Econ 5090, *Introduction to Mathematical Economics I*
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Exam 1

Please write in the space provided, and continue on the back of the page if needed, but mark "Continued on Back" clearly. All questions are of equal weight. You have 90 minutes.

Please print your name clearly in CAPITAL LETTERS: ________________________________

Question 1

1. Write TWO definitions of the linear DEPENDENCE of n vectors, $x_1, x_{2,} ... x_n$.
2. The matrix $A_{n \times n}$ has an inverse. Explain why the columns of A span $\mathbb{R}^n$.

Question 2

1. Show that
$$\frac{d}{dx}u\left(x\right)=u\left(x\right)\frac{d}{dx}\ln\left(u\left(x\right)\right)\ .$$

2. Calculate
$$\frac{d}{dx}\left(x^{x}\right)\ .$$

Question 3
The elasticity of y with respect to x is defined as

$$\sigma_{yx} = \frac{dy/y}{dx/x}$$

and is understood as the percent change in y in response to a 1 percent change in x.

Consider the following Cobb-Douglas production function

$$y = zk^{\alpha}l^{1-\alpha} \ , \ 0 \leq \alpha \leq 1 \ .$$

Define the elasticity of substitution as the percent change in relative factor intensity (k/l) in response to a 1 percent change in the relative factor returns (r/w).

- What is the elasticity of substitution equal to? Assume that factors are paid their marginal product.

Question 4
Consider the following market model

$$
\begin{aligned}
q^d &= d(\overset{-}{p}, \overset{+}{y}) \\
q^s &= s(\overset{+}{p}, \overset{+}{r}) \\
q^s &= q^d \ ,
\end{aligned}
$$

where y is consumer income, r is rainfall and $-/+$ above an argument denotes the direction of the effect of an increase in that variable on the image.

- Use a THREE function system of THREE endogenous variables $F\left(q^d, q^s, p, y, r\right) = 0$ and the implicit function theorem to determine the sign of $\partial q^d / \partial r$ and $\partial p / \partial r$. In doing so, state clearly the premises for the theorem and check them.

Question 4 – continued

Question 5
Suppose that you have a system of n equations in n unknowns, with m exogenous parameters:

$$F\left(y_{n\times 1}, x_{m\times 1}\right) = 0_{n\times 1} \ .$$

Assume that the conditions for the implicit function theorem are satisfied, and that a set of n functions of x exist:

$$y_{n\times 1} = g\left(x_{m\times 1}\right) \ .$$

- Plug g into F and solve for the partial derivatives of y with respect to some x_k. For convenience, just write x instead of x_k. In doing so, state where you are using the premises of the theorem.

Question 6

On the island of Bacchus there are 10 workers, who produce only grapes and wine. In order to produce one kilo of grapes you need 1/2 kilo of grapes (for seeds) and one worker. In order to produce one liter of wine you need 1/2 kilo of grapes, 1/4 liter of wine (think about this as waste) and one worker. The workers demand 1 kilo of grapes and 3 liters of wine. Use an open sector input-output model to calculate the quantities of grapes and wine that are produced. The open sector is labor.

1. This implies solving a system of equations $(I - A)\,x = Tx = d$. Write out the A matrix. State the conditions for this system to have a solution, and then solve, if a solution exists.
2. If there is a solution, is there full employment on the island of Bacchus?

Question 7

The linear regression model is $y_{n\times 1} = X_{n\times k}\beta_{k\times 1} + \varepsilon_{n\times 1}$. The estimated model by OLS is $y = Xb + e$, where $b = (X'X)^{-1} X'y$. Therefore we have predicted values $\widehat{y} = Xb$ and residuals $e = y - \widehat{y}$.

1. Write down what e and $\widehat{y}$ are equal to in terms of X and y.
2. Show that e is orthogonal to $\widehat{y}$, i.e. that their inner product is zero.

Question 8

1. Suppose that the matrix product AB is defined. We know that $(AB)' = B'A'$, if the product $B'A'$ exists. Under what conditions will the product $B'A'$ exist?

2. Show that $AB \neq BA$ for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$