Econ 5090, Introduction to Mathematical Economics I Professor Ariell Reshef University of Virginia Summer 2012

## Exam 2 – Optimization under Constraints

Please write in the space provided, and continue on the back of the page if needed, but mark "Continued on Back" clearly. You have 90 minutes.

Please print your name in capital letters: \_\_\_\_\_

## Question 1 (70 points)

A firm produces output q using skilled labor h and unskilled labor l, according to the following (CES) production function

$$q = \left[\alpha h^{\varphi} + (1 - \alpha) l^{\varphi}\right]^{\frac{1}{\varphi}} ,$$

where  $\varphi < 1$ ,  $\alpha \in (0, 1)$ . The firm faces fixed wages for skilled labor s and unskilled labor w, both strictly positive, so that the cost of employment is

$$c = wl + sh .$$

The goal of this exercise is to characterize the cost function and the competitive price of output.

- 1. Show that the production function is homogenous of degree 1 (constant returns to scale). This implies showing that multiplying all inputs by some factor k multiplies output by k as well.
- 2. The firm wishes to minimize costs, as long as it produces at least q units of output. Write down the cost minimization problem. State the problem explicitly as "choose X to minimize Y subject to Z", where you must fill in X, Y and Z. Ignore non-negativity constraints.
- 3. Set up the Lagrangian of this problem and carefully write down all first order necessary conditions.
- 4. Show that in the optimal solution the constraint binds and that the multiplier is strictly positive.
- 5. Solve for the optimal demand for factors h and l in terms of the wages, q and the multiplier of the constraint. Don't solve for the multiplier yet. In doing so, take advantage of the fact that  $[\alpha h^{\varphi} + (1-\alpha) l^{\varphi}] = q^{\varphi}$ , so that  $[\alpha h^{\varphi} + (1-\alpha) l^{\varphi}]^{\frac{1}{\varphi}-1} = q^{1-\varphi}$ . In addition, you may find it useful to write  $\frac{1}{1-\varphi}$  as  $\sigma$ .
- 6. Given your answer above, what is the relationship between output and demand for labor?
- 7. Now solve for the multiplier. This is done by plugging optimal demand for labor into the production function. Show that the multiplier can be written as

$$[as^r + bw^r]^{\frac{1}{r}}$$

where a, b and r can be expressed using  $\alpha$  and  $\sigma$ .

- 8. Explain in words what is the interpretation of the multiplier.
- 9. In a competitive economy with constant returns to scale production there can be no economic profits. This means that cost equals revenue:

$$C = R = p \cdot q ,$$

where p is the price of output and q is the quantity produced; and C is cost of producing this output. Show that under these conditions p is equal to the multiplier.

## Question 2 (30 points)

You have preferences over two goods, x and y, represented by the following utility function:  $u(x,y) = \alpha \ln (x) + (1 - \alpha) \ln (y)$ . The price of x is p and the price of y is q. You have income I. p, q, I > 0 and  $\alpha \in (0, 1)$ . You wish to maximize your welfare, i.e. u, but you cannot spend more than your income and you cannot consume negative quantities of any good.

- 1. Write down the utility maximization problem. State the problem explicitly as "choose X to maximize Y subject to Z", where you must fill in X, Y and Z.
- 2. Set up the Lagrangian of this problem and carefully write down all first order necessary conditions.
- 3. Show that the non negativity constraints do not bind.
- 4. Show that the share of income I that is spent on x and  $y p \cdot x$  and  $q \cdot y$ , respectively—are  $\alpha$  and  $(1 \alpha)$ , respectively.