Econ 5090, *Introduction to Mathematical Economics I*
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Exam 2 – Optimization under Constraints

Please write in the space provided, and continue on the back of the page if needed, but mark "Continued on Back" clearly. You have 90 minutes.

Please print your name in capital letters: ________________________________

Question 1 (60 points)

The economy has 300 workers and 450 units of land. These can be used to produce either apples or bananas. Here are the technologies:

- Apples require 2 workers and 1 unit of land for each kilo of apples.
- Bananas require 1 worker and 2 unit of land for each kilo of bananas.

And you cannot "produce" negative quantities of fruit. Aggregate social preferences over apples and bananas are represented by the following welfare function:

$$w(a, b) = (1 - \beta)\ln(a) + \beta \ln(b) \ ,$$

where $\beta \in (0, 1)$. To be perfectly clear, $\beta \neq 0$, $\beta \neq 1$. You wish to maximize welfare.

To solve this question, follow these steps:

1. Write down explicitly all **4 inequality constraints**: one for the use of labor, one for the use of land, and two non-negativity constraints. Note that both labor and land may be partially unemployed/idle.
2. Show that the constraint set (i.e. the set of all feasible quantities of apples and bananas) is a convex set. You may use the fact that an intersection of convex sets is a convex set. Then draw the constraint set in the (a, b) space.
3. Show that the welfare function is strictly concave. Recall: a C^2 function is concave if (and only if) its Hessian is negative definite, i.e. if the determinants of the odd principal minors are negative and the even ones are positive.
4. Given that the welfare function is strictly concave, what do you know about the set $S = \{(a, b) : w(a, b) \geq k\}$, where $k \in \mathbb{R}$?
5. State the problem explicitly as "choose X to maximize Y subject to Z", where you must fill in X, Y and Z.
6. Set up the Lagrangian of this problem and carefully write down all first order necessary conditions.
7. Show that in the optimal solution the non-negativity constraints never bind.
8. Show that in the optimal solution if one factor is not fully employed, then the other must be fully employed (i.e. both cannot be less than fully employed at the same time).
9. Show that there are three possible solutions and characterize those solutions in terms of β. Sign the multipliers, but there is no need to fully solve for them.
10. For each solution that you have found, interpret the multipliers as shadow prices for labor and land.

Question 1 – continued

Question 1 – continued

Question 1 – continued

Question 1 – continued

Question 2 (40 points)

You have preferences over two goods, x and y, represented by the following utility function: $u(x, y) = y + a\ln(x)$. The price of x is p and the price of y is 1. You have income I. You cannot spend more than your income and you cannot consume negative quantities of any good.

1. **Completely** characterize the solution of the constrained maximization problem (including the multipliers) in terms of the parameters of the problem: a, p and I.

2. Explain how the optimal (x, y) bundle changes with p and I.

Question 2 – continued

Question 2 – continued

Question 2 – continued