Econ 5090, *Introduction to Mathematical Economics I*
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Final Exam

Please write your answers on the answer sheet, in the spaces provided. Continue on the back of the page if needed, but mark "Continued on Back" clearly.

- You have 3 hours.

Question 1 (20 points)

Let $u(x) \in C^2$ be a utility function over wealth $x > 0$. The Arrow-Pratt measure of relative risk aversion at wealth x is

$$\sigma(x) = -\frac{u''(x)\, x}{u'(x)} \ . \tag{1}$$

1. Express $\sigma(x)$ as an elasticity of marginal utility $u'(x)$ with respect to wealth x.
2. Now suppose that $\sigma(x) = \sigma$, a constant relative risk aversion (CRRA). Denote $u'(x) = v(x)$ and solve for v, using (1) as a separable first order differential equation. Pay attention to the constants of integration.
3. Given your solution for v, now solve for u for all values of $\sigma > 0$. Pay attention to the constants of integration.
4. What happens when $\sigma = 1$?
5. Solve the following differential equation

$$y'' - 2y' + 3y = 9t^2$$

SHOULD HAVE BEEN

$$y'' - 2y' - 3y = 9t^2$$

by using the sum of a particular solution and the solution to the homogenous equation (setting the right hand side to zero). Note that the particular solution needs to be found using the method of undetermined coefficients. Hint: the particular solution is NOT of the form $y_p(t) = kt^2$.

Question 2 (30 points)

You have preferences over two goods, x and y, represented by the following utility function: $u(x, y) = y + a\ln(x)$. The price of x is p and the price of y is 1. You have income I. You cannot spend more than your income and you cannot consume negative quantities of any good. All parameters are strictly positive: $a, p, I > 0$.

1. **Completely** characterize the solution of the constrained maximization problem (including the multipliers) in terms of the parameters of the problem: a, p and I.
2. Explain how the optimal (x, y) bundle changes with p and I.

Question 3 (10 points)

You are about to enter an auction for a painting, where you face an unknown number of bidders that submit one sealed auction each. Your valuation of the painting is v and denote your bid as b. You think that the distribution of your opponents' bids is given by $g(B)$, so that $\Pr\{B < b\} = \int^b g(B)\,dB$. There are two types of auctions.

1. In a *first*-price sealed auction you get to have the painting for the price of your bid if your bid is the highest. If your bid b is the highest, then the probability that you win the auction is

$$\Pr\{win\} = \Pr\{B < b\} = \int^b g(B)\,dB\ .$$

   In case that you win, your gain is the value of the painting minus your bid: $v - b$. If not, you pay nothing. Your expected gain is

$$\begin{aligned} E &= (v-b)\Pr\{win\} + 0\,[1 - \Pr\{win\}] \\ &= (v-b)\int^b g(B)\,dB\ . \end{aligned}$$

   What is your gain-maximizing bid in this auction?

2. In a *second*-price sealed auction you get to have the painting for the price of the *second highest bidder's bid* if your bid is the highest. If your bid b is indeed the highest you pay the bid that is below yours, so your expected gain is

$$E = \int^b (v - B)\,g(B)\,dB\ .$$

   What is your gain-maximizing bid in this auction?

3. In which auction is your bid higher?

Question 4 (15 points)

Consider a standard normal random variable $Z \sim N(0,1)$ with density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \ . \tag{2}$$

(You can answer this question without knowing the formula, but I provide it here for completeness.) Normal random variables are distributed on the entire real line $(-\infty, \infty)$, so that

$$\int_{-\infty}^{\infty} \phi(z)\, dz = 1 \ . \tag{3}$$

(It is not possible to compute this integral but we can prove that it is correct.) In this question I want you to find the density function of a lognormal random variable Y that is given by $Y = e^Z$. To do so, follow these steps:

1. Given $Z \in (-\infty, \infty)$, what values does Y take, i.e. what is the support of Y?
2. Denote the density function of Y as $f(y)$. Use the change of variables formula in (3) to find $f(y)$.
3. Using the density function $f(y)$ that you found above, show that

   $$\int_a^b f(y)\, dy = 1 \ ,$$

   where you must specify correctly a and b.

The change of variables formula: for some $u = u(x)$ we have

$$\int_{u(c)}^{u(d)} g(u)\, du = \int_c^d g(u(x)) \frac{du}{dx} dx \ .$$

(FYI: we are relying on the fact that ln is a continuous monotone transformation).

Question 5 (15 points)

An exponential random variable X has the following cumulative density function: $\Pr[X \leq x] = F(x) = 1 - e^{-\lambda x}$, where $x \geq 0$. Answer the following questions:

1. Derive the probability density function, $f(x) = dF(x)/dx$.
2. Given the $f(x)$ that you found in 1, show that $\int f(x)\, dx = 1$.
3. The expectation of a random variable X is defined as $E(X) = \int x dF(x)$. Find $E(X)$ for the exponential.
4. The variance of a random variable X is defined as $V(X) = \int (x-\mu)^2 dF(x)$, where $\mu = E(X)$ is the expectation. Find $V(X)$ for the exponential.

You may need to use integration by parts:

$$\int H(x)\, g(x)\, dx = H(x)\, G(x) - \int h(x)\, G(x)\, dx \ .$$

Question 6 (10 points)

Consider the linear regression model

$$y_i = x_i'\beta + \varepsilon_i \ ,$$

where y is the dependent variable, x is a vector of k explanatory variables, β are their coefficients and ε is the error term. There are n observations, so that this can be neatly summarized as

$$y = X\beta + \varepsilon \ ,$$

where the dimensions are $y_{n\times 1}$, $X_{n\times k}$ and $\varepsilon_{n\times 1}$. We don't know what β is, but we want to estimate it.

1. What are the dimensions of β?
2. What are the dimensions of $(X'X)$? And what is the maximal rank of $(X'X)$?
3. If $(X'X)$ has its maximal rank, is it invertible? Explain.
4. The OLS estimator of β can be thought of as the estimator that assumes that the errors are orthogonal to the explanatory variables, i.e. $X'\varepsilon = 0$ (FYI: this is called a moment condition). Find b, the OLS estimator of β; this involves solving the moment condition for b.
5. Suppose that one of the variables (that corresponds to a column in X) is linearly dependent on (perfectly colinear with) the other variables. What is the rank of X?
6. Under the conditions in 5, can you compute b? Explain.