is still under debate. In addition to these positive effects, the optimal design of money injection in a liquidity trap is an open question[1].

The contribution of this paper is to identify optimal monetary policy at the ZLB in a model where households have heterogeneous money holdings, based on explicit financial frictions. In

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[1]Developments in 2015 in the Euro area have shown that the nominal interest rate can be negative because of additional opportunity costs to hold money, such as the costs of storing cash for instance. I do not try to estimate these additional costs, and the zero lower bound is defined as a floor 0 for the nominal interest rate, as done by the papers in this literature (Krugman (1998), Eggertsson and Woodford (2003), Eggertsson and Woodford (2003), Werning (2011) among others)

this setup, it is shown that money injections have a real effect even at the ZLB, and that open-market operations (contrary to lump-sum money transfer) can stabilize the economy after financial shocks. Household heterogeneity thus helps to clarify the role of the money injections, and more precisely the role of the central bank balance sheet at the zero lower bound.

The motivation for this model is based on recent developments in households' money demand. First, the distribution of money (M1) in the US is very unequal across households (Erosa and Ventura (2002)), and money inequality is comparable to wealth inequality, and much higher than consumption inequality Ragot (2014)). In addition, financial frictions, such as limited participation in financial markets or credit constraints, are key to understand households' money holdings (Alvarez and Lippi (2009)). As a consequence, the redistributive effects of monetary policy can be expected to have a first-order effect on the economy. In this paper, limited participation in financial markets is thus introduced to generate at the same time a well-defined money demand and money holding heterogeneity. This paper thus departs from the assumption of a representative agent to study monetary policy at the ZLB. The representative agent assumptions generate the result that money injections done by open-market operations are irrelevant for both nominal and real variables, even at the zero lower bound (Eggertsson and Woodford (2003)). Compared to this benchmark, this paper shows that open-market operations have a real effect when agents hold heterogeneous money holdings and that, in addition, these operations have a comparative advantage over other forms of money creation, such as lump-sum money transfers.

Before presenting the model it may be useful to provide some simple intuitions of the new channels generated by financial frictions and agents' heterogeneity at the ZLB. At the ZLB, open-market operations swap two assets with the same zero-nominal return and the same risk (money and public debt). This has a real effect because open-market operations affect the distribution of money and the tax system through an effect on the State budget. The redistributions generated by open-market operations have real effects at the ZLB if some agents face credit constraints at the ZLB. This mechanism is likely to be important, as many analyses of the 2008 crisis in the US identify a tightening of the credit constraint as a key cause of the low observed real interest rate (Guerrieri and Lorenzoni (2011); Eggertsson and Krugman (2012) among others).

The goal of this paper is to explicitly derive optimal monetary policy in a liquidity trap, in a microfounded monetary heterogeneous agents model. These monetary heterogeneous agent models are known to be very hard to solve, and generally require numerical techniques to be simulated, which prevents the clear identification of optimal policies (see the literature review below). For this reason, the strategy of this paper is to provide a simple model to derive optimal monetary policy with paper and pencil, to clearly identify the trade-offs. To justify that this simple model is relevant and can be of independent interest, a careful motivation of the required simplifying assumptions is provided in Section 2. More precisely, a flexible-price endowment economy is considered, where some agents always participate in financial markets whereas other agents don't participate in financial markets and can only hold money to smooth income fluctuations. All agents face credit constraints, but these ones only bind in equilibrium for non-participating households[2]. When the ZLB does not bind, only non-participating agents hold money, because money is a dominated asset. Only when the ZLB binds do participating households hold money.

In this economy, an unanticipated once-for-all tightening of the credit constraints is modeled. The ability of monetary policy to stabilize the economy, i.e. to undo the effect of such a shock, is analyzed. The model yields three results.

First, and as can be expected, the tightening of the credit constraint pushes the economy to the zero lower bound for two reasons. It first decreases the real interest rate because, as gross borrowing is reduced, the market-clearing interest rate must decrease, as discussed by Guerrieri and Lorenzoni (2011) and Eggertsson and Krugman (2012). Second, a tightening of the credit constraint induces deflationary pressures, because it increases real money demand. In this case, non-participating agents must indeed hold more money to smooth consumption.

Second, when the zero lower bound does not bind, monetary policy can stabilize the economy whatever the process of money creation: open-market operations or helicopter drops of money. Indeed, optimal monetary policy improves the consumption smoothing of all agents by affecting the relative wealth of different types of agents. The redistributive effects of monetary policy are key to understand the results, and they should be contrasted with a more direct fiscal policy to transfer wealth across agents. As discussed in the paper, the comparative advantage of monetary policy compared to fiscal policy is that it allows reaching the first-best allocation without any information about the identity of agents (or about their decision to participate in financial markets).

The third and main result of this analysis is that monetary policy can also stabilize the economy if the ZLB is binding, but only if money is created by open-market operations. Helicopter drops of money have real effects but don't allow fully stabilizing the economy. Monetary policy can stabilize the economy, because agents participating in financial markets start holding money at the ZLB, which alters the distribution of money across agents. The optimal use of this new money demand by monetary authorities increases the ability to stabilize the economy. Full stabilization can be obtained by open-market operations only, because of the difference in the redistributive

[2]The model can be seen as a mix of Alvarez and Lippi (2014) and Woodford (1990): Limited participation is modeled as an exogenous share of households not participating in financial markets, and it embeds the role of money as a consumption smoothing device because of incomplete markets.

effect of open-market operations and helicopter drops of money. Open-market operations *de facto* generate a money creation targeted to a sub-group of agents, who participate in financial markets (and sell or buy some financial assets against money), and it has a more complex effect on the State budget, because it affects current and future transfers through the redistribution of the profits of the central bank. At the ZLB the size of the central bank balance sheet is thus key to improving the market allocation. To my knowledge, this paper is the first to identify such a sharp difference between helicopter drops of money and open-market operations at the ZLB. For this reason, the difference between the two processes of money creation is carefully investigated.

This last result has important implications for monetary policy. First, the quantity of money matters, but more importantly the process of money creation is crucial at the ZLB. Second, the ZLB is not a constraint for monetary policy, when the process of money creation is optimally defined (i.e. open-market operations). These sharp results obviously depend on the model simplifying assumptions, and their relevance for more general environments is discussed in the concluding remarks. These results favor the view that periods when the ZLB binds are special times, which need special tools, but that these can be handled by monetary policy if the central bank manages the size of its balance sheet.

In monetary debates, the claim that the size of the central bank balance sheet has real effects is often attributed to Friedman and Schwartz (1963), after their analysis of monetary policy in the Great Depression (see Buera and Nicolini (2014) for a recent paper). The fact that introducing agents' heterogeneity confirms the Friedman and Schwartz view about the role of the central bank balance sheet should not come as a surprise. Indeed, the heterogeneous agents model, after the seminal contribution of Bewley (1983), is explicitly inspired by the informal money theory presented in the analysis by Friedman (1969) of the optimum quantity of money, which explicitly introduces households' heterogeneous money holdings, and considers the redistributive effect of monetary policy.

In the recent literature, the current paper is first related to the literature on heterogeneous agents and limited participation in monetary models. Second, it is related to papers studying the monetary policy at the zero lower bound.

Heterogeneous households in monetary economics. On the modeling side, the current paper is close to the literature analyzing the implications of limited participation for monetary policy in simple environments (Alvarez, Atkeson, and Kehoe (2002); Alvarez, Atkeson, and Edmond (2009); Alvarez and Lippi (2014) among others). In this literature, the contribution of my model is to simplify the environment to be able to derive optimal monetary policy simply both at the ZLB and outside the ZLB.

There is a growing literature on agents' heterogeneity in monetary models to study the redistributive effects of monetary policy in realistic environments (Bewley (1983) for a seminal contribution; Kehoe, levine, and Woodford (1992); Erosa and Ventura (2002); Algan and Ragot (2010); Ragot (2015) among others). In this quantitative literature, there are some technical issues about simulating these models when the economy can temporarily hit the zero lower bound. Recently, Ragot (2014), following some early insight in Erosa and Ventura (2002), has shown that the introduction of limited participation in this type of model is key to reproduce a realistic distribution of money. The modeling strategy of the current paper is to simplify the environment to be able to derive optimal monetary policy at the ZLB.

Monetary policy in a liquidity trap has been studied in representative-agent, cashless economies with sticky prices (Werning (2011); Cochrane (2015)). The current model with heterogeneous money holdings and flexible prices identifies other mechanisms based on the redistributive effects of monetary policy, such as the role of the money-creation process. The interaction of the two frictions (sticky-prices and financial frictions generated the money demand) is discussed in the concluding remarks.

Recent papers have studied the positive effect of money injection in a liquidity trap in models where financial frictions mainly affect firms or entrepreneurs and limit the efficiency of capital allocation (Buera and Nicolini (2014); Bachetta, Benhima, and Kalantzis (2015)). These papers do not consider optimal policies.

Section 2 justifies the simplifying assumptions. Section 3 presents the model. Section analyses the model when the ZLB does not bind. Section 4 derives optimal monetary policy at the ZLB. Section 6 is the conclusion.

2. Financial frictions and the distribution of wealth and money

This paper relies on some sharp simplifications to be able to identify optimal monetary policy. It is assumed that 1) some agents will always participate in financial markets, whereas others never participate in financial markets. 2) Agents not participating in financial markets face fluctuating income and use money to smooth consumption 3) Non-participating agents experience an unexpected tightening of the credit constraint. This section justifies these three key assumptions.

The assumption of limited participation is motivated by the analysis of households' portfolios. Summarizing the Survey of Consumer Finance results, Bricker, Dettling, Henriques, Hsu, Moore, Sabelhaus, Thompson, and Windle (2014) show that roughly half of the population (usually the most wealthy) participates in the stock market either directly or indirectly. The fraction of the population using financial assets to smooth consumption in the business cycle may be smaller than half of the population, because an important part of those financial assets are saved in retirement plans. Recently Kaplan, Violante, and Weidner (2014) concluded that one should consider "wealthy hand-to-mouth" households to account for the response of household consumption to changes in fiscal policy: Although some agents have financial wealth, they do not use it to smooth the effect of a transitory income shock. For this reason, and following the order of magnitude of Challe and Ragot (2016), it will be assumed that 2/3 of the population does not use financial assets (i.e. non-monetary assets) to smooth consumption, whereas 1/3 can use financial assets.

The second assumption is that non-participating agents (denoted as N-agents in the model) use money to smooth fluctuating incomes. This is the Bewley theory of money. To obtain tractability, we elaborate on the Woodford (1990) model in a monetary setting. It is assumed that half of the non-participating households (thus 1/3 of the total population) receives some income in odd periods and wants to consume in even periods, whereas the other half receives some income in even period and consumes in odd periods. This generates a demand for money in all periods for consumption smoothing. Deterministic income fluctuations can be seen as an extreme case of the Bewley model to obtain tractability. These two assumptions together imply that the money is held by the wealth-poor agents when the ZLB does not bind, which is a consistent result. As shown by Erosa and Ventura (2002) and Ragot (2014), the wealth-rich have indeed a much lower share of their wealth in money than do the wealth-rich. In this setup, the participating agents hold money only at the ZLB, i.e. only when the return on money and financial assets is the same.

The third assumption is that non-participating agents face borrowing constraints and experience an unexpected tightening of the credit constraint. Following many contributions in the literature (Guerrieri and Lorenzoni (2011); Eggertsson and Krugman (2012)), this assumption captures the US households' deleveraging process after 2008.

3. THE MODEL

Time is discrete and periods are indexed by $t = 0, 1...$ The model features a closed economy populated by a continuum of households indexed by i and uniformly distributed along the unit interval, as well as a representative firm. Households have a log period utility function $u(c) = \log c$, but the model can be extended to a CRRA utility function at the cost of more algebra. The discount factor is β. It is assumed that the economy is composed of two types of households. There is a fraction $\Omega = \frac{2}{3}$ of agents, denoted as N-households, who must pay a fixed cost κ^N each time they want to participate in financial markets. The remaining fraction $1 - \Omega = \frac{1}{3}$ of households, denoted as $P-$households, don't pay any cost to save in financial markets. The cost

κ^N is determined in Section 3.2 below. It is high enough such that N-households never participate in financial markets. As a consequence, only $P-$households participate in financial markets.[3]

We will assume

$$\beta > 2/3$$

In addition, we assume that all households face a borrowing limit $d_t \leq 1$ in all periods. This borrowing limit is the maximum claims that households can issue in each period. Before period 0, the borrowing limit is d^*, and in period 0, there is an unexpected change in the borrowing limit which follows a path $d_t, t \geq 0$. After period 0, no other shocks hit the economy. As a consequence, the focus of the paper is on the adjustment of the economy after an unexpected change in the borrowing limit.

3.1. **Agents.**

3.1.1. *Non-participating households.* $N-$households are denoted by the upper-script n. A fraction $\Omega/2 = 1/3$ consumes in odd periods and receives one unit of goods in even periods. The other fraction $\Omega/2$ consumes in even periods and receives one unit of goods in odd periods. The price of the final good in period t is denoted P_t and $\pi_t = \frac{P_t}{P_{t-1}} - 1$ is the net inflation rate. In all periods, households may receive a real transfer τ_t from the state.

In period t, households cannot issue some money but they can sell claims on period $t+1$ goods, up to the limit d_t. Denote as Q_t the nominal price of those claims in period t.

It is conjectured (and it is checked below) that 1) $N-$households save in money when they receive some income and 2) N-households hit the borrowing constraint when they consume. Denote as M_t^n the nominal quantity of money saved by N-households when they receive some income. When $N-$households consume, they consume the amount

$$P_t c_t^n = M_{t-1}^n + P_t \tau_t + Q_t d_t$$

The right-hand side is the sum of past saving in money M_{t-1}^n, of the transfer from the State $P_t \tau_t$ and the money obtained by selling the d_t claims in period t. In real terms consumption is

$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \tau_t + q_t d_t \tag{1}$$

where $m_{t-1}^N = M_{t-1}^N / P_{t-1}$ and $q_t = Q_t / P_t$ is the price of a claim in real terms.

[3]This participations costs structure is a simplification of the general framework of Alvarez et al. (2002). It allows studying limited participation in a simple environment such as the one of Alvarez and Lippi (2014). It would be very easy to introduce the same participation cost for P-households at the cost of more algebra.

The fraction $\Omega/2$ who gets one unit of endowment but does not consume save in money the real amount

$$m_t^n = 1 + \tau_t - d_{t-1} \tag{2}$$

The quantity $1 + \tau_t$ is the real value of the endowment after the transfer. The quantity $-d_{t-1}$ is the repayment of the d_{t-1} claims issued in the previous period.

For this allocation to be an equilibrium, $N-$households must hit the credit constraint when they consume. The condition is

$$q_t u'\left(c_t^n\right) > \beta^2 \frac{1}{1+\pi_{t+2}} u'\left(c_{t+2}^n\right) \tag{3}$$

The left-hand side is the marginal gain of issuing one additional claim in period t. The right-hand side is the marginal cost: If households issue one additional claim in period t, they will have less goods in period $t+1$. As the have saved in money (because they do not consume in period $t+1$), these goods generate an income $\frac{1}{1+\pi_{t+2}}$ in period $t+2$, which is valued $\beta^2 u'\left(c_{t+2}^n\right)\frac{1}{1+\pi_{t+2}}$ in period t. When the condition is fulfilled, households would like to issue claims at period t and hit the credit constraint d_t.

An additional condition is that $N-$households don't hold money when they consume (i.e. they consume all their income). Along the same lines, this condition is

$$u'\left(c_t^n\right) > \beta^2 \frac{1}{1+\pi_{t+1}} \frac{1}{1+\pi_{t+2}} u'\left(c_{t+2}^n\right) \tag{4}$$

These two conditions will be fulfilled in the economies studied below, even when the ZLB is binding.

3.1.2. *Participating Households.* Variables concerning $P-$households are indicated by the upperscript p. These households supply one unit of labor every period. They can save in money and in financial markets[4]. $P-$ households can buy three types of assets: money or claims issued either by the State (i.e. government debt) or by $N-$households. In period t, they buy a quantity b_t^g of claims on the government, which pay a nominal interest rate i_t between period t and period $t+1$. As a consequence, the period t real return of the quantity b_{t-1}^g of goverment debt bought in period $t-1$ is $\frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^g$.

$P-$households buy a quantity b_t^h of claims issued by $N-$households at a real price q_t and they may hold a real quantity $m_t^p \geq 0$ of money. The after-transfer income of $P-$households is $1 + \tau_t$ in period t. For symmetry reasons, we also assume that $P-$households can't issue more claims than the borrowing limit d_t, but this constraint will not bind in the equilibrium under consideration.

[4]It is direct to introduce consumption and income every two-periods for $P-$households for them to have the same utility function as non-participating agents. This would only generate more algebra.

The budget constraint of a representative $P-$households and the conditions on money holdings are:

$$q_t b_t^h + b_t^g + m_t^p + c_t^p = 1 + \tau_t + b_{t-1}^h + \frac{m_{t-1}^p}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^g \tag{5}$$

$$m_t^p \geq 0 \tag{6}$$

$$b_t^h \geq -d_t \tag{7}$$

where c_t^p is real consumption. The right-hand side is the after-tax income $1+\tau_t$ plus the return on the three assets. The left-hand side is private consumption plus the cost of investment in the three assets. Standard intertemporal utility maximization yields the three first-order conditions, for $t \geq 0$:

$$q_t u'\left(c_t^p\right) \geq \beta u'\left(c_{t+1}^p\right) \tag{8}$$

$$u'\left(c_t^p\right) \geq \beta \frac{1+i_t}{1+\pi_{t+1}} u'\left(c_{t+1}^p\right) \tag{9}$$

$$u'\left(c_t^p\right) \geq \beta \frac{1}{1+\pi_{t+1}} u'\left(c_{t+1}^p\right) \tag{10}$$

The first condition is an equality when P-households do not face the credit constraint, which will always be the case. The second condition is an equality when P-households hold government debt, which will also again be the case in the equilibrium under consideration. Considering the first two conditions with equalities, one finds the no-arbitrage condition

$$\frac{1}{q_t} = \frac{1+i_t}{1+\pi_{t+1}} \tag{11}$$

The real return on government bonds is the real return on claims, because there is no aggregate shock after period 0, and there are thus no risk premia. The third inequality (10) is an equality if $P-$households hold money. Observing the second and third inequality, this cannot be the case when $i_t > 0$. In other words, only when $i_t = 0$, i.e. at the ZLB, can $P-$households hold money. At the ZLB, considering the first and the third conditions with equalities, one finds the condition on the inflation

$$1+\pi_{t+1} \geq q_t \text{ and } m_t^p = 0 \text{ if } 1+\pi_{t+1} > q_t$$

Monetary policy and the State budget. The monetary authorities issue a real quantity m_t^{CB} of money at each period t. To consider a general case, it is assumed that the monetary authorities issue a fraction θm_t^{CB} of money by lump-sum transfers to all households and a fraction $(1-\theta)\, m_t^{CB}$ by open-market operations. The money issued by open-market operations is used to buy a real quantity of public debt $x_t \equiv (1-\theta)\, m_t^{CB}$ in period t, which will generate a real profit for the central bank $x_t \frac{1+i_t}{1+\pi_{t+1}} \equiv (1-\theta)\, m_t^{CB} \frac{1+i_t}{1+\pi_{t+1}}$ in period $t+1$. As is standard, it is assumed that this profit is given back to the State at period $t+1$.

Denote as M_t^{tot} the nominal quantity of money in circulation at the end of period t. As the (nominal) new money created in period t is $P_t m_t^{CB}$, the nominal quantity of money in period t is $M_t^{tot} = M_{t-1}^{tot} + P_t m_t^{CB}$. Denote $m_t^{tot} = M_t^{tot}/P_t$, the real quantity of money in circulation. We have the law of motion

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1+\pi_t} + m_t^{CB} \tag{12}$$

It is assumed that the State issues a quantity of debt $\bar{d}^g$ at each period t, to finance a public good $\bar{g}$. The budget of the State is

$$\bar{g} + \bar{d}^g \frac{1+i_{t-1}}{1+\pi_t} + \lambda_t = \bar{d}^g + (1-\theta)\, m_{t-1}^{CB} \frac{1+i_{t-1}}{1+\pi_t}$$

where λ_t is the (net) fiscal transfer to each household, which is different from the total transfer τ_t, which also includes the new money created by lump-sum transfers. As a normalization, and without loss of generality, I set $\bar{g} = \bar{d}^g = 0$. As a consequence, $\lambda_t = (1-\theta)\, m_{t-1}^{CB} \frac{1+i_{t-1}}{1+\pi_t}$. In other words, the profits generated by monetary operations are given back to households. The total transfer τ_t received by each household is thus the sum of the lump-sum money transfer θm_t^{CB} and $\lambda_t = (1-\theta)\, m_{t-1}^{CB} \frac{1+i_{t-1}}{1+\pi_t}$. Using the equality (11), we obtain

$$\tau_t = \theta m_t^{CB} + \frac{(1-\theta)\, m_{t-1}^{CB}}{q_{t-1}} \tag{13}$$

3.2. **Market equilibrium.** There are four market equilibria in this economy. First, the equilibrium of the money market implies that the real quantity of money m_t^{tot} is held either by the fraction $\Omega/2 = 1/3$ of $N-$households who save in money or by the fraction $1-\Omega = 1/3$ of $P-$households, if they hold some money. Hence $m^{tot} = \frac{1}{3} m_t^n + \frac{1}{3} m_t^p$, or

$$3 m_t^{tot} = m_t^n + m_t^p \tag{14}$$

As only $P-$households participate in financial markets, the equilibrium on the market for claims is $(1-\Omega)\, b_t^h = \frac{\Omega}{2} d_t$, or

$$b_t^h = d_t$$

Public debt (in zero supply) is bought either by the $(1-\Omega)$ $P-$households or by the monetary authorities for a quantity $x_t = (1-\theta)\, m_t^{CB}$:

$$\frac{1}{3} b_t^g + (1-\theta)\, m_t^{CB} = 0 \tag{15}$$

Finally, the goods market equilibrium stipulates that the total quantity of goods $(1-\Omega)+\frac{\Omega}{2}$ in each period is consumed either by $P-$households or by $N-$households who consume. Hence, $(1-\Omega)\,c_t^p+\frac{\Omega}{2}c_t^n=(1-\Omega)+\frac{\Omega}{2}$ or, as $\Omega=2/3$:

$$c_t^p+c_t^n=2 \tag{16}$$

3.3. **Summary of the model.** It may be useful to summarize the equations defining the equilibrium under consideration, using market equilibria to substitute for b_t^h and b_t^g.

$$m_t^n=1+\tau_t-d_{t-1} \tag{17}$$

$$c_t^n=\frac{m_{t-1}^n}{1+\pi_t}+\tau_t+q_t d_t \tag{18}$$

$$\begin{aligned} q_t\left(d_t-3\left(1-\theta\right)\frac{m_t^{CB}}{q_t}\right)+m_t^p+c_t^p &= 1+\tau_t \\ &+\left(d_{t-1}-3\frac{\left(1-\theta\right)m_{t-1}^{CB}}{q_{t-1}}\right)+\frac{m_{t-1}^p}{1+\pi_t} \end{aligned} \tag{19}$$

$$q_t u'\left(c_t^p\right)=\beta u'\left(c_{t+1}^p\right) \tag{20}$$

$$m_t^p=0 \text{ if } 1+\pi_{t+1}>q_t \tag{21}$$

$$m_t^n+m_t^p=\frac{m_{t-1}^n+m_{t-1}^p}{1+\pi_t}+3m_t^{CB} \tag{22}$$

$$\tau_t=\theta m_t^{CB}+\left(1-\theta\right)\frac{m_{t-1}^{CB}}{q_{t-1}} \tag{23}$$

Together with the inequalities

$$c_t^n,c_t^p,m_t^n,m_t^p\geq 0 \tag{24}$$

$$1+\pi_{t+1}\geq q_t \tag{25}$$

and the condition

$$q_t u'\left(c_t^n\right)>\beta^2\frac{1}{1+\pi_{t+2}}u'\left(c_{t+2}^n\right) \tag{26}$$

Equation (17) is the money demand of $N-$households in periods when they work and don't consume. Equation (**??**) is the consumption of these $N-$households when they consume but don't work. These two equations assume that N-households don't hold money when they consume, and thus that condition (4) is fulfilled. It is checked that it is the case, in all equilibria under consideration. Equation (19) is the budget constraint of $P-$households. The financial market equilibrium (15) has been used to substitute for b_{t+1}^g. Equation (20) is the Euler equation of $P-$households. Equation (21) is the Euler equation of $P-$households for money holdings: $P-$households only hold money if the zero-lower bound is binding. Equality (22) is the money market equilibrium, and equality (23) is the process of money creation.

Inequality (24) stipulates that consumption can't be negative and that households cannot issue money. Inequality (25) is the zero lower bound.

Finally, condition (26) is the condition for the equilibrium under consideration to exist (i.e. the credit constraint binds for $N-$households). Note that inequality (25) implies that when this inequality is satisfied, inequality (4) is also satisfied, and N-households never save in money in periods where they consume.

3.4. **Steady State.** The steady-state allocation is defined as an allocation where no money is created ($m^{CB} = 0$), where the credit limit is constant and equal to d^*, and where real variables are constant. Steady-state values are denoted with a star and, for instance, c^{n*} is the steady-state consumption of $N-$households. One easily finds that $\tau^* = 0$ and that the financial market equilibrium implies $b^{h*} = d^*$ and that the money held by $N-$households who don't consume is $m^{n*} = 1 - d^*$. Moreover, the price of the financial asset is simply $q^* = \beta$. Hence, the consumption of the two types of agents is, from (**??**) and (19):

$$c^{n*} = 1 - (1-\beta)\, d^*$$
$$c^{p*} = 1 + (1-\beta)\, d^*$$

Finally, note that in steady state, the inflation rate is 0, $\pi^* = 0$. As a consequence, conditions (3) and (4) are fulfilled.

It will be assumed that the economy is in steady state and that at period 0 that it is hit only once by an un-anticipated shock on the process $d_t, t \geq 0$. This implies that

$$d_{-1} = d^*, m^n_{-1} = m^{n*}, m^p_{-1} = 0, m^{CB}_{-1} = 0 \tag{27}$$

Equilibrium definition: For a deterministic sequence of credit constraint $\{d_t\}_{t\geq 0}$ and monetary policy $\{m^{CB}_t\}_{t\geq 0}$, an equilibrium of this economy is a sequence $\{c^n_t, c^p_t, m^n_t, m^p_t, \tau_t, q_t, \pi_t\}_{t\geq 0}$ which satisfies the seven equations (17)-(23), the inequality constraints (24)-(25), and the initial conditions (27).

3.5. **Optimal allocation.** The first-best allocation is defined as follows. It is assumed that the planner gives a weight ω_p to $P-$households and a weight 1 to $N-$households (without loss of generality). The tilde is used to indicate the optimal allocation. For instance $\tilde{c}^n_t$ is the optimal consumption of a $N-$household in period t. The intertemporal social welfare function is

$$\max \sum_{t=0}^{\infty} \beta^t \left(u\left(\tilde{c}^n_t\right) + \omega_p u\left(\tilde{c}^p_t\right)\right) \tag{28}$$

and the resource constraint of the planner is $\tilde{c}^n_t + \tilde{c}^p_t = 2$.

Using again asterisk to denote stationary values, the optimal allocation is thus simply defined by the two equations $u'\left(\tilde{c}^{n*}\right) = \omega_p u'\left(\tilde{c}^{p*}\right)$ and $\tilde{c}^{n*} + \tilde{c}^{p*} = 2$. In what follows, we use the specific values of the Pareto weight ω^p such that the steady-state allocation of the market economy is optimal, which is summarized by by the following Assumption.

ASSUMPTION 1: *The Pareto weight is chosen such that* $\omega_p = \frac{u'(c^{n*})}{u'(c^{p*})}$. *The optimal allocation is thus* $\tilde{c}^n_t = c^{n*}$ *and* $\tilde{c}^p_t = c^{p*}$.

In words, it is assumed that the goal of the planner is to stabilize the economy such that consumption levels are equal to their steady-state value. As the only exogenous process is variations in the borrowing limit d_t, which does not affect aggregate resources $(1-\Omega) + \Omega/2 = 2/3$, this allocation is always feasible. The rest of the paper is the analysis of the ability of monetary policy to neutralize the effect on consumption of the credit constraint shock $\{d_t\}_{t\geq 0}$.

4. The economy when the ZLB doesn't bind

When the ZLB does not bind, to understand the structure of the model, one may first consider the economy when monetary policy is inactive $m^{CB} = 0$ and thus $\tau_t = 0$. In this case, equations (17)-(23) can be simply written as, for $t \geq 0$

$$c^p_t = 1 - q_t d_t + d_{t-1} + \Theta_t$$
$$q_t u'\left(c^p_t\right) = \beta u'\left(c^p_{t+1}\right)$$

with

$$\Theta_t \equiv (3-2\theta)\, m^{CB}_t - 2\,(1-\theta)\,\frac{m^{CB}_{t-1}}{q_{t-1}}$$

The first equation is the budget constraint of $P-$households, and the second equation is the Euler equation. The third equation is the definition of Θ_t which summarizes the effect of money creation. These three equations form a system that can be solved to find c^p_t, q_t and Θ_t. The consumption level of $N-$households is simply $c^n_t = 2 - c^p_t$ from the goods market equilibrium.

In general, equilibrium money creation is only a transfer between N and $P-$households. When money is created as a lump-sum transfer, $\theta = 1$, $\Theta_t = m^{CB}_t$ the transfer goes directly from N-households (who hold money at the end of period) to $P-$households who sell all the money they receive and do not hold money at the end of the period.

When money is created by open-market operations $\theta = 0$, we have $\Theta_t \equiv 3m^{CB}_t - 2\frac{m^{CB}_{t-1}}{q_{t-1}}$. In this case, the redistributive effect to $P-$households of the money created in the period (i.e. m^{CB}_t) is bigger because the monetary authorities transfer all the new money to participating households buying some of their assets. Moreover, all agents benefit from the redistribution of the profits of the monetary authorities coming from the purchase of assets in the previous period. This appears as a reduction in the transfer to P-households because a part of the return on the debt issued by $N-$households is given back to these same households, by the monetary authorities, instead of being fully received by P-households. These redistributive effects of lump-sum money creation and open-market operations are different. This difference will be important only at the ZLB.

As can be seen from the previous system, money creation redistributes income across households, and can be used to undo the effect of a change in credit conditions. Indeed, the next Proposition states that monetary policy can implement the first best, whatever the process of money creation, i.e. whatever the value of θ.

Proposition 1. *For $\theta \in [0,1]$ the first best can be implemented when the ZLB doesn't bind. Money creation must follow the rule, for $t \geq 0$*

$$m_t^{CB} = \frac{(d^* - d_{t-1}) - \beta\,(d^* - d_t)}{3 - 2\theta} + 2\frac{1-\theta}{3-2\theta}\frac{m_{t-1}^{CB}}{\beta} \tag{29}$$

Proof. I first provide the necessary conditions. Assume that the ZLB does not bind ($1 + \pi_{t+1} > q_t$ and $m_t^p = 0, t \geq 0$) and that the first best is achieved, $c_t^n = c^{n*}$ and $c_t^p = c^{p*} = 1 + (1-\beta)\,\bar{d}^*$. As a consequence, $q_t = \beta$, from equation (20). Then using (19) and (23) to substitute for the value of τ_t, one finds the quantity of money created in all periods that satisfies equality (29).

Second, assuming the money creation follows (29) and using (23), it is easy to check that the first-best allocation is an equilibrium. In this case, the inflation rate is (with $d_{-2} = d_{-1} = d^*$)

$$1 + \pi_t = \frac{1 + m_{t-1}^{CB} - d_{t-2}}{1 - d_{t-1} - 2m_t^{CB}}$$

Finally, note that the assumption $\frac{2}{3} < \beta$, ensures that $\frac{2(1-\theta)}{3-2\theta}\frac{1}{\beta} < 1$ and that money creation is not a diverging process for all values of $\theta \in [0,1]$, what ends the proof. $\square$

When money is created by lump-sum transfers ($\theta = 1$), the optimal process for money creation is simply $m_t^{CB} = (d^* - d_{t-1}) - \beta\,(d^* - d_t)$. Money creation reacts to current and past values of the credit constraint to undo the effect of a change in the ability to borrow. When money is created by open-market operations ($\theta = 0$), then optimal money creation is $m_t^{CB} = ((d^* - d_{t-1}) - \beta\,(d^* - d_t))\,/3 + \frac{2}{3\beta}m_{t-1}^{CB}$. Money creation now depends on the past money creation m_{t-1}^{CB}. In this case, the central bank redistributes its profits generated by past money creation. Optimal monetary policy must first undo this transfer and then implement the optimal redistribution across agents.

Finally, the assumption that the ZLB does not bind is an implicit assumption about the credit constraint process d_t. This assumption is valid if d_t remains close to d^* and a threshold is given in the next Section for a specific process.

Monetary and fiscal policy. Optimal monetary policy redistributes wealth across agents who have different money holdings. One could argue that this should be the role of fiscal policy. Obviously, if fiscal policy could generate lump-sum transfers between $P-$households and $N-$households who are not working, the first-best allocation could be implemented. The relative advantage of monetary policy is that it does not rely on the identity of households. Money transfers are the same for all agents, and depends only on aggregate variable such as financial conditions (which is here the severity of the financial constraint d_t). Thus monetary policy is efficient even when households are anonymous.

5. Optimal monetary policy at the ZLB

To further investigate the properties of the model, I now focus on a simple deleveraging shock: Assume that $d_0 < d^*$ and $d_t = d^*$ for $t \geq 1$. The credit contraint is more stringent than its steady-state value only in period 0. This simplification, which is frequent in the literature (Eggertsson and Woodford (2003) among others), allows analytical characterization of the ZLB and optimal monetary policy. The next Proposition first presents the effect of this deleveraging shock, when monetary policy is inactive, i.e. $m_t^{CB} = 0$. The proofs of the following Propositions are in the Appendix.

Proposition 2. *If $m_t^{CB} = 0$:*
1) The first-best allocation is an equilibrium after period 2 : $c_t^p = c^{p*}$ and $c_t^n = c^{n*}$, for $t \geq 2$
2) Period 0 interest rate and inflation rate are

$$1 + r_0 = \frac{1}{q_0} = \frac{1}{\beta} + \left(\frac{1}{1+\bar{d}^*} + \frac{1}{\beta}\right)\frac{d_0 - \bar{d}^*}{1+\bar{d}^*} \text{ and } \pi_1 = \frac{\bar{d}_0 - \bar{d}^*}{1 - \bar{d}_0}$$

as consequence, a deleveraging shock (a decrease in d_0) decreases both the real interest rate and the inflation rate.
3) There is a threshold d_0^{thres}

$$d_0^{thres} \equiv \frac{1 + \frac{1-\bar{d}^*}{\beta}\left(\left(\frac{\beta}{1+\bar{d}^*} + 1\right)\frac{\bar{d}^*}{1+\bar{d}^*} - 1\right)}{1 + \frac{1-\bar{d}^*}{\beta}\left(\frac{\beta}{1+\bar{d}^*} + 1\right)\frac{1}{1+\bar{d}^*}}$$

such that the ZLB binds in period 0 if $d_0 < d_0^{thres}$. In this case $m_0^P > 0$.

The first part of the Proposition states the the interesting dynamic occurs between period 0 and period 1. When $m_t^{CB} = 0$, it is easy to see that $c_t^p = c^{p*}$ and $c_t^n = c^{n*}$, when $t \geq 2$ (the economy is back at its steady-state equilibrium in period 2).

The second part of the Proposition states that the more severe the credit constraint in period 0 (in other words, the smaller d_0) the lower the real interest rate and the inflation rate between periods 0 and period 1. Indeed, when d_0 decreases, there are less assets for $P-$households to save. As a consequence, the price of the assets increases and the real interest rate decreases. This result is also found in Guerrieri and Lorenzoni (2011). The new part concerns the inflation rate. When d_0 decreases, $N-$households borrow less in period 0. This implies that their income is higher in period 1. Their period 1 saving in money increases, which contributes to decrease the period 1 price level and thus the inflation rate between period 0 and period 1. As a consequence a deleveraging shock is likely to push the economy to the ZLB, as both the real interest rate and the inflation rate decreases.

The third part of the Proposition presents the threshold below which the economy hits the ZLB in period 0. When the ZLB binds in period 0, we have $q_0 = 1 + \pi_1$ (in other words, the nominal interest rate is 0). Moreover, only in period 0 do P-households hold money: $m_0^P > 0$, whereas $m_t^P = 0$, for $t = 1$... The values of all variables as a function of d_0 are provided in the proof of the Proposition in the Appendix.

When the ZLB does not bind, in period 0, the previous Section has proven that optimal monetary policy can implement the first best allocation. The interesting question is now the ability of monetary policy to implement the first-best allocation, with a binding ZLB.

5.1. **Lump-sum money transfers.** The next Proposition first investigates the case of lump-sum money creation.

Proposition 3. *Assume $d_0 < d_0^{Threshold}$. If $\theta = 1$ (lump-sum money creation):*
1) Period 0 lump-sum money creation $m_0^{CB} > 0$ changes consumption levels
2) The first-best allocation cannot be implemented.

The formal proof is in the Appendix, but the intuition can be provided. When money is created by lump-sum transfers, the Proposition first shows that money creation is not neutral even at the ZLB. The reason is that N-households who don't work consume any additional transfer, because the credit constraint is binding even at the ZLB. The interesting result is the second part of the Proposition: Although money is not neutral, the first-best allocation cannot be implemented if money is created by lump-sum transfers. To understand why, first note from Proposition 1 that the optimal lump-sum money transfer ($\theta = 1$) is 0, after period 2: $m_t^{CB} = 0$, for $t \geq 2$. As a consequence, the instruments to improve the allocation are only m_0^{CB} and m_1^{CB}. These two instruments must be set to obtain 3 objectives. First, a period 0 transfer between N and P-households to compensate for the decrease in the period 0 ability to borrow. Second, a period 1 transfer between the same households to change the allocation (as N-households enter period 1 wilth less debt). In addition, if the ZLB binds, to obtain $c_0^p = c_1^p = c^{p*}$ we must have $q_0 = 1 + \pi_1 = \beta$, from the

Euler equation of $P-$households (20). The two instruments are not enough to match these three objectives. As a final step of the proof, one can show that when $d_0 < d_0^{thres}$ the quantity of money to get out of the ZLB can't implement the first best. The conclusion is thus that the first best can't be implemented by lump-sum money transfers. The next Section shows that this result crucially depends on the process of money creation.

5.2. **Open market operations.**

Proposition 4. *Assume $d_0 < d_0^{Threshold}$. If $\theta = 0$ (open–market operations), the first-best allocation can be implemented.*
The process of money creation is

$$m_0^{CB} = \left(\frac{1}{\beta} - 1\right)(d^* - 1), \, m_1^{CB} = -\frac{m_0^{CB}}{3\beta}$$
$$\text{and } m_t^{CB} = \frac{2m_{t-1}^{CB}}{3\beta}, t \geq 2 \tag{30}$$

Open-market operations can implement the first-best allocation. What is the difference between open-market operations and lump-sum money transfers? When the monetary authorities create money in period t by open-market operations, they generate some distributional effect in period t, because they create some money to buy bonds, but they also create some redistribution in period $t+1$ because they redistribute the profits generated by the holdings of bonds. As a consequence, if the monetary authorities create some money in period 0 and period 1 to implement the first best, they have to perform some operations to undo redistributive effects in period 2, which will create some redistributive effects in period 3, which have to be undone, and so on. As a consequence, the central bank instrument is now the whole sequence of money creation, $m_t^{CB} \neq 0$, for $t \geq 0$, as can be seen from equation (30). With this whole sequence, optimal money creation can now implement $c_t^p = c^{p*}$ and $c_t^n = c^{n*}$, when $t \geq 2$ and $q_0 = 1 + \pi_1 = \beta$.

In other words, open-market operations generate a more complex redistributive effect than lump-sum money creation, which can be used to implement the first-best allocation. To my knowledge, this paper is the first to show such a sharp difference between the two processes of money creation. More generally, one should thus be careful about the process of money creation in the growing literature about the redistributive effect of monetary policy, for both positive and normative analysis when the ZLB is considered.

5.3. **Tightening of the credit constraint for many periods.** To further investigate the properties of the model, this section provides a numerical example to exhibit optimal monetary policy when the economy experiences a long-lasting period of tightening of the credit constraint. It is now assumed that the credit constraint unexpectedly decreases in period 0 to $d_t = 0$ for $t = 0..T$ and then goes back to $d_t = d^*$ for $t \geq T + 1$. The whole sequence of credit constraints is known by all agents at period 0.

In the simple numerical example, it is assumed that $\beta = 0.99$, and that the credit limit is initially at $d^* = 1\%$ and decreases 0% for 20 periods. At period 21 the credit constraint moves back to $d_t = d^*$. Before providing the optimal monetary policy, Figure 1, Part A first plots the outcome of the market economy when there is no money creation $m_t^{CB} = 0$.

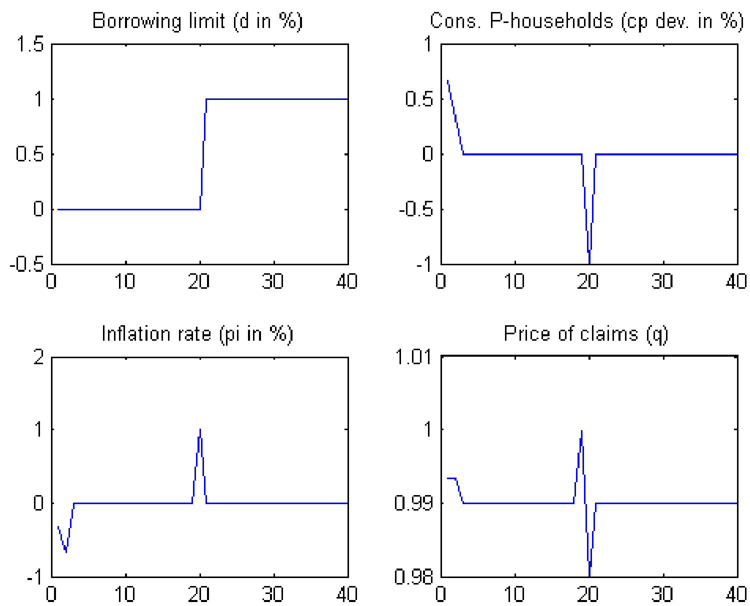
The first panel plots the path of the borrowing limit d_t. It goes back to its steady state value at period 21. The second panel of the first line plots the consumption of $P-$households, in percentage deviation from steady state. The first-best allocation is thus a 0 for this variable ($c_t^p - c^{p*} = 0\%$). The tightening of the borrowing limit generates an increase in the consumption of P-households, as $N-$households cannot borrow to consume (the consumption of $N-$households is simply $c_t^n = 2 - c_t^p$). After the initial increase, the effect of the tightening of the borrowing limit is modest, because N households increase their money demand to undo the expected credit limit. When the credit limit goes back to its steady-state value, the effect is an increase in the consumption of the N-households, who can borrow more, and thus a decrease in the consumption of P-households. The third panel of Part A plots the inflation rate. The tightening of the borrowing limit generates an increase of the money demand to undo the credit constraint. As a consequence, the price level falls for the real quantity of money to increase. When the borrowing limit goes back to its steady-state value, the demand for money decreases and the price level goes back to its steady-state.

The fourth panel plots the price q_t of the claim on one unit of goods. This price $q_t = \beta c_t^p / c_{t+1}^p$ follows the expected inverse of the ratio of the consumption of P-households (due to the Euler equation of $P-$households). One can observe that the ZLB is binding in period 0. Indeed, $1 + \pi_1 = q_0 = 0.9933$ and $m_0^p = 0.0033 > 0$. The price of claims follows the consumption growth of $P-$households.

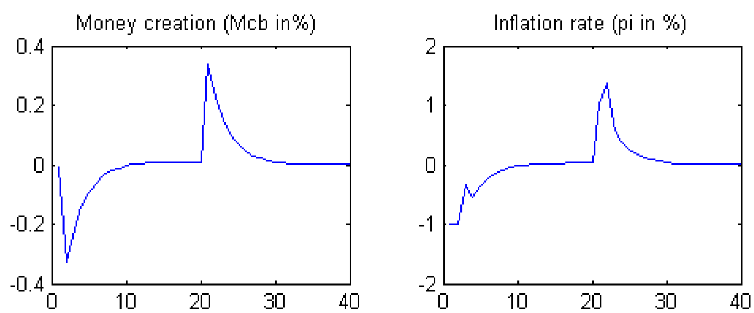
Part B of Figure 1 plots the outcome for optimal money creation, when money is created by open-market operations. The description of the procedure to find the optimal money creation is described in Appendix D. In this case, the first-best allocation can be implemented and $c_t^p = c^{p*}$, $c_t^n = c^{n*}$ and $q_t = \beta$ or $t \geq 0$. As a consequence, only the path of money creation and of the inflation rate are represented. One first observes that the ZLB is binding in period 0. Optimal monetary policy does not avoid the ZLB. An initial decrease in the money supply ensures that $q_0 = 1 + \pi_1 = \beta$ and that initial wealth is transferred to $N-$households to undo the intial tightening of the credit constraint. As money is created by open-market operations, the contraction of the money supply is progressively undone, as was discussed after Proposition 4. When the credit constraint goes back

FIGURE 1. Model outcome

Part A. No Money Creation



Part B. Optimal Money Creation



to its intial value d^*, money is created to generate a transfer toward $P-$households (as can be seen from the path of $P-$households, when no money is created). Again, a decrease in the quantity of new money is necessary to undo the initial money creation.

Although the change in the level of the credit contraint pushes the economy to the ZLB, optimal monetary policy doesn't avoid the ZLB nor deflation.

5.4. **Production economy.** Although the results have been derived in an endowment economy (to study the simplest model), all the results directly apply to a production economy. Indeed, as open-market operations fully stabilize the economy (and the real interest rate), the marginal product of capital, and hence the capital stock, can be stabilized at their first-best value. To consider this more formally, the full model of the production economy is provided in Appendix E.

6. Concluding remarks

This paper provides a simple model where agents hold heterogeneous money holdings due to explicit financial frictions. Optimal monetary policy depends on the redistributive effects of monetary policy and on the ability of agents to smooth consumption. At the ZLB, money injections have real effects, but the surprising result is that the set of feasible allocations depends on the process of money creation: Some desirable allocations can be achieved only by open-market operations and not by lump-sum money transfers. Open-market operations indeed transfer money to the group of agents participating in financial markets. In addition, they have an intertemporal effect on the State budget because this affects future taxes. For instance, optimal policy induces a long-lasting process of money injections, even after a short-lived tightening of the credit constraint.

This framework has been studied with flexible prices to provide analytical proof, in line with many contributions considering financial frictions (Buera and Nicolini (2014); Bachetta et al. (2015) or Azariadis, Bullard, Singh, and Suda (2015) among others). The consideration of sticky prices is a natural extension and would need a more quantitative analysis. Obviously, the result about the difference between open-market operations and lump-sum money creation would not be so clear-cut. Nevertheless, the additional gain of open-market operations compared to lump-sum money transfers (which is to transfer money only to participating households) would remain. This additional gain could emerge as a lower quantity of created money to reach a similar allocation. This is left for future research.

APPENDIX A. PROOF OF PROPOSITION 2

A.1. **Allocation when the ZLB does not bind and existence conditions.** We first define the allocation and the threshold for an equilibrium to exist when the ZLB does not bind. It is thus assumed that $m_t^{CB} = \tau_t = m_t^P = 0$ for $t \geq 0$.

We solve for the allocation backward, from period 2 to period 1 and finally, to period 0.

For $t \geq 2$ (as $d_t = d^*$ for t$\geq$ 1), equations (19) and (20) are

$$q_t d^* + c_t^p = 1 + d^*$$
$$q_t u'\left(c_t^p\right) = \beta u'\left(c_{t+1}^p\right)$$

The only equilibrium sequence in this case is, for $t \geq 2$

$$c_t^p = c^{p*}$$
$$q_t = \beta$$

The budget constraint (19) and the Euler equation (20) in period 1 are (using $u\left(c\right) = \log c$)

$$c_1^p = 1 - \left(q_1 \bar{d}^* - \bar{d}_0\right)$$
$$q_1 c^{p*} = \beta c_1^p$$

Solving for q_1, one finds

$$q_1 = \beta \frac{1 + \bar{d}_0}{1 + \bar{d}^*}$$

The budget constraint (19) and the Euler equation (20) in period 0 are

$$c_0^p = 1 - \left(q_0 \bar{d}_0 - \bar{d}^*\right)$$
$$q_1 c_1^p = \beta c_0^p$$

Solving for q_0, using the expression of q_1 in the value of c_1^p :

$$q_0 = \frac{\beta}{1 + \left(\frac{\beta}{1+\bar{d}^*} + 1\right) \frac{d_0 - \bar{d}^*}{1+\bar{d}^*}}$$

Under these assumptions, equations (17) and (22) are $m_t^n = 1 - d_{t-1}$ and $m_t^n = m_{t-1}^n / (1+\pi_t)$ for $t \geq 0$, from which one easily deduces the inflation rate

$$\pi_0 = 0$$
$$\pi_1 = \frac{\bar{d}_0 - \bar{d}^*}{1 - \bar{d}_0}$$
$$\pi_2 = \frac{\bar{d}^* - \bar{d}_0}{1 - \bar{d}^*}$$
$$\pi_t = 0, t \geq 3$$

Note first that the ZLB cannot bind in period 1. Indeed, we can check

$$q_1 = \beta \frac{1 + \bar{d}_0}{1 + \bar{d}^*} \leq 1 \leq 1 + \pi_2 = \frac{1 - \bar{d}_0}{1 - \bar{d}^*}$$

The question is to check if the ZLB binds in period 0.

Using the previous values, one can check that $1 + \pi_1 > q_0 \Leftrightarrow d_0 \geq d_0^{thres}$ where

$$d_0^{thres} = \frac{1 + \frac{1-\bar{d}^*}{\beta}\left(\left(\frac{\beta}{1+\bar{d}^*} + 1\right)\frac{\bar{d}^*}{1+\bar{d}^*} - 1\right)}{1 + \frac{1-\bar{d}^*}{\beta}\left(\frac{\beta}{1+\bar{d}^*} + 1\right)\frac{1}{1+\bar{d}^*}}$$

As a consequence, the ZLB binds necessarily in period 0 when $d_0 < d_0^{thres}$.

A.2. **Allocation when** $d_0 < d_0^{thres}$**.** We now compute the equilibrium allocation when the ZLB binds in period 0 to show that the equilibrium exists and that in particular $m_0^p > 0$. Binding ZLB in period 0 implies $q_0 = 1 + \pi_1$. We show that q_0, q_1 and m_0^p solve a system of three equations.

As before the economy is in steady state after period 2. The budget constraint (19) and the Euler equation (20) in period 1 are now

$$c_1^p = 1 + d_0 - q_1 d^* + \frac{m_0^p}{1 + \pi_1}$$
$$q_1 c^{p*} = \beta c_1^p$$

Substituting c_1^p with the two previous equations and using $q_0 = 1 + \pi_1$, one finds a first equation

$$q_1\left(\bar{d}_1 + \frac{c^{p*}}{\beta}\right) = 1 + \bar{d}_0 + \frac{m_0^p}{q_0} \tag{31}$$

Using the period 0 and period 1 money market equilibrium (22), one finds

$$q_0 = 1 + \pi_1 = \frac{m_0^n + m_0^p}{m_1^n} = \frac{m^{n*}}{1 + \pi_0}\frac{1}{1 - \bar{d}_0}$$

Using this expression in the budget constraint of consuming $N-$households (**??**) in period 0, one finds

$$c_0^n = q_0 \text{ and } c_0^p = 2 - q_0$$

Using the budget constraint of $P-$households in period 0 (19), one finds

$$q_0 \left(\bar{d}_0 - 1\right) + m_0^p + 2 = 1 + \bar{d}^* \tag{32}$$

Finally, the Euler equation of $P-$households in period 0 and in period 1 gives

$$c_0^p = \frac{q_0 q_1}{\beta^2} = 2 - q_0 \tag{33}$$

We can solve for q_1 using the three equations (31)-(33)

$$q_1 = \frac{\frac{\bar{d}^*-1}{2} + 2}{\bar{d}_1 + \frac{c^{p*}}{\beta}\left(1 - \frac{\bar{d}^*-1}{2\beta}\right)} = \frac{\frac{\bar{d}^*-1}{2} + 2}{\bar{d}^* + \frac{c^{p*}}{\beta}\left(1 - \frac{\bar{d}^*-1}{2\beta}\right)}$$

Then we have

$$q_0 = \frac{2}{\frac{q_1}{\beta^2} c^{p*} + 1}$$

and

$$q_0 \left[q_1 \left(\bar{d}^* + \frac{c^{p*}}{\beta} \right) - \left(1 + \bar{d}_0\right) \right] = m_0^p$$

One can check that $m_0^p \left(d_0^{thres}\right) = 0$ and $m_0^P \left(d_0\right)$ is decreasing, which concludes the proof.

Appendix B. Proof of Proposition 3

B.1. **Proof of 1).** It is assumed that $\theta = 1$ (lump-sum money creation). We show that when the ZLB binds in period 0 a small money injection has to change the consumption level. We investigate the effect of a small money injection in period 0 $m_0^{CB} > 0$ and $m_t^{CB} = 0$ for $t \geq 1$.

First, the variables in the allocation with $m_0^{CB} = 0$ are denoted with a hat. For instance, period 0 (respec. period 1) consumption is $\hat{c}_0^n$ (respec. $\hat{c}_1^n$). The variables in the allocation with a positive money creation $m_0^{CB} > 0$ are noted without a hat. It is shown that we cannot have $\hat{c}_t^n = c_t^n$ and $\hat{c}_t^p = c_t^p$ for $t \geq 0$.

The proof is made by contradiction. Assume that $\hat{c}_t^n = c_t^n$ and $\hat{c}_t^p = c_t^p$ for $t \geq 0$. As

$$\hat{q}_0 \hat{c}_1^p = \beta \hat{c}_0^p$$
$$\hat{q}_1 \hat{c}_2^p = \beta \hat{c}_1^p$$
$$1 + \hat{\pi}_1 = \hat{q}_0$$

As a consequence, we must have

$$q_0 = \hat{q}_0 = 1 + \pi_1$$
$$q_1 = \hat{q}_1$$
$$\pi_1 = \hat{\pi}_1$$

But as

$$c_1^n = \frac{m_0^n}{1+\pi_1} + q_1 \bar{d}_1 = \frac{m_0^n}{1+\hat{\pi}_1} + \hat{q}_1 \bar{d}_1 = \hat{c}_1^n$$

we must have

$$m_0^n = \hat{m}_0^n$$

As $m_0^n = 1 - \bar{d}^* + m_0^{CB}$ and $\hat{m}_0^n = 1 - \bar{d}^*$, this implies $m_0^{CB} = 0$. This is a contradiction.

B.2. **Proof of 2).**

B.2.1. *Step 1.* It is assumed that $\theta = 1$. The proof is made in two steps. First I show that when $d_0 < d_0^{threshold}$, if monetary policy can implement the first best, then the ZLB must bind in period 0. The proof is made by contradiction.

Assume that monetary policy implements the first best, and that the ZLB doesn't bind in period 0. Then the result of Proposition 1 implies that we must have

$$m_0^{CB} = \beta \left(\bar{d}_0 - \bar{d}^*\right)$$

$$m_1^{CB} = \left(\bar{d}_0 - \bar{d}^*\right)$$

With these expressions, the money demand of $N-$households in period 0 and period 1 (17) with $(\tau_t = m_t^{CB})$ are

$$m_1^n = 1 - \bar{d}_0 \text{ and } m_0^n = 1 - \bar{d}^* + \beta \left(\bar{d}_0 - \bar{d}^*\right)$$

Using the period 1 money market equilibrium (22), one finds the inflation rate

$$1 + \pi_1 = \frac{1 - \bar{d}^* + \beta \left(\bar{d}_0 - \bar{d}^*\right)}{1 - \bar{d}^* - 3 \left(\bar{d}_0 - \bar{d}^*\right)}$$

The condition $1 + \pi_1 > \beta$ is equivalent to

$$d_0 > d^* - \frac{(1-\beta)\left(1 - \bar{d}^*\right)}{4\beta}$$

As $\beta > 2/3$, one can check that $d_0^{threshold} < d^* - \frac{(1-\beta)\left(1-\bar{d}^*\right)}{4\beta}$. As a consequence, the previous inequality is a contradiction as we assumed $d_0 < d_0^{threshold}$.

B.2.2. *Step 2.* We now show that the first-best allocation cannot be implemented when $\theta = 1$ and the ZLB binds in period 0.

Again, the proof is made by contradiction. Assume that the first best is implemented. In this case, the period 0 and period 1 Euler equations of $P-$households (20) imply

$$q_0 = q_1 = \beta$$

As the ZLB is binding in period 0, $1 + \pi_1 = q_0 = \beta$. Using the period 0 and period 1 budget constraint (19), one finds

$$m_0^p = \beta \left(\bar{d}^* - \bar{d}_0 \right) + m_0^{CB}$$
$$m_0^p + \beta m_1^{CB} = \beta \left(\bar{d}^* - \bar{d}_0 \right)$$

This implies

$$m_0^{CB} + \beta m_1^{CB} = 0 \tag{34}$$

Using the period 1 money market (22) together with the money demand (17), one finds that $q_0 = 1 + \pi_1$ implies

$$m_0^p = \beta \left(1 - \bar{d}_0 - 2m_1^{CB} \right) - \left(1 - \bar{d}^* \right) - m_0^{CB}$$

Using this expression in the period 1 budget constraint of the P-households (19), one finds

$$m_0^{CB} + \beta m_1^{CB} = (1 - \beta) \left(\bar{d}^* - 1 \right)$$

We cannot have at the same time the previsous equality and equality (34), which proves the contradiction.

Appendix C. Proof of Proposition 4

It is assumed that $\theta = 0$. It is proven that the money creation provided in the text implements the first best, that the ZLB binds at period 0, and that all equilibrium conditions are fulfilled.

First, using the money creation process given in the text in the budget constraint of $P-$households, one can check that $c_t^p = c^{p*}$and thus $c_t^n = c^{n*}$ because of the goods market equilibrium. As a consequence, the Euler equation (20) implies $q_0 = q_1 = \beta$.

Using the money demand (17), and the budget constraint (19) in the period 1 money market equilibrium (22), one finds that $1 + \pi_1 = \beta$.

Using the period 2 money market, one finds that

$$\pi_2 = \frac{\bar{d}^* - \bar{d}_0 + \Delta_0^{CB} + \Delta_1^{CB}}{1 - \bar{d}^* - \Delta_1^{CB}}$$

One can check the two conditions

$$1 > \beta^2 \frac{1}{1+\pi_1} \frac{1}{1+\pi_2} \text{ and } q_0 > \beta^2 \frac{1}{1+\pi_2}$$

Finally, one can check that $m_0^p > 0$ (because $\beta > 2/3$). As a consequence, the first-best allocation is an equilibrium.

Appendix D. Optimal money creation for long-lasting tightening of the credit constraints

The optimal monetary policy is derived in three steps. First, we derive the optimal monetary policy assuming the ZLB binds in period 1, and we show that existence conditions are satisfied. Second, we show that the optimal allocation cannot be implemented if the ZLB does not bind.

Step 1: Assuming that $c_t^n = c^{n*}$ and that $c_t^p = c^{p*}$ and that $1 + \pi_1 = \beta$, the system (17) - (23) gives the following process for money creation.

$$\begin{aligned}
m_0^{CB} &= \beta c^{n*} - (1 - d^*) \\
m_0^p &= c^{n*} + 3m_0^{CB} + d^* - 1 \\
m_1^{CB} &= \frac{c^{p*}}{3} - \frac{1}{3}\left(1 - 2\frac{m_0^{CB}}{\beta} + \frac{m_0^p}{\beta}\right) \\
m_t^{CB} &= \frac{(d^* - d_{t-1}) - \beta\,(d^* - d_t)}{3} + \frac{2}{3}\frac{m_{t-1}^{CB}}{\beta} \text{ for } t \geq 2
\end{aligned}$$

The inflation rate is $1 + \pi_t = \frac{m_{t-1}^n}{m_t^n - 3m_t^{CB}}, t \geq 2$. One can check numerically that existence conditions (4) and (3) are fulfilled and that $1 + \pi_t > \beta$ for $t \neq 1$.

Step 2: Assuming the ZLB does not bind at any period, one can use the results of Proposition (1) to determine the optimal money supply. Simulating the inflation rate with this path of the money supply, one finds that $1 + \pi_1 < \beta$, which proves that the equilibrium does not exist.

Appendix E. Production economy

We now present the model with a production sector. Denote, as before q_{t-1} as the price in period $t-1$ of a unit of goods in period t (hence the real interest rate is $1 + r_{t-1} = 1/q_{t-1}$). K_t is the capital invested in period $t-1$ to produce in period t. The profit maximization of the representative firm in real terms is

$$\max_{K_t, L_t^d} K_t^\alpha \left(L_t^d\right)^{1-\alpha} - w_t L_t^d - K_t/q_{t-1} + (1-\delta)\, K_t$$

gives

$$\frac{1}{q_{t-1}} = \alpha \left(\frac{K_t}{L_t^d}\right)^{\alpha-1} + 1 - \delta$$
$$w_t = (1-\alpha)\left(\frac{K_t}{L_t^d}\right)^{\alpha}$$

The rest of the model is exactly as before: The households and the State are modeled as in Section 3.1. There is no risk on any asset. There is one single financial market for the claims issued by households and the representative firm. The financial market equilibrium is (claims are substitutable)

$$b_t^h = d_t + 3K_{t+1}$$

The labor market equilibrium is $L_t^d = 2/3$. The summary of the full model is now

$$m_t^n = w_t + \tau_t - d_{t-1} \tag{35}$$
$$c_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \tau_t + q_t d_t$$
$$q_t \left(d_t + 3K_{t+1} - 3(1-\theta)\frac{m_t^{CB}}{q_t}\right) + m_t^p + c_t^p = w_t + \tau_t + \left(d_{t-1} + 3K_t - 3\frac{(1-\theta)\, m_{t-1}^{CB}}{q_{t-1}}\right) + \frac{m_{t-1}^p}{1+\pi_t}$$
$$q_t u'\left(c_t^p\right) = \beta u'\left(c_{t+1}^p\right)$$
$$m_t^p = 0 \text{ if } 1+\pi_{t+1} > q_t$$
$$w_t = (1-\alpha)\left(3K_t/2\right)^{\alpha}$$
$$\frac{1}{q_{t-1}} = \alpha\left(3K_t/2\right)^{\alpha-1} + 1 - \delta$$
$$m_t^n + m_t^p = \frac{m_{t-1}^n + m_{t-1}^p}{1+\pi_t} + 3m_t^{CB}$$
$$\tau_t = \theta m_t^{CB} + (1-\theta)\frac{m_{t-1}^{CB}}{q_{t-1}}$$

Together with the inequalities

$$c_t^n, c_t^p, m_t^n, m_t^p \geq 0$$
$$1+\pi_{t+1} \geq q_t$$

and the condition

$$q_t u'\left(c_t^n\right) > \beta^2 \frac{1}{1+\pi_{t+2}} u'\left(c_{t+2}^n\right) \tag{36}$$

Given a deterministic sequence of credit constraint $\{d_t\}_{t\geq 0}$, a monetary policy $\{m_t^{CB}\}_{t\geq 0}$, and an initial capital stock K_0 an equilibrium of this economy is a sequence $\{c_t^n, c_t^p, m_t^n, m_t^p, \tau_t, q_t, \pi_t, K_t\}_{t\geq 0}$ which satisfies the equations (35)-(36).

The steady state is defined as before as an economy where $\pi = \tau = m^{CB} = 0$, where real variables are constant and where $d_t = d^*$. The steady state is defined by the following values (denoted with a star)

$$q^* = \beta, w^* = 1, K^* = \frac{2}{3}\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha - 1}}, w^* = (1-\alpha)\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}},$$

$$c^{n*} = w^* - (1-\beta)\, d^*, c^{p*} = 3\,(K^*)^{\alpha}\left(\frac{2}{3}\right)^{1-\alpha} - 3\delta K^* - c^{n*}$$

Following, the same steps as in (3.5), one finds that the steady state is optimal when $\omega^p = u'(c^{n*})\,/u'(c^{p*})$. If it is assumed that the initial capital stock is $K_0 = K^*$, the optimal allocation is $c_t^p = c^{p*}$, $c_t^n = c^{n*}$ and $K_t = K^*$.

The proof of Propositions 1 to 4 can easily be adapted to this environment. Indeed, the Euler equations are the same for both agents. The only difference between the production economy and the endowment economy is that the real wage is different (but it is a normalization) and participating agents now have an additional income $3K_t$ in each period. But this extra income does not affect the structure of the proof.

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