Econ 509, *Introduction to Mathematical Economics I*
Professor Ariell Reshef
University of Virginia
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Exam 2

Please write in the space provided, and continue on the back of the page if needed, but mark "Continued on Back" clearly. You have 120 minutes.

Please print your name in capital letters: ________________________________

Question 1 (20 points)
Consider the Keynesian macro model

$$\begin{aligned} Y &= C(Y) + I(i) + G \\ M &= kY + L(i) \ , \end{aligned}$$

where

$$\begin{aligned} C'(Y) &> 0 \text{ and } C'(Y) < 1 \\ I'(i) &< 0 \\ L'(i) &< 0 \ , \end{aligned}$$

where $k > 0$, and M and G are exogenous.

By invoking the Implicit Function Theorem, find the comparative statics derivatives, $\partial Y/\partial M$, $\partial i/\partial M$ and determine their signs if possible. In doing so, clearly state the necessary preconditions for its application.

Question 1, continued

Question 2 (20 points)
The cost function of a firm, $C(q, w_1, w_2)$, is defined as the minimal cost of optimally producing q units of output, given prices $w_1, w_2 > 0$. Suppose that the production function is

$$q = x_1^{\alpha} x_2^{\beta} \ ,$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

1. Find the cost function.
2. What conditions must hold for this to be a meaningful question? Think of conditions for $C(q, w_1, w_2)$ to be a function.

Question 2, continued

Question 3 (20 points)
Consider the consumer's utility maximization problem when the utility function, $U(x, y)$ with positive marginal utilities, with $U_{xx} < 0$, $U_{yy} < 0$, and $U_{xy} = 0$, and with prices of p_x for x and p_y for y, and with money income B.

1. Set this up as a maximization problem with an equality constraint for expenditures (assume an interior solution, i.e. $x, y > 0$).
2. Write the Lagrangian and the first-order necessary conditions.
3. Are the second order conditions for a maximum satisfied?
4. Find an expression for dx/dB and determine its sign if possible.

Question 3, continued

Question 4 (8 points)
Please provide definitions of the following:

1. A convex set in $\mathcal{R}^n$.
2. A function that is homogeneous of degree r.
3. A concave function defined on $\mathcal{R}^n$.
4. Let $f(x)$ be quasi-concave, i.e. $\forall x^1, x^2 \in$domain of f and $\forall \theta \in (0,1)$ we have

$$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] \geq f\left(x^1\right)$$

   or, more simply put

$$f\left[\theta x^1 + (1-\theta)\, x^2\right] \geq \min\left\{f\left(x^2\right), f\left(x^1\right)\right\} \ .$$

   Show that $S = \{x | f(x) \geq k\}$ is a convex set $\forall k \in \mathcal{R}$.

Question 5 (8 points)
Find the following limits (or show that they do not exist):

1. $\lim_{x\to\infty} xe^{-x}$
2. $\lim_{\alpha\to 1} \left(x^{1-\alpha} - 1\right) / (1-\alpha)$

Question 6 (8 points)

An initial amount of \$100 at time 0 is invested and grows continuously at an interest rate of r. At what time T will this initial investment reach a value of \$1,000?

Question 7 (8 points)
The demand for a monopolist's product is a function of price: $q = Ap^{-\epsilon}$. Output is produced at a constant cost of c per unit produced, so total cost is $c \cdot q$.

1. Derive the price elasticity of demand. What is the implication of the expression that you found?
2. Find an expression for the profit maximizing price,

$$\pi = R - C = p \cdot q - c \cdot q \ .$$

   What is the implication of the expression that you found?

Question 8 (8 points)
Consider $X_2 > X_1 > 0$. Define the percent difference between X_2 and X_1 as x:

$$x = \frac{X_2 - X_1}{X_1} \ .$$

1. Use a Taylor expansion to show that $\ln(X_2) - \ln(X_1)$ is an approximation for x.
2. Show that this approximation is a good one only for small x. I.e., show that this approximation underestimates the true x, as x increases. You may use a graph to help explain this, but you will also need an analytical proof.