Leibnitz's Rule

Let $f \in \mathbb{C}^1$ (i.e. $F \in \mathbb{C}^2$). Then

$$\frac{\partial}{\partial \theta}\int\limits_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial \theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial \theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x,\theta)\,dx\ .$$

Proof: let $f(x,\theta) = dF(x,\theta)/dx$. Then

$$\begin{aligned}
\frac{\partial}{\partial \theta}\int\limits_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx &= \frac{\partial}{\partial \theta}\left[F(x,\theta)\right|_{a(\theta)}^{b(\theta)} \\
&= \frac{\partial}{\partial \theta}\left[F(b(\theta),\theta) - F(a(\theta),\theta)\right] \\
&= F_x(b(\theta),\theta)\frac{\partial b(\theta)}{\partial \theta} + F_\theta(b(\theta),\theta) - F_x(a(\theta),\theta)\frac{\partial a(\theta)}{\partial \theta} - F_\theta(a(\theta),\theta) \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial \theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial \theta} + \left[F_\theta(b(\theta),\theta) - F_\theta(a(\theta),\theta)\right] \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial \theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial \theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{d}{dx} F_\theta(x,\theta)\,dx \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial \theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial \theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x,\theta)\,dx\ .
\end{aligned}$$

The last line follows by Young's Theorem. Clearly, if the integration limits do not depend on θ, then

$$\frac{\partial}{\partial \theta}\int\limits_{a}^{b} f(x,\theta)\,dx = \int\limits_{a}^{b} \frac{\partial}{\partial \theta} f(x,\theta)\,dx\ ,$$

and if f does not depend on θ, then

$$\frac{\partial}{\partial \theta}\int\limits_{a(\theta)}^{b(\theta)} f(x)\,dx = f(b(\theta))\frac{\partial b(\theta)}{\partial \theta} - f(a(\theta))\frac{\partial a(\theta)}{\partial \theta}\ .$$