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Allais' trading process and the dynamic evolution of a market economy*

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Summary. We construct a simple trading process that is based on the maximization, at each stage, of the total distributable surplus. We show that this process converges to a Pareto optimal allocation.

Keywords and Phrases: Trading process, Distributable surplus, Benefit function.

JEL Classification Numbers: D000, D500.

1 Introduction

This note takes up the issue of the optimality of a trading process in a market economy from a benefit viewpoint. We show that the trading process consisting in maximizing at each stage the total benefit in the economy (ie, that ensures the maximal gains to exchange at each point in time) is efficient in the sense that it is individually rational at each stage and converges to a Pareto optimal allocation. Allais (1981) introduced the idea of total distributable surplus as a way to analyze the efficiency properties of a market economy. Luenberger (1992a, b) extend Allais' analysis and proves a series of result linking distributable surplus (the benefit function in his terminology) to efficiency properties in particular (see also Luenberger, 1996).

More precisely, Allais (1943) defined a market economy as an economy in which agents make all possible advantageous transactions. In contrast to the walrasian theory of markets, agents do not trade through a single price system. A stable equilibrium is then defined as a situation in which no further trade is done, *i.e.*, where no further surplus can be distributed. Allais (1968) stated

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(without proofs) two "fundamental theorems for market economy". The first one, which assesses that an equilibrium (in his sense) is Pareto optimal was formally proved using the notion of benefit function by Luenberger (1992b). Allais' second fundamental theorem states that a market economy will converge to a "situation of stable equilibrium". A first attempt to prove this theorem may be found in De Montbrial (1970), where it is shown that Allais' second theorem is not true in full generality. The purpose of this note is to give a proof of Allais' second theorem by formalizing a particular trading mechanism, that maximizes the total distributable surplus at each stage. We then find conditions under which such a mechanism converges to a Pareto optimal allocation.

2 Set-up

We consider a pure exchange economy with C goods and I agents ($i = 1, \ldots, I$). Agent i's endowments are denoted w_i. We assume $w_i >> 0$ for all i. Denote $w = (w_1, \ldots, w_I)$. Each agent has a utility function $u_i : \mathbb{R}_+^C \to \mathbb{R}$ that will be assumed continuous, strictly increasing and strictly quasi-concave. Let

$$FA(w) = \left\{ x \in \mathbb{R}_+^{CI} \mid \sum_{i=1}^{I} x_i = \sum_{i=1}^{I} w_i \right\}$$

be the set of feasible allocations.

We now introduce the concept of distributable surplus (Allais, 1943, 1981) or of benefit function (Luenberger, 1992a):

Definition. *Let $g \in \mathbb{R}_+^C$, $g \neq 0$, be the "reference" bundle. The benefit function b_i corresponding to utility function u_i is defined as:*

$$b_i(x, u; g) = \max\{\beta \mid u_i(x - \beta g) \geq u, x - \beta g \in \mathbb{R}_+^C\}$$

If the constraint is not feasible, set $b_i(x, u; g) = -\infty$.

Taking the reference bundle as fixed, we'll omit it as an argument of the benefit function and write $b_i(x, u)$. The benefit function measures the maximum an individual i is willing to give up of a bundle g to move from a utility level of u to the point x. If x is "above" the indifference curve of level u, $b_i(x, u)$ is positive, while it is negative if the point x is "below" the indifference curve u.

3 A trading process

We now consider a trading process, based on the maximization of the total distributable surplus or total benefit, which leads to a Pareto optimal allocation. Define the set of individually rational allocations as follows:

$$IR(y) = \{x \in FA(w) \mid u_i(x_i) \geq u_i(y_i) \ \forall i\} \quad \text{for all} \ \ y \in \mathbb{R}_+^{CI}$$

The allocation $x^1 = (x_1^1, \ldots, x_I^1)$ resulting from the first round of exchanges is a solution to the following problem:

$$\max_{x^1} \sum_{i=1}^{I} b_i \left(x_i^1, u_i(w_i)\right) \text{ s.t. } x^1 \in IR(w)$$

At the nth stage, the allocation x^n is a solution of:

$$\max_{x^n} \sum_{i=1}^{I} b_i \left(x_i^n, u_i(x_i^{n-1})\right) \text{ s.t. } x^n \in IR(x^{n-1}) \qquad (\star)$$

Hence, at each stage of the trading process each individual's utility increases. Further, the allocation maximizes the total benefit function. We now show that this trading process converges to a Pareto optimal allocation.

Proposition. *Let $\{x^n\}_n$ be a sequence of allocations such that x^n is a solution to ($\star$) for all n. Then $\{x^n\}_n$ converges to an allocation $\bar{x}$. Furthermore, $\bar{x}$ is Pareto optimal.*

Proof. The proof is decomposed in four steps.

Step one. Let $\{u_i^n\}$ be i's utility along the sequence $\{x^n\}$, and $u^n = (u_1^n, \ldots, u_I^n)$. Then, there is a $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_I)$ such that $u^n \to \bar{u}$.

Proof. $\{u^n\}$ is an increasing sequence that lies in

$$\mathscr{U}(w) \equiv \left\{u \in \mathbb{R}^I \mid \exists x \in IR(w), u_i(x_i) = u_i \ \forall i\right\}$$

This set is compact as it is the image of the compact set $IR(w)$ by a continuous function. Hence, $\{u^n\}$ converges.

Step two. There exists a unique $\bar{x} \in FA(w)$ s.th. $\bar{u}_i = u_i(\bar{x}_i)$ for all i. Furthermore, $x^n \to \bar{x}$.

Proof. Existence is ensured since $\bar{u} \in \mathscr{U}(w)$. Suppose it is not unique, i.e., there exist $\bar{x}, \bar{x}'$ such that $\bar{u}_i = u_i(\bar{x}_i) = u_i(\bar{x}_i')$ and $\bar{x} \neq \bar{x}'$. Then, by strict quasi-concavity,

$$u_i(\lambda\bar{x}_i + (1-\lambda)\bar{x}_i') \geq u_i(\bar{x}_i)$$

for all $\lambda \in (0,1)$, and for all i, with a strict inequality for at least one i. Hence, $\sum_{i=1}^{I} b_i(\lambda\bar{x}_i + (1-\lambda)\bar{x}_i', \bar{u}_i) > 0$ since $b_i(\lambda\bar{x}_i + (1-\lambda)\bar{x}_i', \bar{u}_i) \geq 0$ for all i with a strict inequality for at least one i. But then $\bar{u}$ cannot be a limit utility allocation of the trading sequence since $\lambda\bar{x} + (1-\lambda)\bar{x}'$ yields a higher benefit and is in $IR(\bar{x})$. Hence, $\bar{x}$ is unique.

Finally, since $u_i^n = u_i(x_i^n) \to \bar{u}_i$ and there exists a unique $\bar{x} \in FA(w)$ such that $\bar{u}_i = u_i(\bar{x}_i)$, $x_i^n \to \bar{x}_i$.

Step three. $IR(.)$ is a continuous correspondence on $FA(w)$.

Proof. $IR(.) : FA(w) \to FA(w)$ and $FA(w)$ is compact. Furthemore, $IR(.)$ is closed. Hence it is u.h.c.

We now prove it is l.h.c. as well. Let $y^n \to y$ and $x \in IR(y)$. We want to show $\exists x^n \in IR(y^n)$ s.th. $x^n \to x$ (Feldman (1973)).

If $x = y$, then choose $x^n = y^n$.

Suppose now $x \neq y$ and suppose $IR(.)$ is not l.h.c. at y. Then, there exists $\varepsilon > 0$ such that, for an infinite number of n, $IR(y^n) \cap B_\varepsilon(x) = \emptyset$, where $B_\varepsilon(x)$ is a closed ball of radius ε centered on x. Let $\{y^q\}$ be a sub-sequence of $\{y^n\}$ s.th.

$$IR(y^q) \cap B_\varepsilon(x) = \emptyset \quad \forall q$$

Consider now $\widehat{x} = \lambda y + (1 - \lambda)x$ with $\lambda \in (0, 1)$. Observe that by strict quasi-concavity of u_i, $u_i(\widehat{x}_i) \geq u_i(y_i)$ for all i and with a strict inequality for at least one i. Construct from $\widehat{x}$ an allocation $\widetilde{x}$ s.th. $u_i(\widetilde{x}_i) > u_i(y_i)$ for all i. This is always possible since u_i is strictly increasing and continuous.

Now, since u_i is strictly increasing and continuous for all i, it is possible to pick λ and $\widetilde{x}$ such that the distance between x and $\widetilde{x}$ is $\varepsilon/2$. Choose $\varepsilon' > 0$ small enough (less than $\varepsilon/2$) so that

$$\forall z \in B_{\varepsilon'}\left(\widetilde{x}\right) \text{ and } \forall y^q \in B_{\varepsilon'}(y), u_i(z_i) > u_i(y_i^q) \quad \forall i$$

and $B_{\varepsilon'}\left(\widetilde{x}\right) \subset B_\varepsilon(x)$.

Since $y^q \to y$, there exists N s.th. $\forall q \geq N, y^q \in B_{\varepsilon'}(y)$ and therefore $u_i(y_i^q) < u_i(z_i)$, $\forall i, \forall z \in B_{\varepsilon'}\left(\widetilde{x}\right)$. Thus, $IR(y^q) \cap B_{\varepsilon'}\left(\widetilde{x}\right) \neq \emptyset$ $\forall q \geq N$. Hence, since $B_{\varepsilon'}\left(\widetilde{x}\right) \subset B_\varepsilon(x)$, one gets

$$\forall q \geq N, IR(y^q) \cap B_\varepsilon(x) \neq \emptyset$$

but this is a contradiction since we constructed y^q so that $\forall q \geq N, IR(y^q) \cap B_\varepsilon(x) = \emptyset$.

Step four. $\overline{x}$ is a Pareto optimal allocation.

Proof. Define

$$V(x^{n-1}) = \max_x \left\{ \sum_{i=1}^{I} b_i(x_i, u_i(x_i^{n-1})) \text{ s.t. } x \in IR(x^{n-1}) \right\}$$

Since $IR(.)$ is a compact-valued, and continuous correspondence, $V(.)$ is continuous. Hence, $V(x^n) \to V(\overline{x})$. By definition, $b_i(\overline{x}_i, u_i(\overline{x}_i)) = 0$ for all i, and hence $V(\overline{x}) = 0$.

Thus, $\overline{x}$ solves $\max_x\{\sum_{i=1}^{I} b_i(x_i, u_i(\overline{x}_i))$ s.t. $x \in IR(\overline{x}\}$, and $\sum_{i=1}^{I} b_i(\overline{x}_i, u_i(\overline{x}_i)) = 0$.

Assume now $\overline{x}$ is not Pareto optimal. Then, there exists $y \in FA(w)$ such that $u_i(y_i) > u_i(\overline{x}_i)$ for all i. Furthermore, since utility functions are strictly increasing, it is possible to choose $y_i \gg 0$ (Luenberger (1996), Theorem 2) for all i. Then, there exists $y \in IR(\overline{x})$ s.th. $\sum_{i=1}^{I} b_i(y_i, u_i(\overline{x}_i)) > 0$, a contradiction. □

References

Allais, M.: A la recherche d'une discipline économique. Imprimerie Nationale (1943)
Allais, M.: The conditions of efficiency in the economy. Economia Internazionale, 399–419 (1968)
Allais, M.: La théorie générale des surplus. Economies et Sociétés **XV**, 1–716 (1981)
De Montbrial, T.: Economie théorique. Presses Universitaires de France 1970
Feldman, A.: Bilateral trading processes, pairwise optimality and Pareto optimality. Review of Economic Studies, 463–473 (1973)
Luenberger, D.: Benefit functions and duality. Journal of Mathematical Economics **21**, 461–481 (1992a)
Luenberger, D.: New optimality principles for economic efficiency and equilibrium. Journal of Optimization Theory and Applications **75**, 221–263 (1992b)
Luenberger, D.: Welfare from a benefit viewpoint. Economic Theory **7**(3), 445–462 (1996)