Econ 5090, *Introduction to Mathematical Economics I*
Professor Ariell Reshef
University of Virginia
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Final Exam

Please write your answers on the answer sheet, in the spaces provided. Continue on the back of the page if needed, but mark "Continued on Back" clearly.

- You have 3 hours.

Question 1 (10 points)

1. Solve the following differential equation
$$\dot{y} + ay = b \ ,$$
with $y(0) = y_0$, a constant.
2. Solve the following differential equation
$$y'' - 2y' + y = 3$$
by using the sum of a particular solution and the solution to the homogenous equation (setting the right hand side to zero), with $y(0) = 4$ and $y'(0) = 2$. Is the system stable? If so, show that it converges to the particular solution that you found.

Question 2 (30 points)

The cost function of a firm, $C(q, w_1, w_2)$, is defined as the minimal cost of optimally producing q units of output, given prices $w_1, w_2 > 0$. Suppose that the production function is
$$q = x_1^{\alpha} x_2^{1-\alpha} \ ,$$
where $\alpha \in (0, 1)$.

1. Find the cost function. To do this, follow these steps:
   (a) Find the demands x_1^* and x_2^* given (q, w_1, w_2) and then
   (b) Plug those demands into the objective function to get the value of the problem.
2. What conditions must hold for this to be a meaningful question? I.e., what are the conditions for $C(q, w_1, w_2)$ to be a *function*?
3. Verify Shephard's Lemma. I.e. show that given the cost function that you found above,
$$\frac{\partial C(q, w_1, w_2)}{\partial w_i} = x_1^* \ ,$$
where x_1^* is what you computed above in 1(a).

Question 3 (10 points)

A Pareto random variable X has the following distribution function: $\Pr[X \leq x] = F(x) = 1 - (b/x)^p$, where $x \geq b > 0$ and $F(x) = 0$ for $x < b$. The term of art is that the *support* of X is $[b, \infty)$. To prevent any confusion: x is defined only for values greater or equal to b and b is strictly positive. Answer the following questions:

1. Derive the density function of the Pareto, i.e. $f(x) = dF(x)/dx$, and show that it is indeed a density function. I.e. prove that
$$\int_b^\infty f(x)\,dx = 1 \ .$$
2. When is the expected value of X finite? I.e. compute
$$\mu = \int_b^\infty x f(x)\,dx$$
and give a condition under which this integral is convergent.

Question 4 (20 points)

Consider the Pareto random variable from the previous question. In this question I want you to show that if X is distributed Pareto, then $Y = \ln(X)$ is an exponential random variable. An exponential random variable Y has the following distribution function: $\Pr[Y \leq y] = G(y) = 1 - ke^{-\lambda y}$ for some k and some λ. I want you to find k and λ in terms of b and p (usually $k = 1$ and $Y \geq 0$, but now I want you to do something more general). To do so, follow these steps:

1. Given $X \in [b, \infty)$, what values does Y take, i.e. what is the support of Y?
2. Since $Y = \ln(X)$, it follows that $X = e^Y$. Plug this into $F(x)$ to derive $G(y)$.
3. You proved above that for the Pareto $\int_b^\infty f(x)\,dx = 1$, where $f(x) = dF(x)/dx$. Use the change of variables formula to show that
$$\int_m^\infty g(y)\,dy = 1$$
for some m, which you need to find, and where $g(y) = dG(y)/dy$. The change of variables formula: for some $u = u(z)$ we have
$$\int_c^d f(u)\frac{du}{dz}dz = \int_c^d f(u)\,u'dz = \int_{u(c)}^{u(d)} f(u)\,du \ .$$
4. Express the expected value of Y
$$E(Y) = \int_m^\infty y g(y)\,dy$$
in terms of b and p. You may need to use integration by parts:
$$\int U(x)\,v(x)\,dx = U(x)\,V(x) - \int V(x)\,u(x)\,dx \ .$$

(FYI: we are relying on the fact that $y = \ln(x)$ is a continuous monotone transformation for $x \geq b > 0$).

Question 5 (10 points)

1. Suppose that $x^* \in \mathbb{R}$ maximizes $f(x, \theta)$ without constraints. Prove the Envelope Theorem, i.e.

$$\frac{df(x^*, \theta)}{d\theta} = \frac{\partial f(x^*, \theta)}{\partial \theta} .$$

2. Explain the meaning of the last result.

3. Suppose that if $x^* \in \mathbb{R}$ maximizes $f(x, \theta)$ under the constraint $g(x, \theta) \leq c$. Prove the Envelope Theorem for constrained optimization, i.e.

$$\frac{df(x^*, \theta)}{d\theta} = \frac{\partial f(x^*, \theta)}{\partial \theta} - \lambda \frac{\partial g(x^*, \theta)}{\partial \theta} ,$$

where λ is a Lagrange multiplier.

4. Explain the meaning of the last result if $f(x)$ does not depend on θ, asuming that the constraint $g(x, \theta) = c$ binds.

Question 6 (10 points)

Consider $X_2 > X_1 > 0$. Define the percent difference between X_2 and X_1 as x:

$$x = \frac{X_2 - X_1}{X_1} .$$

1. Use a first order Taylor expansion to show that $\ln(X_2) - \ln(X_1)$ is an approximation for x.

2. Show that this approximation is a good one only for small x. I.e., show that this approximation underestimates the true x, as x increases. You may use a graph to help explain this, but you will also need an analytical proof.

Question 7 (10 points)

Let v_1, v_2 and v_3 be three (column) vectors in $\mathbb{R}^3$. Define the matrix V as $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$. Show that if V is nonsingular, then (v_1, v_2, v_3) are a base for $\mathbb{R}^3$.