Econ 509, *Introduction to Mathematical Economics I*
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Summer 2010

Final Exam

Please write in the space provided, and continue on the back of the page if needed, but mark "Continued on Back" clearly. You have 3 hours.

Please print your name in capital letters: ________________________________

Question 1 (10 points)
The cost function of a firm, $C(q, w_1, w_2)$, is defined as the minimal cost of optimally producing q units of output, given prices $w_1, w_2 > 0$. Suppose that the production function is

$$q = x_1^{\alpha} x_2^{\beta} \ ,$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

1. Find the cost function.

2. What conditions must hold for this to be a meaningful question? I.e., what are the conditions for $C(q, w_1, w_2)$ to be a function?

Question 1, continued

Question 2 (20 points)
A consumer has preferences over consumption bundles of apples and bananas, (a, b), that can be represented by the following utility function:

$$U(a, b) = u(a) + b \ ,$$

where $u' > 0$, $u'' < 0$.

Answer the following questions:

1. Is $U(a, b)$ quasi concave? Is it concave? Is it strictly concave?

2. An indifference set is defined as a set of consumption bundles that give the same level of utility. Characterize the indifference set for $U(a, b) = k$ as a function $b = i(a)$.

3. The consumer wants to maximize her happiness (read: utility). She has an income of m and faces a set of prices (p_a, p_b). She cannot spend more than her income and cannot consume negative amounts of fruit. Solve for the optimal consumption bundle. Follow these steps

   (a) Write the maximization problem explicitly.

   (b) Write down the Lagrangian function and interpret the Lagrange coefficients.

   (c) Write down the FONCs.

   (d) Solve for the optimal bundle.

4. Is it possible to have zero consumption of any kind of fruit?

Question 2, continued

Question 3 (20 points) – Lifetime Saving
A worker is born at time $t = 0$ and dies with certainty at $t = T$. The worker's objective is to maximize lifetime welfare, given by

$$W = \int_0^T e^{-\rho t} \ln(c) \, dt \ ,$$

where c denotes consumption and ρ is the subjective discount rate. Wages are fixed at w. Capital markets give a return r on savings, denoted by a (assets). Therefore, instant income at any period is $w + ra$. The budget constraint, and hence the law of motion for assets, is

$$\dot{a} + c = w + ra \ .$$

The worker is born with an inheritance of

$$a(0) = a_0 \ .$$

Capital markets will not tolerate debt at death (there will be no one around to pay it), so

$$a(T) \geq 0 \ .$$

Solve the planning problem for saving and consumption. Do this in the following steps.

1. State the problem clearly (choose X to maximize Y s.t....)
2. Set up the *current value* Hamiltonian, generally written as

$$H = F(u, y, t) + \lambda g(u, y, t)$$

   and write down the FONCs

$$\begin{aligned} H_u &= 0 \\ H_\lambda &= \dot{y} \\ H_y &= \rho\lambda - \dot{\lambda} \end{aligned}$$

3. Completely characterize the saving and consumption plans. HELP: for some constant k and some variable x we have

$$\frac{d}{dt}\left[x e^{kt}\right] = e^{kt}\left(\dot{x} + kx\right) \ .$$

4. Is consumption increasing or decreasing? Why?

Question 3, continued

Question 4 (10 points)

A Pareto random variable X has the following cumulative density function: $\Pr[X \leq x] = F(x) = 1 - (b/x)^p$, where $x \geq b > 0$. To prevent any confusion: x is defined only for values greater or equal to b and b is strictly positive. $F(x) = 0$ for $x < b$. Answer the following questions:

1. Derive the probability density function, $f(x) = dF(x)/dx$.

2. The expectation of a random variable X is defined as $\int x dF(x)$. Find a restriction on p that ensures that the expectation exists for the Pareto distribution.

3. The variance of a random variable X is defined as $\int (x - \mu_x)^2 dF(x)$, where μ_x is the expectation. Find a restriction on p that ensures that the variance exists for the Pareto distribution.

Question 4, continued

Question 5 (10 points)

1. Suppose that $x^* \in \mathbb{R}$ maximizes $f(x, \theta)$ without constraints. Prove the Envelope Theorem, i.e.

$$\frac{df(x^*, \theta)}{d\theta} = \frac{\partial f(x^*, \theta)}{\partial \theta} .$$

2. Suppose that if $x^* \in \mathbb{R}$ maximizes $f(x, \theta)$ under the constraint $g(x, \theta) = c$. Prove the Envelope Theorem for constrained optimization, i.e.

$$\frac{df(x^*, \theta)}{d\theta} = \frac{\partial f(x^*, \theta)}{\partial \theta} - \lambda \frac{\partial g(x^*, \theta)}{\partial \theta} ,$$

   where λ is a Lagrange multiplier.

3. What does this imply if $g(x)$ does not depend on θ? How does this square with your result from 1?

Question 6 (10 points)

1. Provide a *second order* Taylor expansion (not approximation) of $F(x, y)$.

2. A maximizer of $F(x, y)$ can be defined as a point from which movement can only reduce the value of $F(x, y)$. Suppose that (x^*, y^*) satisfies the necessary conditions for a maximizer of $F(x, y)$. Use your Taylor expansion to provide a sufficient condition that ensures that (x^*, y^*) is indeed a maximizer.

3. Is it a better deal to receive an annual return on investment of r, or an instantaneous return of r for every instant during one year? Use a Taylor expansion to substantiate your answer and quantify it.

Question 6, continued

Question 7 (10 points)
Let v_1, v_2 and v_3 be three (column) vectors in $\mathbb{R}^3$. Define the matrix V as $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$. Show that if V is nonsingular, then (v_1, v_2, v_3) are a base for $\mathbb{R}^3$.