Econ 5090, *Introduction to Mathematical Economics I*
Professor Ariell Reshef
University of Virginia
Summer 2011

Implicit Function Theorem Application

The closed economy is populated by H skilled workers and L unskilled workers; (relative) skill abundance is defined as

$$h = \frac{H}{L} \ .$$

There are two technologies available – which define two sectors – for producing goods (G) and services (S). These are

$$\begin{aligned} G &= \left[(L_g)^{\rho_g} + \left(\beta_g H_g\right)^{\rho_g}\right]^{1/\rho_g} \\ S &= \left[(L_s)^{\rho_s} + (\beta_s H_s)^{\rho_s}\right]^{1/\rho_s} \ , \end{aligned}$$

where $\rho_i \leq 1$ and the elasticities of substitution are given by $\sigma_i = 1/(1-\rho_i)$ $i \in \{g, s\}$. σ_s need not equal σ_g. This gives rise to relative factor demands for each sector

$$\begin{aligned} h_g &= \omega^{-\sigma_g} \beta_g^{\sigma_g - 1} \\ h_s &= \omega^{-\sigma_s} \beta_s^{\sigma_s - 1} \ , \end{aligned}$$

where $\omega = w_H/w_L$ is the relative wage of skilled workers, $h_i = H_i/L_i$ is skill intensity and β_i is relative factor efficiency of skilled workers versus unskilled. Workers are freely mobile across sectors, so w_H and w_L are the same in both sectors.

Workers of both types supply labor inelastically and their income is their wage. Their preferences over goods and services are represented by

$$U(S, G) = \left[\mu S^{\psi} + (1-\mu) G^{\psi}\right]^{1/\psi} \ ,$$

where $\psi \leq 1$.

The solution to this model is obtained by solving for ω in the following function:

$$\begin{aligned}
\Phi\left(\omega, h, \beta_g, \beta_s, \mu\right) &= \left(\frac{h-h_g}{h_s-h}\right)\frac{(1+\omega h_s)^{(\varphi-\sigma_s)/(1-\sigma_s)}}{(1+\omega h_g)^{(\varphi-\sigma_g)/(1-\sigma_g)}} - \left(\frac{\mu}{1-\mu}\right)^{\varphi} \\
&= \left(\frac{h-\omega^{-\sigma_g}\beta_g^{\sigma_g-1}}{\omega^{-\sigma_s}\beta_s^{\sigma_s-1}-h}\right)\frac{\left(1+\omega^{1-\sigma_s}\beta_s^{\sigma_s-1}\right)^{(\varphi-\sigma_s)/(1-\sigma_s)}}{\left(1+\omega^{1-\sigma_g}\beta_g^{\sigma_g-1}\right)^{(\varphi-\sigma_g)/(1-\sigma_g)}} - \left(\frac{\mu}{1-\mu}\right)^{\varphi} \\
&= 0\ ,
\end{aligned}$$

where $\varphi = 1/\left(1-\psi\right)$ is the elasticity of substitution in demand. The function Φ defines an implicit function in ω and all the exogenous parameters. Solving for the unique ω completely determines the equilibrium in the economy. All comparative statics can be computed by applying the implicit function theorem. To make things concrete, assume that $\sigma_s < \sigma_g$ and that $h_s > h > h_g$ always holds (although Φ is completely general).

Questions:

While σ_i define within sector elasticities of substitution (percent changes in h_i invoked by one percent change in ω), they do not take into account general equilibrium restrictions, which are implicit in Φ.

- Use the implicit function theorem to derive $d\omega/dh$.

Define the pseudo aggregate elasticity of substitution as the change in skill abundance that would be invoked by one percent change in ω (holding all other parameters fixed). This is given by

$$\sigma = \frac{dh/h}{d\omega/\omega} = \frac{dh}{d\omega}\frac{\omega}{h}\ .$$

This is a pseudo elasticity because h is actually fixed.

- Show that σ can be written a convex combination of σ_s, σ_g and φ (this is a weighted average with non-negative weights).

Once you get this, you will realize that an aggregate elasticity of substitution combines both subtitution within sectors (technology) and across sectors (demand).