Technical appendix for "Asset returns, idiosyncratic and aggregate risk exposure"

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December 18, 2014

This technical appendix completes the paper "Asset returns, idiosyncratic and aggregate risk exposure". It offers a step-by-step derivation of the proof of Proposition 4 (*Small positive volumes*) in Section 4.

All equations numbers refer to equations in the main text.

Proof of Proposition 4

Equity premium and bond holdings. In this setup, wages ω^i and δ^i are constant, while securities are in (small) positive supplies. Because the dividend process is IID, the stock price is constant and Because the dividend process is IID, the stock and bond prices are constant, as well as the bond holdings of both types: $b_t^1 = b^1$ and $b_t^i = b^i$. Since the type-2 agents cannot be excluded from bond markets due to condition (18), we deduce the Euler equations for both securities:

$$P^{PV} = \beta E^{\tilde{z}}\left[\left(\alpha^1 + (1-\alpha^1)\frac{u'(\delta^1 + (P_{t+1} + y_{t+1})\frac{V_X}{\eta^1} + b^1)}{\lambda^1}\right)(P^{PV} + y(\tilde{z}))\right],$$
$$Q^{PV} = \beta(\alpha^2 + (1-\alpha^2)\frac{u'(\delta^2 + b^2)}{\lambda^2}).$$

We solve for the price expressions in the neighborhood of zero volume (so as to obtain closed-form expressions). We assume that $0 < V_X \ll 1$ and $0 < V_B \ll 1$. Since bonds cannot be short-sold, we also have $0 \leq b^1, b^2 \ll 1$. We begin with the stock price:[1]

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[1] The approximation sign $\approx$ refers to a first order development with respect to security volumes.

$$
\begin{aligned}
P^{PV} \approx \beta E^{\tilde z}&\left[\left(\alpha^1 + (1-\alpha^1)\frac{1}{\lambda^1}u'(\delta^1)\right)(P^{PV} + y(\tilde z))\right] \\
&+ (1-\alpha^1)u''(\delta^1)\frac{1}{\lambda^1}\frac{V_X}{\eta^1}\beta E^{\tilde z}\left[(P^{PV} + y(\tilde z))^2\right] \\
&+ (1-\alpha^1)\frac{1}{\lambda^1}u''(\delta^1)b^1\beta E^{\tilde z}\left[P^{PV} + y(\tilde z)\right].
\end{aligned}
$$

Using a guess-and-verify strategy, we assume that there exists two real values π_x and π_b such that $P^{PV} \approx P^{ZV} + \pi_x \frac{V_X}{\eta^1} + \pi_b b^1$, where P^{ZV} defined in equation (47) is the stock price in zero volume and with constant wages:

$$
\begin{aligned}
P^{ZV} + \pi_x\frac{V_X}{\eta^1} + \pi_b b^1 = \beta&\left(\alpha^1 + (1-\alpha^1)\frac{1}{\lambda^1}u'(\delta^1)\right)E^{\tilde z}\left[(y(\tilde z) + P^{ZV} + \pi_x\frac{V_X}{\eta^1} + \pi_b b^1)\right] \\
&+ (1-\alpha^1)\frac{1}{\lambda^1}u''(\delta^1)\frac{V_X}{\eta^1}\beta E^{\tilde z}\left[(P^{ZV} + y(\tilde z))^2\right] \\
&+ (1-\alpha^1)\frac{1}{\lambda^1}u''(\delta^1)b^1\beta E^{\tilde z}\left[P^{ZV} + y(\tilde z)\right]
\end{aligned}
$$

or:

$$
\begin{aligned}
\pi_x\frac{V_X}{\eta^1} + \pi_b b^1 = \beta&\left(\alpha^1 + (1-\alpha^1)\frac{1}{\lambda^1}u'(\delta^1)\right)(\pi_x\frac{V_X}{\eta^1} + \pi_b b^1) \\
&+ (1-\alpha^1)\frac{1}{\lambda^1}u''(\delta^1)\frac{V_X}{\eta^1}\beta E^{\tilde z}\left[(P^{ZV} + y(\tilde z))^2\right] \\
&+ (1-\alpha^1)\frac{1}{\lambda^1}u''(\delta^1)b^1\beta E^{\tilde z}\left[P^{ZV} + y(\tilde z)\right],
\end{aligned}
$$

which implies:

$$
\pi_x(1-\beta\kappa^1) = \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}E^{\tilde z}\left[(P^{ZV} + y(\tilde z))^2\right] \tag{a}
$$

$$
\pi_b(1-\beta\kappa^1) = \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}E^{\tilde z}\left[P^{ZV} + y(\tilde z)\right] \tag{b}
$$

$$
\text{with: } \kappa^i = \alpha^i + (1-\alpha^i)\frac{1}{\lambda^i}u'(\delta^i),\ i = 1, 2. \tag{c}
$$

For the bond price, we obtain:

$$
Q^{PV} \approx Q^{ZV} + \beta(1-\alpha^2)\frac{u''(\delta^2)}{\lambda^2}b^2 \tag{d}
$$

The Euler equation for bonds of type-1 agents can be expressed as follows:

$$
\begin{aligned}
Q^{PV} &\geq \beta(\alpha^1 + (1-\alpha^1)\frac{1}{\lambda^1}E^{\tilde z}[u'(\delta^1 + \frac{V_X}{\eta^1}(P^{ZV} + y(\tilde z)) + b^1)]) \\
&\gtrsim \beta\kappa^1 + \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}b^1 + \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}\frac{V_X}{\eta^1}E^{\tilde z}[P^{ZV} + y(\tilde z)].
\end{aligned} \tag{e}
$$

If type-1 agents do not participate to the bond market, the previous inequality is strict and we have $b^1 = 0$ and $b^2 = \frac{V_B}{\eta^2}$. If type-1 agents trade bonds, the previous inequality is an equality and noticing that $b^1 = \frac{V_B}{\eta^1} - \frac{\eta^2}{\eta^1}b^2$, we deduce using (d) and (e) the bond expressions (24) and (25). Because of condition (18), type-2 agents cannot be credit-constrained. Otherwise, we would have $(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}(\frac{V_B}{\eta^1} + \frac{V_X}{\eta^1}E^{\tilde{z}}[P^{ZV} + y(\tilde{z})]) > \kappa^2 - \kappa^1 > 0$, which contradicts positive volumes.

We therefore deduce the following expressions for bond and stock returns:

$$R_f \approx \frac{1}{\kappa^2}(1 - \beta(1-\alpha^2)\frac{u''(\delta^2)}{\lambda^2}b^2) \tag{f}$$

$$\begin{aligned} R_s &\approx \frac{E^{\tilde{z}}[y(\tilde{z})] + P^{ZV} + \pi_V V_X + \pi_b b^1}{P^{ZV} + \pi_V V_X + \pi_b b^1} \\ &\approx 1 + \frac{E^{\tilde{z}}[y(\tilde{z})]}{P^{ZV}}(1 - \frac{\pi_V}{P^{ZV}}V_X + \frac{\pi_b}{P^{ZV}}b^1) \end{aligned}$$

Since we have $P^{ZV} = \frac{\kappa_1}{1-\kappa_1}E[y(\tilde{z})]$, we deduce that stock returns can be expressed as follows:

$$R_s \approx \frac{1}{\kappa^1}\left(1 - \beta(1-\alpha^1)\frac{u''(\delta^1)}{\lambda^1}\left(\frac{E^{\tilde{z}}\left[(P^{ZV} + y(\tilde{z}))^2\right]}{P^{ZV}}V_X + \frac{E^{\tilde{z}}\left[P^{ZV} + y(\tilde{z})\right]}{P^{ZV}}b^1\right)\right) \tag{g}$$

From (f) and (g), we have the expression (22) of the equity premium.

Average consumptions. Since idiosyncratic and aggregate shocks are independent, $\bar{c}_i^{PV} = E[\tilde{c}_i^{PV}] = E^{\xi}[E^{\tilde{z}}[\tilde{c}_i^{PV}]]$, where $E[\cdot]$ is the total expectation, $E^{\xi}[\cdot]$ the expectation with respect to idiosyncratic risk and $E^{\xi}[\cdot]$ the expectation with respect to idiosyncratic risk. Let us start with computing the different realizations of $E^{\tilde{z}}[\tilde{c}_i^{PV}]$. Agents of type 1 consume:

- $\omega^1 - \omega^1\tau + E[y(\tilde{z})]\frac{V_X}{\eta^1} + (1-Q^{PV})b^1$ with (unconditional) probability $\alpha^1\eta^1$ (i.e., *pp* agents);
- $\omega^1 - \omega^1\tau - P^{PV}\frac{V_X}{\eta^1} - Q^{PV}b^1$ with probability $(1-\alpha^1)\eta^1$ (i.e., *pu* agents);
- $\delta^1 + (P^{PV} + E[y(\tilde{z})])\frac{V_X}{\eta^1} + b^1$ with probability $(1-\rho^1)(1-\eta^1)$ (i.e., *up* agents);
- δ^1 with probability $\rho^1(1-\eta^1)$ (i.e., *uu* agents);

  Noting that $(1-\alpha^1)\eta^1 = (1-\rho^1)(1-\eta^1)$ and $\tau = \frac{1-Q^{ZV}}{\omega^1\eta^1+\omega^2\eta^2}V_B$, the expression (27) is then straightforward to derive. By the same token, we can easily obtain the expression (28) for type-2 agents.

Variance of consumption growth. For type 1 agents, we denote $\tilde{z}$ the current aggregate state and $\tilde{z}'$ the future one. At the first order in asset volumes, the consumption growth rates of type-1 agents are as follows (w.p. stands for *with probability*):

$$\begin{array}{rcl}
1+(y(\tilde{z}')-y(\tilde{z}))\dfrac{V_X}{\eta^1}\dfrac{1}{\omega^1} & \text{w.p.} & \dfrac{1-\rho^1}{2-\alpha^1-\rho^1}\alpha^1\alpha^1 \\
\dfrac{\delta^1}{\omega^1}\left(1+\tau+(P+y(\tilde{z}'))\dfrac{V_X}{\eta^1\delta^1}+\dfrac{b^1}{\delta^1}-y(\tilde{z})\dfrac{V_X}{\eta^1\omega^1}-(1-Q)\dfrac{b^1}{\omega^1}\right) & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\alpha^1 \\
1+(P+y(\tilde{z}'))\dfrac{V_X}{\eta^1}\dfrac{1}{\omega^1}+b^1\dfrac{1}{\omega^1} & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\alpha^1 \\
\dfrac{\delta^1}{\omega^1}\left(1+\tau+(P+y(\tilde{z}'))\dfrac{V_X}{\eta^1}\dfrac{1}{\delta^1}+b^1\dfrac{1}{\delta^1}+P\dfrac{V_X}{\eta^1}\dfrac{1}{\omega^1}+Qb^1\dfrac{1}{\omega^1}\right) & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}(1-\alpha^1) \\
1-(P+y(\tilde{z}))\dfrac{V}{\eta^1}\dfrac{1}{\delta^1}-b^1\dfrac{1}{\delta^1} & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\rho^1 \\
\dfrac{\omega^1}{\delta^1}\left(1-\tau-P\dfrac{V_X}{\eta^1}\dfrac{1}{\omega^1}-Qb^1\dfrac{1}{\omega^1}-(P+y(\tilde{z}))\dfrac{V_X}{\eta^1}\dfrac{1}{\delta^1}-b^1\dfrac{1}{\delta^1}\right) & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}(1-\rho^1) \\
1 & \text{"} & \dfrac{1-\alpha^1}{2-\alpha^1-\rho^1}\rho^1\rho^1 \\
\dfrac{\omega^1}{\delta^1}(1-\tau-P\dfrac{V_X}{\eta^1}\dfrac{1}{\omega^1}-Qb^1\dfrac{1}{\omega^1}) & \text{"} & \dfrac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\rho^1
\end{array}$$

We use the law of total variance to express the (total) variance $V[\tilde{\gamma}_c^{1,PV}]$ of consumption growth for type-1 agents: $V[\tilde{\gamma}_c^{1,PV}] = E^\xi V^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}] + V^\xi E^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]$, where the ξ (resp. $\tilde{z}$) exponent refers to idiosyncratic (resp. aggregate) moments.

We start with the expression of $E^\xi V^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]$:

$$\begin{aligned}
\frac{E^\xi V^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]}{\frac{V_X}{\eta^1}V^{\tilde{z}}[y(\tilde{z})]} &= \frac{1-\rho^1}{2-\alpha^1-\rho^1}\alpha^1\alpha^1 2(\frac{1}{\omega^1})^2 \\
&+ \frac{1-\rho^1}{2-\alpha^1-\rho^1}\alpha^1(1-\alpha^1)(\frac{\delta^1}{\omega^1})^2\left((\frac{1}{\delta^1})^2+(\frac{1}{\omega^1})^2\right) \\
&+ \frac{1-\alpha^1}{2-\alpha^1-\rho^1}(1-\rho^1)\alpha^1(\frac{1}{\omega^1})^2 \\
&+ \frac{1-\alpha^1}{2-\alpha^1-\rho^1}(1-\rho^1)(1-\alpha^1)(\frac{\delta^1}{\omega^1})^2(\frac{1}{\delta^1})^2 \\
&+ \frac{1-\rho^1}{2-\alpha^1-\rho^1}(1-\alpha^1)\rho^1(\frac{1}{\delta^1})^2 \\
&+ \frac{1-\rho^1}{2-\alpha^1-\rho^1}(1-\alpha^1)(1-\rho^1)(\frac{\omega^1}{\delta^1})^2(\frac{1}{\delta^1})^2
\end{aligned}$$

or after some manipulations:

$$\begin{aligned}
\frac{E^\xi V^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]}{\frac{V_X}{\eta^1}V^{\tilde{z}}[y(\tilde{z})]} = \nu_{13} &= \eta^1(\frac{1}{\omega^1})^2+\eta^1\alpha^1(\frac{1}{\omega^1})^2\left(\alpha^1+(1-\alpha^1)(\frac{\delta^1}{\omega^1})^2\right) \\
&+\eta^1(1-\alpha^1)(\frac{1}{\delta^1})^2(\rho^1+(1-\rho^1)(\frac{\omega^1}{\delta^1})^2)
\end{aligned} \tag{h}$$

Second, we express $E^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]$. We have

$$E^{\xi}E^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}] = \frac{1-\rho^1}{2-\alpha^1-\rho^1}\alpha^1\left(1+(1-\alpha^1)(P+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\omega^1}+(1-\alpha^1)\frac{b^1}{\omega^1}\right)$$
$$+\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\left((P+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\omega^1}+\frac{b^1}{\omega^1}\right)$$
$$+\frac{1-\alpha^1}{2-\alpha^1-\rho^1}\rho^1\left(1-(1-\rho^1)(P+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\delta^1}-(1-\rho^1)\frac{b^1}{\delta^1}\right)$$
$$+\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\frac{\omega^1}{\delta^1}\left(-(1-\rho^1)(P+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\delta^1}-(1-\rho^1)b^1\frac{1}{\delta^1}\right)$$
$$+\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\frac{\delta^1}{\omega^1}\left(1+\tau+P\frac{V}{\eta^1}\frac{1}{\omega^1}-\alpha^1(P+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\omega^1}-(\alpha^1-Q)b^1\frac{1}{\omega^1}\right)$$
$$+\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\frac{\omega^1}{\delta^1}\left(1-\tau-P\frac{V_X}{\eta^1}\frac{1}{\omega^1}-Qb^1\frac{1}{\omega^1}\right)$$

or after some algebra manipulation:

$$E^{\xi}E^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}] = E[\tilde{\gamma}_c^{1,ZV}]+\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}$$
$$\times\left(\left(1+\alpha^1(1-\frac{\delta^1}{\omega^1})-\frac{\omega^1}{\delta^1}(\rho^1+(1-\rho^1)\frac{\omega^1}{\delta^1})\right)\left((P+Ey(\tilde{z}'))\frac{V_X}{\eta^1}\frac{1}{\omega^1}+\frac{b^1}{\omega^1}\right)\right.$$
$$\left.+\left(\frac{\delta^1}{\omega^1}-\frac{\omega^1}{\delta^1}\right)\left(P\frac{V_X}{\eta^1}\frac{1}{\omega^1}+Q\frac{b^1}{\omega^1}+\tau\right)\right)$$

where: $E[\tilde{\gamma}_c^{i,ZV}] = 1+\frac{(1-\rho^i)(1-\alpha^i)}{2-\alpha^i-\rho^i}(\frac{\delta^i}{\omega^i}+\frac{\omega^i}{\delta^i}-2)$. (i)

We deduce that

$$(E^{\xi}E^{\tilde{z},\tilde{z}'}[\tilde{\gamma}_c^{1,PV}])^2 = E[\tilde{\gamma}_c^{1,ZV}]^2+2E[\tilde{\gamma}_c^{1,ZV}]\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}$$
$$\times\left(K_A^1(P^{ZV},P^{ZV}+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\omega^1}+K_A^1(Q^{ZV},1)\frac{b^1}{\omega^1}+\tau\left(\frac{\delta^1}{\omega^1}-\frac{\omega^1}{\delta^1}\right)\right), \quad \text{(j)}$$

where K_A^1 is a function of price p and payoff π:

$$K_A^1(p,\pi) = (\alpha^1-\rho^1\frac{\omega^1}{\delta^1})\pi+\frac{\delta^1}{\omega^1}\left(p+\pi(\frac{\omega^1}{\delta^1}-\alpha^1)\right)-\frac{\omega^1}{\delta^1}\left(p+\pi(1-\rho^1)\frac{\omega^1}{\delta^1}\right). \quad \text{(k)}$$

Moreover, we similarly deduce:

$$E^{\xi}[E^{\tilde{z}\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]^2] = E[(\tilde{\gamma}_c^{1,ZV})^2]+2\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}$$
$$\times\left(K_B^1(P^{ZV},P^{ZV}+E^{\tilde{z}}[y(\tilde{z})])\frac{V_X}{\eta^1}\frac{1}{\omega^1}+K_B^1(Q^{ZV},1)\frac{b^1}{\omega^1}+\tau\left((\frac{\delta^1}{\omega^1})^2-(\frac{\omega^1}{\delta^1})^2\right)\right), \quad \text{(l)}$$

where K_B^1 is a function of price p and payoff π:

$$K_B^1(p,\pi) = (\alpha^1-\rho^1\frac{\omega^1}{\delta^1})\pi+(\frac{\delta^1}{\omega^1})^2\left(p+\pi(\frac{\omega^1}{\delta^1}-\alpha^1)\right)-(\frac{\omega^1}{\delta^1})^2\left(p+\pi(1-\rho^1)\frac{\omega^1}{\delta^1}\right). \quad \text{(m)}$$

We finally obtain grouping equations (j) and (l):

$$V^{\xi}[E^{\tilde{z}\tilde{z}'}[\tilde{\gamma}_c^{1,PV}]] = V[\tilde{\gamma}_c^{1,ZV}]-\nu_{10}\tau-\nu_{11}b^1-\nu_{12}\frac{V_X}{\eta^1}, \quad \text{(n)}$$

where we deduce from (k) and (m) that there exists a function $K^1 = -2\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\frac{1}{\omega_1}(K_B^1 - E[\tilde{\gamma}_c^{1,ZV}]K_A^1)$ such that:

$$\nu_{11} = K^1(Q^{ZV}, 1), \tag{o}$$

$$\nu_{12} = K^1(P^{ZV}, P^{ZV} + E^{\tilde{z}}[y(\tilde{z})]), \tag{p}$$

$$\text{where: } K^i(p,\pi) = -2\frac{(1-\rho^i)(1-\alpha^i)}{2-\alpha^i-\rho^i}\frac{1}{\omega^i} \times \Bigg((1 - E[\tilde{\gamma}_c^{i,ZV}])(\alpha^i - \rho^i\frac{\omega^i}{\delta^i})\pi \tag{q}$$

$$+\frac{\delta^i}{\omega^i}(\frac{\delta^i}{\omega^i} - E[\tilde{\gamma}_c^{i,ZV}]) \left(\pi(\frac{\omega^i}{\delta^i} - \alpha^i) + p\right) - \frac{\omega^i}{\delta^i}(\frac{\omega^i}{\delta^i} - E[\tilde{\gamma}_c^{i,ZV}]) \left(p + \pi(1-\rho^i)\frac{\omega^i}{\delta^i}\right)\Bigg)$$

and

$$\nu_{10} = 2\frac{(1-\rho^1)(1-\alpha^1)}{2-\alpha^1-\rho^1}\left(\frac{\omega^1}{\delta^1} - \frac{\delta^1}{\omega^1}\right)\left(1 + \frac{1-\alpha^1\rho^1}{2-\alpha^1-\rho^1}(\frac{\delta^1}{\omega^1} + \frac{\omega^1}{\delta^1} - 2)\right) > 0. \tag{r}$$

We finally deduce from (h) and (n) that the total consumption growth of type-1 agents is:

$$V[\tilde{\gamma}_c^{1,PV}] = V[\tilde{\gamma}_c^{1,ZV}] - \nu_{10}\tau - \nu_{11}b^1 - \nu_{12}\frac{V_X}{\eta^1} + \nu_{13}\frac{V_X}{\eta^1}V^{\tilde{z}}[y(\tilde{z})], \tag{s}$$

where the expressions of ν_{13}, ν_{11} and ν_{12} can be found in equations (h), (o) and (p), respectively (and expression of $E[\tilde{\gamma}_c^{1,ZV}]$ in (i)). Using Lemma A below, it is straightforward to deduce that $\nu_{13} > 0$, $\nu_{11} > 0$ and $\nu_{12} > 0$.

Lemma A (Sign of κ) *Let us consider two probabilities $0 \le \alpha, \rho \le 1$, two positive variables $0 < \delta \le \omega$, and two positive scalars $p, \pi \ge 0$. We have then:*

$$(1-\gamma)\left(\alpha - \rho\frac{\omega}{\delta}\right)\pi + \frac{\delta}{\omega}(\frac{\delta}{\omega} - \gamma)\left(\pi(\frac{\omega}{\delta} - \alpha) + p\right) - \frac{\omega}{\delta}(\frac{\omega}{\delta} - \gamma)\left(p + \pi(1-\rho)\frac{\omega}{\delta}\right) \le 0,$$

$$\textit{where: } \gamma = 1 + \frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(\frac{\delta}{\omega} + \frac{\omega}{\delta} - 2).$$

Proof.

Once we group terms in p and π, for proving the result, it is sufficient to have:

$$\frac{\delta}{\omega}(\frac{\delta}{\omega} - \gamma) - \frac{\omega}{\delta}(\frac{\omega}{\delta} - \gamma) \le 0 \tag{t}$$

$$(1-\gamma)\left(\alpha - \rho\frac{\omega}{\delta}\right) + \frac{\delta}{\omega}(\frac{\delta}{\omega} - \gamma)(\frac{\omega}{\delta} - \alpha) - (\frac{\omega}{\delta})^2(\frac{\omega}{\delta} - \gamma)(1-\rho) \le 0 \tag{u}$$

We denote $X = \frac{\omega}{\delta} \ge 1$. We start with inequality (t) and we denote I_1 the left hand-side. After rearrangement, we have

$$I_1 = X^{-2} - X^2 + (X - X^{-1})(1 + \frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(X + X^{-1} - 2)).$$

It is increasing in $\frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho} \le 1$, so we have:

$$I_1 \le X^{-2} - X^2 + (X - X^{-1})(X + X^{-1} - 1) = \frac{(1-X)(1+X)}{X} \le 0,$$

which proves (t).

We now turn to (u) and we denote I_2 the left hand-side. After rearrangement:

$$\begin{aligned} I_2 &= (\alpha - X + X(1-\rho))\,(1-\gamma) - X^{-1}(X^{-1}-\gamma)(\alpha - X) - X^2(X-\gamma)(1-\rho) \\ &= \left(X^{-1}(X^{-1}-\gamma)+\gamma-1\right)(X-\alpha) - \left(X(X-\gamma)+\gamma-1\right)X(1-\rho). \end{aligned}$$

We have

$$\begin{aligned} X^{-1}(X^{-1}-\gamma)+\gamma-1 &= X^{-1}(X^{-1}-1-(\gamma-1))+\gamma-1 \\ &= (X^{-1}-1)(X^{-1}+1-\gamma), \\ X(X-\gamma)+\gamma-1 &= (X-1)(X+1-\gamma). \end{aligned}$$

We can also simplify the expression of γ:

$$\gamma - 1 = \frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(X+X^{-1}-2) = \frac{1}{X}\frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(X-1)^2$$

So, after some arrangement, we get:

$$\frac{I_2}{(1-X)X^{-2}} = (1-\frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(X-1)^2)(X-\alpha) + X^2(X^2-\frac{(1-\rho)(1-\alpha)}{2-\alpha-\rho}(X-1)^2)(1-\rho)$$

Since $X \geq 1 \geq \alpha$ and noting that $X-\alpha = X-1+1-\alpha$, we obtain that $\frac{I_2}{(1-X)X^{-2}}$ is a decreasing function of $1-\alpha \leq 1$. We thus deduce:

$$\begin{aligned} \frac{I_2}{(1-X)X^{-3}} &\geq 1 - \frac{1-\rho}{2-\rho}(X-1)^2 + X(X^2-\frac{1-\rho}{2-\rho}(X-1)^2)(1-\rho) \\ &\geq 1 + \left(X^3 - \frac{1}{2-\rho}(X-1)^2(1+(1-\rho)X)\right)(1-\rho) \end{aligned}$$

Since $X \geq 1$, we further have $1+(1-\rho)X \leq X(2-\rho)$ and:

$$\frac{I_2}{(1-X)X^{-3}} \geq 1 + X\left(X^2-(X-1)^2\right)(1-\rho) \geq 0.$$

where the last inequality comes from the fact that $X^2 \geq (X-1)^2$ and $1-\rho \geq 0$. We finally conclude that $I_2 \leq 0$, which proves (u). ■

For type-2 agents, we find an expression very similar to (s) for type-1 agents, and

$$V[\tilde{\gamma}_c^{2,PV}] = V[\tilde{\gamma}_c^{2,ZV}] - \nu_{20}\tau - \nu_{21}b^2$$

where the expression of ν_{21} is symmetric to the one of ν_{11} in (o) –the expression of K^i is in equation (q):

$$\nu_{21} = K^2(Q^{ZV}, 1), \tag{v}$$

and where ν_{20} is similar to ν_{20} in (r):

$$\nu_{20} = 2\frac{(1-\rho^2)(1-\alpha^2)}{2-\alpha^2-\rho^2}\left(\frac{\omega^2}{\delta^2}-\frac{\delta^2}{\omega^2}\right)\left(1+\frac{1-\alpha^2\rho^2}{2-\alpha^2-\rho^2}(\frac{\delta^2}{\omega^2}+\frac{\omega^2}{\delta^2}-2)\right) > 0.$$

It is indeed straightforward to deduce $\nu_{21} > 0$ using Lemma A.