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Diversification, convex preferences and non-empty core in the Choquet expected utility model*

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Summary. We show, in the Choquet expected utility model, that preference for diversification, that is, convex preferences, is equivalent to a concave utility index and a convex capacity. We then introduce a weaker notion of diversification, namely "sure diversification." We show that this implies that the core of the capacity is non-empty. The converse holds under concavity of the utility index, which is itself equivalent to the notion of comonotone diversification, that we introduce. In an Anscombe-Aumann setting, preference for diversification is equivalent to convexity of the capacity and preference for sure diversification is equivalent to non-empty core. In the expected utility model, all these notions of diversification are equivalent and are represented by the concavity of the utility index.

Keywords and Phrases: Diversification, Choquet expected utility, Capacity, Convex preferences, Core.

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1 Introduction

Dekel (1989) made the point that having a preference for portfolio diversification is an important feature when modelling markets of risky assets. He also observed that the relationship between risk aversion and preference for diversification is

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trivial in the expected-utility model, and much more complicated in alternative models. More precisely, the equivalence between these two properties established in the EU framework does not hold in more general models. There, diversification implies risk-aversion but the converse is false.

This note takes up the study of diversification to the case of uncertainty, that is, non-probabilized risk, focussing on the Choquet-expected-utility model (CEU model henceforth).

We first consider a finite state space setting in which consequences are real number, say, monetary payoffs. We establish there that preference for portfolio diversification (*i.e.* convexity of the decision maker (DM henceforth) preferences) is equivalent to the agent having a convex capacity and a concave utility index. We then introduce a weaker notion of preference for diversification, *i.e.* preference for sure diversification. This property simply says that when indifferent between several assets, an agent should prefer a combination of these assets that yields a constant act to any of the ones used in the combination. We show that preference for sure diversification implies that the core of the capacity is non-empty. The converse holds true under the assumption that the utility index is concave.

This leads us to find conditions under which the utility index is concave. As it turns out, the concavity of the utility index is equivalent to a property we name comonotone diversification. This states that if two assets are indifferent and comonotone, then an agent prefers a combination of these assets to any of them. A CEU agent might exhibit preference for sure diversification but not comonotone diversification, as we make clear with an example. Conversely, it is clear that an agent exhibiting preference for comonotone diversification does not necessarily exhibit a preference for sure diversification.

A corollary to the previous result is that comonotone diversification and sure diversification is equivalent to the capacity having a non-empty core and the utility index being concave.

We then place ourselves in an Anscombe and Aumann (1963) setup, following the lead of Schmeidler (1989). In that setup consequences are lotteries. Schmeidler (1989) established that convexity of preferences (which is equivalent to preference for diversification as we defined it) is equivalent to convexity of the capacity. Schmeidler called this property *uncertainty aversion*. We show that preference for sure diversification, slightly adapted to fit the Anscombe-Aumann setting, is equivalent to non-empty core of the capacity. Preference for sure diversification in that particular setting is best understood (and labelled) as preference for sure “expected” utility diversification. Indeed, the axiom says that if indifferent among acts that yield lotteries and if a combination of these acts gives rise to an act paying off lotteries in different states among which the decision maker is indifferent, then the DM should prefer this combination to any of the initial acts. Hence, a CEU DM prefers an act which gives him, in expected utility terms, the same utility state by state. However, in each state, he still bears some risk, *i.e.*, which eventual (monetary) payoff the lottery will give.

Finally, we show that these different notions of diversification cannot be distinguished in the EU model, and are all equivalent to the concavity of the utility index.

Our contribution has some links with the recent debate around the definition and measurement of uncertainty aversion. Schmeidler (1989) provided an axiomatic definition of uncertainty aversion for his model (in an Anscombe-Aumann setting), showing that it is characterized by the convexity of the capacity. Wakker (1990) and Chateauneuf (1991) subsequently derived convexity of the capacity from axioms respectively labelled pessimism-independence and strong uncertainty aversion, that are strengthenings of comonotone independence used in the derivation of CEU. Ghirardato and Marinacci (1997) defined ambiguity aversion identifying *a priori* uncertainty neutrality with expected utility. They then show that this notion of ambiguity aversion is equivalent to non-empty core. Epstein (1999) based his definition of uncertainty aversion on the *a priori* identification of uncertainty neutrality with probabilistic sophistication. His notion of uncertainty aversion however cannot be directly linked to convexity of the capacity or non-emptiness of its core.

We view our contribution as a complement to these results. First, our notion of preference for sure diversification reflects a general notion of uncertainty aversion encompassing both aversion towards risk and towards ambiguity, *i.e.*, we do not attempt to disentangle risk from ambiguity. Although such a distinction is theoretically important, one could argue that in most “real-life” situations that distinction is not so clear.

Second, in an Anscombe-Aumann setting, where risk is treated via lotteries in the second stage, we find that preference for sure “expected” utility diversification is equivalent to non-empty core for a CEU DM. This property also characterizes ambiguity aversion according to Ghirardato and Marinacci (1997) where ambiguity aversion is defined comparatively, by factoring out risk attitudes. By giving a direct characterization of the non-emptiness of the core (in the Anscombe-Aumann setting), we provide another justification for this property.

Finally, although non-empty core characterizes a notion of preference for sure “expected” utility diversification in the Anscombe-Aumann setting, this is not the case for the notion of preference for sure diversification in a Savage like setup (that is, in which the consequence space is the set of real numbers). As recalled in Epstein (1999), the difference between the two setups should carefully be taken into account when transposing notions of uncertainty aversion from one to another. Indeed, in a Savage like setup, preference for sure diversification (which could be viewed as a notion of aversion towards uncertainty broadly defined) is stronger than non-empty core.

The paper is constructed as follows. We introduce the notation and recall some definitions in Section 2. Section 3 contains our main results in a Savage framework, while Section 4 studies notions of diversification in an Anscombe-Aumann setting. In Section 5 we characterize all three types of preference for diversification introduced in Section 3 in the expected utility model. Section 6 concludes. Proofs are gathered in an appendix.

2 Notation and definitions

There are k possible states of the world, indexed by superscript j. Let S be the set of states of the world and $\mathscr{A}$ the set of subsets of S.

Let $\succeq$ be the preference relation of a decision maker, defined on the set D of non-negative random variables on S. Say that two random variables C and C' are indifferent, that is $C \sim C'$, if $C \succeq C'$ and $C' \succeq C$. $C^j \in \mathbb{R}_+$ is wealth in state j.

As usual, say that an agent's preferences are

- *convex* if $\forall C, C' \in D,\ \forall \alpha \in [0,1],\quad C \succeq C' \Rightarrow \alpha C + (1-\alpha)C' \succeq C'$
- *continuous* if for all $x \in \mathbb{R}_+^k$, $\{C \in \mathbb{R}_+^k \mid C \succeq x\}$ and $\{C \in \mathbb{R}_+^k \mid x \succeq C\}$ are closed.
- *monotone* if $\forall C, C' \in D,\ C \geq C' \Rightarrow C \succeq C'$.

We focus on Choquet-Expected-Utility (Schmeidler, 1989). Preferences are then represented by the Choquet integral of a utility index u with respect to a capacity ν. The function u is cardinal *i.e.* defined up to a positive affine transformation.

A capacity is a set function $\nu : \mathscr{A} \to [0,1]$ such that $\nu\left(\emptyset\right) = 0$, $\nu(S) = 1$, and, for all $A, B \in \mathscr{A}, A \subset B \Rightarrow \nu(A) \leq \nu(B)$. We assume throughout that there exists $A \in \mathscr{A}$ such that $1 > \nu(A) > 0$.

A capacity ν is convex if for all $A, B \in \mathscr{A}$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$.

The core of a capacity ν is defined as follows

$$\text{core}(\nu) = \left\{ \pi \in \mathbb{R}_+^k \mid \sum_j \pi^j = 1 \text{ and } \pi(A) \geq \nu(A),\ \forall A \in \mathscr{A} \right\}$$

where $\pi(A) = \sum_{j \in A} \pi^j$. core($\nu$) is a compact, convex set which may be empty.

We now define the Choquet integral of $f \in \mathbb{R}^S$:

$$\int f d\nu \equiv E_\nu(f) = \int_{-\infty}^0 (\nu(f \geq t) - 1)\, dt + \int_0^\infty \nu(f \geq t)\, dt$$

Hence, if $f^j = f(j)$ is such that $f^1 \leq f^2 \leq \ldots \leq f^k$:

$$\int f d\nu = \sum_{j=1}^{k-1} \left[\nu\left(\{j, \ldots, k\}\right) - \nu\left(\{j+1, \ldots, k\}\right)\right] f^j + \nu\left(\{k\}\right) f^k$$

and, if we assume that an agent has wealth C^j in state j, and that $C^1 \leq \ldots \leq C^k$, then his preferences are represented by:

$$V(C) = [1-\nu(\{2,..,k\})]u(C^1) + ...[\nu(\{j,..,k\}) \\ -\nu(\{j+1,..,k\})]u(C^j) + ...\nu(\{k\})u(C^k)$$

It is well-known that when ν is convex, its core is non-empty and the Choquet integral of any random variable f is given by $\int f d\nu = \min_{\pi \in \text{core}(\nu)} E_\pi f$ (see Shapley, 1967, 1971; Rosenmueller, 1972; Schmeidler, 1986).

3 Convexity and the core

We now study the implications of different forms of diversification. We first define a natural notion of diversification (see also Dekel, 1989).

Definition 1. *$\succeq$ exhibits preference for diversification if for any $C_1, C_2, \ldots, C_n \in D$, and $\alpha_1, \ldots \alpha_n \geq 0$ such that $\sum_{i=1}^n \alpha_i = 1$.*

$$[C_1 \sim C_2 \sim \ldots \sim C_n] \Rightarrow \sum_{i=1}^n \alpha_i C_i \succeq C_\ell \ \ \forall \ell$$

For sake of completeness we recall that this notion of diversification is equivalent to convexity of preferences, that is, in our setup, equivalent to the quasi-concavity of V.

Proposition 1. *Let $\succeq$ be continuous and monotone. Then, the following two assertions are equivalent:*
(i) $\succeq$ exhibits preference for diversification
(ii) $\succeq$ is convex

The following result provides a characterization of CEU agents that are diversifiers. We establish that convexity of preferences is equivalent to the capacity being convex and the utility index being concave.

Theorem 1. *Assume $u : \mathbb{R}_+ \to \mathbb{R}$ to be continuous, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following statements are equivalent*
(i) $\succeq$ exhibits preference for diversification
(ii) V is concave
(iii) V is quasi-concave
(iv) u is concave and ν is convex.

This notion of diversification might seem fairly strong and we now introduce a weaker notion.

Definition 2. *$\succeq$ exhibits preference for sure diversification if for any $C_1, C_2, \ldots, C_n \in D$, $\alpha_1, \ldots, \alpha_n \geq 0$ such that $\sum_{\ell=1}^n \alpha_\ell = 1$, and $b \geq 0$:*

$$\left[C_1 \sim C_2 \sim \ldots \sim C_n, \text{ and } \sum_{\ell=1}^n \alpha_\ell C_\ell = b\mathbf{1}_S\right] \Rightarrow b\mathbf{1}_S \succeq C_\ell \ \ \forall \ell$$

Thus, sure diversification means that if the decision maker can attain certainty by a convex combination of equally desirable random variables, then he prefers certainty to any of these random variables. This axiom can be interpreted as an axiom of uncertainty aversion at large, reflecting the fact that the DM prefers total certainty. Observe that this axiom embodies a notion of aversion to ambiguity (*i.e.* imprecise probability) as well as a notion of aversion to risk.

Theorem 2. *Let a decision maker be a CEU maximizer with capacity ν and contινous, strictly increasing utility index u, differentiable on $\mathbb{R}_{++}$. Then,*

(i) $\succeq$ exhibits preference for sure diversification $\Rightarrow$ core(ν) $\neq \emptyset$.

(ii) If u is concave, core(ν) $\neq \emptyset \Rightarrow \succeq$ exhibits preference for sure diversification.

This theorem falls short of a complete characterization of sure diversification. Indeed, if the DM has a convex utility index, he might or might not be a sure diversifier even though *core*(ν) $\neq \emptyset$. The following two examples illustrate this point. In example 1, the DM has a capacity with a non-empty core and a convex utility index and is not a sure diversifier. In example 2, the DM also has a capacity with a non-empty core and a convex utility index, but this time he is a sure diversifier.

Example 1. Assume there are two states. Let $\nu^1 = \nu^2 = \frac{1}{3}$ and $u(x) = x^2$. core(ν) is obviously non-empty. However, $(1, 11) \sim (11, 1)$ and $\frac{1}{2}(1, 11) + \frac{1}{2}(11, 1) = (6, 6)$ but $v(6, 6) = 36 < v(1, 11) = 41$. $\Diamond$

Example 2. Assume there are two states, 1 and 2. Let $u(x) = 3x + \frac{1}{1+x}$ and $\nu^1 = \nu^2 = \frac{1}{4}$. u is strictly increasing, strictly convex.

We show that the set $\mathscr{C} = \{C = (C^1, C^2) \in \mathbb{R}_+^2 \mid C \sim a\mathbf{1}_S\}$ is above the hyperplane $\mathscr{H} = \left\{C = (C^1, C^2) \in \mathbb{R}_+^2 \mid \frac{1}{2}C^1 + \frac{1}{2}C^2 = a\right\}$. We then conclude that any sure convex combination of elements of $\mathscr{C}$ is preferred to $a\mathbf{1}_S$.

In order to show that the set $\mathscr{C}$ is above the hyperplane $\mathscr{H}$, it is enough to note that the indifference curve $\mathscr{C}$ consists of two concave curves, $\mathscr{C}_1 : C^2 = g_1(C^1),\ 0 \leq C^1 \leq a$ and $\mathscr{C}_2 : C^2 = g_2(C^1),\ a \leq C^1 \leq b$, such that the slope of the tangent to $\mathscr{C}_1$ for $C^1 = 0$ is smaller than -1, and, symmetrically, the slope of the tangent to $\mathscr{C}_2$ for $C^1 = b$ is greater than -1.

Notice that the existence of b and c such that $(0, c)$ and $(b, 0)$ belong to $\mathscr{C}$ follows from strict increasingness, continuity and unboundedness of u; concavity of g_1 and g_2 comes from convexity of u. Finally, straightforward computations yield that $g_1'(0) = \frac{-6}{u'(c)}$ and $g_2'(0) = \frac{-1}{6}u'(b)$. Since $u'(x) \leq 3\ \forall x \in \mathbb{R}_+$, it comes $g_1'(0) \leq -2$ and $g_2'(0) \geq \frac{-1}{2}$, and hence $g_1'(0) \leq -1$ and $g_2'(0) \geq -1$. Figure 1 illustrates this example. $\Diamond$

Now, the concavity of the utility index can be shown to be equivalent to a different form of diversification, from which any hedging is eliminated.

To define this notion of diversification, we first need to recall the definition of comonotony of random variables. Say that two random variables x and x' are comonotone if there is no s and s' such that $x(s) > x(s')$ and $x'(s') > x'(s)$.

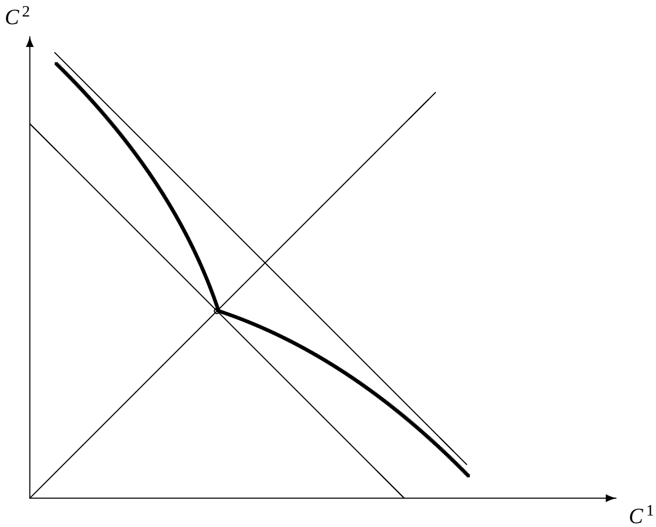

Figure 1

Definition 3. *A decision maker exhibits preference for comonotone diversification if for all comonotonic C and C' such that $C \sim C'$ one has $\lambda C + (1 - \lambda) C' \succeq C$ for all $\lambda \in (0, 1)$.*

Hence, comonotone diversification is nothing but convexity of preferences restricted to comonotone random variables. Note that any hedging (in the sense of Wakker, 1990) is prohibited in this diversification operation.

This type of diversification turns out to be equivalent, in the CEU model, to the concavity of u.

Theorem 3. *Let a decision maker be a CEU maximizer with capacity ν and continuous utility index u, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following two assertions are equivalent:*

(i) $\succeq$ *exhibits preference for comonotone diversification.*

(ii) u *is concave.*

Corollary 1. *Let a decision maker be a CEU maximizer with capacity ν and continuous utility index u, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following two assertions are equivalent:*

(i) $\succeq$ *exhibits preference for comonotone and sure diversification.*

(ii) u *is concave and core(ν) is non-empty.*

4 Diversification in an Anscombe and Aumann setting

We now take up the issue of diversification in an Anscombe-Aumann setting. Let Y be the set of distributions with finite support over $\mathbb{R}_+$. Let L be the set of acts, *i.e.*, functions from S to Y. In this framework, convex combinations in L are performed pointwise, *i.e.* for $f, g \in L$ and $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g = h$ where $h(s) = \alpha f(s) + (1-\alpha)g(s)$ on S.

The decision maker has preferences over L, which are represented, in the CEU model, by a unique capacity ν on $\mathscr{A}$ and an affine real valued function U on Y, that is, letting f and g be two acts

$$f \succeq g \;\Leftrightarrow\; W(f) = \int U(f(.))d\nu \geq W(g) = \int U(g(.))d\nu$$

Observe that $U(\alpha f(s)+(1-\alpha)g(s)) = \alpha U(f(s))+(1-\alpha)U(g(s))$. Finally, define $u : \mathbb{R}_+ \to \mathbb{R}$ by $u(x) = U(\delta_x)$ where δ_x is the degenerate lottery yielding the outcome $x \in \mathbb{R}_+$ for sure.

The following result, due to Schmeidler (1989), asserts that uncertainty aversion, defined as convexity of the preferences, is, in this Anscombe-Aumann setting, equivalent to the convexity of the capacity.

Theorem 4. *(Schmeidler, 1989) A binary relation $\succeq$ on L exhibits uncertainty aversion (i.e., for any $f, g \in L$ and $\alpha \in [0,1]$, if $f \succeq g$ then $\alpha f + (1-\alpha)g \succeq g$) if and only if the capacity ν is convex.*

Define now the adaptation of our preference for sure diversification to the present setup. Before that, notice that the relation $\succeq$ on L induces a relation, also denoted $\succeq$ on Y: if f is an act, say that $f(s) \succeq f(t)$ for $s, t \in S$ if the constant act yielding lottery $f(s)$ in all states is preferred to the constant act yielding lottery $f(t)$ in all states.

Definition 4. *$\succeq$ exhibits preference for sure "expected" utility diversification if for any $f_1, \ldots, f_r \in L$, $\alpha_1, \ldots, \alpha_r \geq 0$ such that $\sum_{\ell=1}^{r} \alpha_\ell = 1$:*

$$\left[f_1 \sim f_2 \sim \ldots \sim f_r, \text{ and } \sum_{\ell=1}^{r} \alpha_\ell f_\ell = f \text{ s.th. } f(s) \sim f(t)\ \forall s, t \in S \right] \Rightarrow f \succeq f_\ell \;\; \forall \ell$$

Observe that, in essence, preference for sure "expected" utility diversification is stronger than preference for sure diversification (that is, in which the condition $f(s) \sim f(t)$ is replaced by $f(s) = f(t)$), as the DM does not achieve a constant outcome but rather only a constant (expected) utility from the lotteries[1]. A decision maker who satisfies preference for sure "expected" diversification also satisfies preference for sure diversification, while the converse is obviously false. The following theorem proves that this notion is equivalent to non-empty core.

Theorem 5. *Let a decision maker be a CEU maximizer with capacity ν and an affine real valued utility function U defined on Y. Assume u is continuous and strictly increasing on $\mathbb{R}_+$. Then, the following assertions are equivalent:*

(i) $\succeq$ exhibits preference for sure "expected" utility diversification

(ii) core(ν) is non-empty

[1] Note that in the Savage like setup of the previous Section, in which consequences are real numbers, these two axioms are equivalent, since the DM is then indifferent between tw o consequences if and only if they are equal.

Observe that "risk-attitude", as reflected by properties of u does not play any role in this theorem.

5 Diversification with expected utility

We briefly discuss the implications of the different forms of diversification in the (subjective) expected utility model, in the setup of Section 3. It is well-known [although may be not in the finite case, for which the proof is more intricate, see Debreu and Koopmans (1982) and Wakker (1989)] that diversification (*i.e.* preference convexity) is equivalent to the concavity of the utility index. One can also deduce from Theorem 3 that comonotone diversification is equivalent to the concavity of the utility index in the EU model as well. Finally, sure diversification is also equivalent, in the EU model, to concavity of the utility index.

Proposition 2. *Let a decision maker be an EU maximizer with utility index* u*,* C^2 *on* $\mathbb{R}_{++}$*, strictly increasing and continuous on* $\mathbb{R}_+$*. Then, the following assertions are equivalent:*

(i) $\succeq$ *exhibits preference for diversification*
(ii) $\succeq$ *exhibits preference for sure diversification*
(iii) $\succeq$ *exhibits preference for comonotone diversification*
(iv) u *is concave*

In the EU model, the two forms of diversification we introduced, namely sure and comonotone diversification, are both represented by concavity of the utility index and consequently cannot be distinguished. Furthermore, they cannot be distinguished from the usual notion of diversification (*i.e.* convexity of the preferences).

6 Concluding remarks

Our goal in this paper is not to give here yet another definition of aversion to ambiguity, with the loose meaning that the DM would prefer to bet on events with known probability rather than on "ambiguous" events. In particular, the present study has not much to say on how to disentangle risk attitude from ambiguity attitude.

Schmeidler (1989) established in an Anscombe-Aumann setting that convexity of the preferences is equivalent to convexity of the capacity. He named that convexity property "uncertainty aversion" since it means that "smoothing or averaging *utility* distributions makes the decision maker better off" (Schmeidler, 1989, p.582, italics by the author). When moving away from the Anscombe-Aumann setup, our convexity axiom has a slightly different flavor since it means here that smoothing *consumption* distributions makes the decision maker better off. That convexity axiom is therefore stronger when applied to a Savage like setup. The characterization we obtain is also stronger than his since we *derive* concavity of the utility function as well.

Ghirardato and Marinacci (1997) defined a notion of ambiguity aversion and characterized it in the CEU model (among other models) and found that this notion is equivalent to non-empty core. Our sure diversification axiom also yields non-empty core, although it is stronger than this, since non-empty core may not in general imply sure diversification. Sure diversification is indeed stronger as it also embodies some notion of risk attitude as well as ambiguity attitude, whereas risk attitudes are factored out in Ghirardato and Marinacci (1997). The exact characterization of sure diversification in this setup is still an open issue. In an Anscombe-Aumann setting, in which risk is dealt with in the second stage, our adaptation of sure diversification, namely, preference for sure "expected" utility diversification, is equivalent to non-empty core.

Appendix : Proofs

Proof of Proposition 1

$(ii) \Rightarrow (i)$ Let $C_i \in D$, $i = 1, \ldots, n$ be such that $C_1 \sim \ldots \sim C_n$, and let us prove by induction on n that $\sum_i \alpha_i C_i \succeq C_1$. The result is straightforwardly true for $n = 2$. Assume it holds true for $n \geq 2$, and let us show it is true for $n + 1$. Let $C_1 \sim \ldots \sim C_n \sim C_{n+1}$ and $\alpha_i > 0$, $i = 1, \ldots, n+1$, $\sum_{i=1}^{n+1} \alpha_i = 1$. Define $\beta_i = \frac{\alpha_i}{1-\alpha_{n+1}}$, $i = 1, \ldots, n$. From the induction hypothesis, $\sum_{i=1}^{n} \beta_i C_i \succeq C_1$ and hence $\sum_{i=1}^{n} \beta_i C_i \succeq C_{n+1}$. Now, $\succeq$ convex implies $(1 - \alpha_{n+1}) \left(\sum_{i=1}^{n} \beta_i C_i\right) + \alpha_{n+1} C_{n+1} \succeq C_{n+1}$ that is $\sum_{i=1}^{n+1} \alpha_i C_i \succeq C_1$.

$(i) \Rightarrow (ii)$ What remains to be proved is that

$$C \succ C' \Rightarrow \alpha C + (1 - \alpha)C' \succeq C' \quad \text{where } \alpha \in [0, 1]$$

$\{\alpha \mid 0 \leq \alpha \leq 1, C' \succeq (1 - \alpha)C\} \neq \emptyset$ since $C' \geq 0$ implies $C' \succeq 0$ by monotonicity. Let $\varepsilon \in \mathbb{R}_+$ be defined by $\varepsilon = \inf\{\alpha, 0 \leq \alpha \leq 1, C' \succeq (1 - \alpha)C\}$. $\varepsilon > 0$ since $C \succ C'$. Let us show now that $(1 - \varepsilon)C \sim C'$. Let (ε_n) be a strictly increasing sequence converging towards ε. From the definition of ε, $(1 - \varepsilon_n)C \succ C'$, and from continuity $(1 - \varepsilon)C \succeq C'$. Therefore, $(1 - \varepsilon)C \sim C'$. Applying (i) gives $\alpha(1 - \varepsilon)C + (1 - \alpha)C' \succeq C'$ and hence by monotonicity $\alpha C + (1 - \alpha)C' \succeq C'$. □

Proof of Theorem 1

$(i) \Leftrightarrow (iii)$ follows from Proposition 1.
$(ii) \Rightarrow (iii)$ is well-known.

We now establish that $(iii) \Rightarrow (iv)$. We first show V quasi-concave implies ν convex. Convexity of ν is equivalent (see Shapley, 1971) to:

$$\forall A, B, E \in \mathscr{A} \text{ s.th. } B \subset A \text{ and } E \cap A = \emptyset,$$
$$\nu(A \cup E) - \nu(A) \geq \nu(B \cup E) - \nu(B) \tag{1}$$

Assume (1) is false, and let $A, B, E \in \mathscr{A}$ be such that:

$B \subset A$, $E \cap A = \emptyset$, and $(\nu(A \cup E) - \nu(A)) - (\nu(B \cup E) - \nu(B)) + \alpha < 0$ for some $\alpha > 0$.

Let $c \in \mathbb{R}_{++}$ be such that $u'(c) > 0$ and let $a, b \in \mathbb{R}_+$ satisfy $a < c < b$. Finally, let $F, E, A \setminus B, B$ be a partition of S and consider the following random variables:

	F	E	$A \setminus B$	B
C	a	$c - \varepsilon\alpha_1$	$c + \varepsilon\alpha_2$	b
C'	a	$c + \varepsilon\beta_1$	$c - \varepsilon\beta_2$	b

where $\varepsilon > 0$ is sufficiently small so that a and b are respectively the smallest and the largest value of C and C', and where

$$\alpha_1 = \nu(A) - \nu(B) \qquad \alpha_2 = \nu(A \cup E) - \nu(A) + \alpha$$
$$\beta_1 = \nu(A \cup E) - \nu(B \cup E) + \alpha \qquad \beta_2 = \nu(B \cup E) - \nu(B)$$

Let us assume, w.l.o.g., that $u(a) = 0$ and $u(b) = 1$. A straightforward computation yields, knowing that u is strictly increasing:

$$V\left(\frac{C + C'}{2}\right) < (\nu(A \cup E) - \nu(B))\, u(c) + \nu(B)$$

Now, one gets the following expression for $V(C)$:

$$\begin{aligned} V(C) &= (\alpha_2 - \alpha)\, u(c - \varepsilon\alpha_1) + \alpha_1 u(c + \varepsilon\alpha_2) + \nu(B) \\ &= (\alpha_2 - \alpha)[u(c) \\ &\quad -\varepsilon\alpha_1 u'(c) + \varepsilon\alpha_1(\varepsilon)] + \alpha_1\left[u(c) + \varepsilon\alpha_2 u'(c) + \varepsilon\alpha_2(\varepsilon)\right] + \nu(B) \\ &= (\nu(A \cup E) - \nu(B))\, u(c) + \nu(B) + \varepsilon\left[u'(c)\,\alpha_1\alpha + \alpha_3(\varepsilon)\right] \end{aligned}$$

where $\alpha_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ for $i = 1, 2, 3$. $\alpha_1 > 0$ since if it were not then (1) would be true by monotony of ν. Hence, $u'(c)\,\alpha_1\alpha > 0$, and therefore:

$$V(C) > (\nu(A \cup E) - \nu(B))\, u(c) + \nu(B)$$

for ε small enough.

A similar argument would establish the same inequality for $V\left(C'\right)$, and therefore we get:

$$V\left(\frac{C + C'}{2}\right) < \min\left(V(C), V\left(C'\right)\right)$$

that is, V not quasi-concave, a contradiction. We conclude that ν is convex.

Let us now show that V quasi-concave implies that u is concave. Recall first Theorem 2 in Debreu and Koopmans (1982):

> Let I and J be open intervals in $\mathbb{R}$, f and g functions that are non-constant on I and J and such that $F : I \times J \to \mathbb{R}$ defined by $F(x, y) = f(x) + g(y)$ is quasi-convex. Then, at least one of the two functions f or g is convex.

Let $a > 0$ and $A \in \mathscr{A}$ be chosen such that $0 < \nu(A) < 1$. Let $I \equiv]0, a[$ and $J \equiv]a, +\infty[$ and define F on $I \times J$ by $F(x, y) = V(x\mathbf{1}_{A^c} + y\mathbf{1}_A)$.

Clearly, F is quasi-concave and $F(x, y) = (1 - \nu(A))\,u(x) + \nu(A)\,u(y)$. Therefore, u is concave on $]0, a[$ or on $]a, +\infty[$ for all $a > 0$, hence on $]0, +\infty[$ since u is differentiable, and on $\mathbb{R}_+$ since u is continuous.

Finally, $(iv) \Rightarrow (ii)$. Indeed, $V(C)$ is then equal to $\min_{Q \in \text{core}(\nu)} \int u(C)\, dQ$ and is therefore concave being the minimum of a family of concave functions.□

Proof of Theorem 2

(i) Recall first (see Shapley, 1997) that core$(\nu) \neq \emptyset$ is equivalent to

$$\left[\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S,\ a_\ell \geq 0\right] \Rightarrow \sum_{\ell=1}^{r} a_\ell \nu(A_\ell) \leq 1, \quad \text{where } A_\ell \in \mathscr{A}$$

Let $A_\ell \in \mathscr{A}$, $a_\ell \geq 0$ be such that $\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S$. W.l.o.g., assume $a_\ell > 0$.

Assume there exists $x > 0$ such that $\sum_\ell a_\ell \nu(A_\ell) > 1 + x$. $\mathscr{L}$ will denote the set $\{\ell \mid \nu(A_\ell) > 0\}$. Let $a > 0$ be such that $u'(a) > 0$ and choose $\varepsilon > 0$ such that

$$\varepsilon(1 + x) \leq a \tag{2}$$

Define now the following positive random variables:

$$D_{\ell,\varepsilon} = [a - \varepsilon\nu(A_\ell)]\,\mathbf{1}_{A_\ell^c} + [a + \varepsilon(1 + x - \nu(A_\ell))]\,\mathbf{1}_{A_\ell}$$

Let $\alpha_\ell = \dfrac{a_\ell}{\sum_\ell a_\ell}$. A straightforward computation (recall that $\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S$) yields: $\sum_\ell \alpha_\ell D_{\ell,\varepsilon} = d(\varepsilon)\,\mathbf{1}_S$ where

$$d(\varepsilon) = a + \frac{\varepsilon}{\sum_\ell a_\ell}\left(1 + x - \sum_\ell a_\ell \nu(A_\ell)\right) < a$$

If $\ell \notin \mathscr{L}$, $\nu(A_\ell) = 0$ and clearly $V(D_{\ell,\varepsilon}) = u(a)$.

If $\ell \in \mathscr{L}$, a computation similar to the one of Theorem 1 yields:

$$V\left(D_{\ell,\varepsilon}\right) = u(a) + u'(a)\,\varepsilon\,(x\nu(A_\ell) + \alpha_\ell(\varepsilon))$$

where $\alpha_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$. By assumption, $u'(a) > 0$. Hence, $V\left(D_{\ell,\varepsilon}\right) > u(a)$ for all $\ell \in \mathscr{L}$ if $\varepsilon > 0$ is sufficiently small.

Let ε_0 be such an ε. For all $\ell \in \mathscr{L}$, consider the following random variables:

$$D'_{\ell,t_\ell} = D_{\ell,\varepsilon_0} - t_\ell \mathbf{1}_S$$

where $t_\ell \geq 0$ is chosen such that $D'_{\ell,t_\ell} \geq 0$. Let

$$\begin{aligned} g_\ell(t_\ell) \quad \equiv \quad & V\left(D'_{\ell,t_\ell}\right) = (1 - \nu(A_\ell))\,u(a - t_\ell - \varepsilon_0\nu(A_\ell)) \\ & + \nu(A_\ell)\,u(a - t_\ell + \varepsilon_0(1 + x - \nu(A_\ell))) \end{aligned}$$

g_ℓ is continuous, strictly decreasing, and $g_\ell(0) = V\left(D_{\ell,\varepsilon_0}\right) > u(a)$ as previously shown.

Let us now prove that there exists $t_\ell \geq 0$ such that $D'_{\ell,t_\ell} \geq 0$ and $g_\ell(t_\ell) \leq u(a)$. It is enough to show that there exists t_ℓ satisfying:

$$a - t_\ell - \varepsilon_0 \nu(A_\ell) \geq 0 \quad (3)$$
$$a - t_\ell + \varepsilon_0 (1 + x - \nu(A_\ell)) \geq 0 \quad (4)$$
$$-t_\ell + \varepsilon_0 (1 + x - \nu(A_\ell)) \leq 0 \quad (5)$$

Since (3) implies (4), it is enough to note that there exists $t_\ell \geq 0$ satisfying (3) and (5), *i.e.*, $t_\ell \geq 0$ such that $\varepsilon_0 (1 + x - \nu(A_\ell)) \leq t_\ell \leq a - \varepsilon_0 \nu(A_\ell)$. This proves to be true from (2).

Hence, for all $\ell \in \mathscr{L}$, there exists $\overline{t_\ell} > 0$ such that $D'_{\ell,\overline{t_\ell}} \geq 0$ and $D'_{\ell,\overline{t_\ell}} \sim a\mathbf{1}_S$. Let $C_\ell = D_{\ell,\varepsilon_0}$ if $\ell \notin \mathscr{L}$, and $C_\ell = D'_{\ell,\overline{t_\ell}}$ if $\ell \in \mathscr{L}$. Then $C_\ell \sim a\mathbf{1}_S$ for all ℓ, and $\sum_\ell \alpha_\ell C_\ell = b\mathbf{1}_S$, where $b \geq 0$, and $b = d(\varepsilon_0) - \sum_{\ell \in \mathscr{L}} \alpha_\ell \overline{t_\ell} < a$, a contradiction.

(ii) Suppose u concave and assume core$(\nu) \neq \emptyset$. Let C_ℓ, $\ell = 1, \ldots, n$ be such that $C_1 \sim C_2 \sim \ldots \sim C_n$ and $\sum_{\ell=1}^n \alpha_\ell C_\ell = b\mathbf{1}_S$, $\alpha_\ell \geq 0$, $\sum_{\ell=1}^n \alpha_\ell = 1$. Let $\pi \in \text{core}(\nu)$. Then, $\int u(C_\ell)\, \mathrm{d}\nu \leq E_\pi u(C_\ell)$ for all ℓ (see, *e.g.*, Proposition 2.1 in Chateauneuf, Dana and Tallon, 2000). Hence,

$$\sum_{\ell=1}^n \alpha_\ell \int u(C_\ell)\, \mathrm{d}\nu \leq \sum_{\ell=1}^n \alpha_\ell E_\pi u(C_\ell) \leq E_\pi u\left(\sum_{\ell=1}^n \alpha_\ell C_\ell\right) = u(b)$$

Therefore, $u(b) \geq \int u(C_\ell)\, \mathrm{d}\nu$ for all ℓ, *i.e.* $b\mathbf{1}_S \succeq C_\ell$ for all ℓ. □

Proof of Theorem 3

$[(i) \Rightarrow (ii)]$ The same argument as in the end of the proof of $(ii) \Rightarrow (iii)$ of Theorem 1 applies, since the random variables considered there, *i.e.* $x\mathbf{1}_{A^c} + y\mathbf{1}_A$ are comonotone.

$[(ii) \Rightarrow (i)]$ Let C and C' be two comonotone random variables such that $C \sim C'$, and $\lambda \in (0,1)$. Then, $\lambda E_\nu u(C) + (1-\lambda) E_\nu u\left(C'\right) = E_\nu[\lambda u(C) + (1-\lambda) u\left(C'\right)]$. This last expression is less than $E_\nu u\left(\lambda C + (1-\lambda) C'\right)$ by concavity of u and hence $\lambda C + (1-\lambda) C' \succeq C$. □

Proof of Theorem 5

$[(i) \Rightarrow (ii)]$ The argument of the proof is essentially the same as that of (i) of Theorem 2, and we only sketch the argument. Using the same notation as there, define the acts $f_{\ell,\varepsilon}$ as follows (recall u is strictly increasing and continuous, and take w.l.o.g. $u(0) = 0$):

$$f_{\ell,\varepsilon} = \begin{cases} \delta_{u^{-1}(a - \varepsilon\nu(A_\ell))} & \text{if } s \in A_\ell^c \\ \delta_{u^{-1}(a + \varepsilon(1 + x - \nu(A_\ell)))} & \text{if } s \in A_\ell \end{cases}$$

where $a > 0$ and $\varepsilon > 0$ such that $\varepsilon(1+x) \leq a$. Let $f_\varepsilon = \sum_\ell \alpha_\ell f_{\ell,\varepsilon}$ where $\alpha_\ell = \frac{a_\ell}{\sum_\ell a_\ell}$. A straightforward computation yields $U(f_{\ell,\varepsilon}(s)) = D_{\ell,\varepsilon}(s)$ for all $s \in S$ and hence $U(f_\varepsilon(s)) = d(\varepsilon)$ for all $s \in S$ and $f_\varepsilon(s) \sim f_\varepsilon(t)$ for all $s,t \in S$.

Observe that if $\ell \notin \mathscr{L}$, $W(f_{\ell,\varepsilon}) = a$ and if $\ell \in \mathscr{L}$

$$W(f_{\ell,\varepsilon}) = a - \varepsilon\nu(A_\ell) + \varepsilon(1+x)\nu(A_\ell) = a + \varepsilon x\nu(A_\ell)$$

Define next

$$f'_{\ell,t_\ell} = \begin{cases} \delta_{u^{-1}(a-\varepsilon\nu(A_\ell)-t_\ell)} & \text{if } s \in A_\ell^c \\ \delta_{u^{-1}(a+\varepsilon(1+x-\nu(A_\ell))-t_\ell)} & \text{if } s \in A_\ell \end{cases}$$

for $t_\ell > 0$ sufficiently small such that $f'_{\ell,t_\ell} \in L$. Note that $W(f'_{\ell,t_\ell}) = a + \varepsilon x\nu(a_\ell) - t_\ell$.

Define $\overline{t_\ell} = \varepsilon x\nu(a_\ell)$ and note that $W(f'_{\ell,\overline{t_\ell}}) = a$.

Now, let $f_\ell = f_{\ell,\varepsilon}$ if $\ell \notin \mathscr{L}$ and $f_\ell = f'_{\ell,\overline{t_\ell}}$ if $\ell \in \mathscr{L}$. Then, $W(f_\ell) = a$ for all ℓ. Letting $f = \sum_\ell \alpha_\ell f_\ell$, one gets that $U(f(s)) = b$ for all $s \in S$, where $b = d(\varepsilon) - \sum_{\ell\in\mathscr{L}} \alpha_\ell \overline{t_\ell} < a$. Hence, $W(f) < W(f_\ell)$ for all ℓ, a contradiction.

[$(ii) \Rightarrow (i)$] Suppose core$(\nu) \neq \emptyset$. Let $f_\ell \in L$, $\ell = 1,\ldots,r$ be such that $f_1 \sim \ldots \sim f_r$ and $f = \sum_{\ell=1}^r \alpha_\ell f_\ell$ be such that $f(s) \sim f(t)$ for all $s,t \in S$. Let $\pi \in$ core(ν). Then, $\int U(f_\ell)d\nu \leq \int U(f_\ell)d\pi$ for all ℓ. Hence,

$$\begin{aligned} \sum_{\ell=1}^r \alpha_\ell \int U(f_\ell)d\nu \leq \sum_{\ell=1}^r \alpha_\ell \int U(f_\ell)d\pi &= \int U(\sum_{\ell=1}^r \alpha_\ell f_\ell)d\pi \\ &= \int U(f)d\pi = \int U(f)d\nu \end{aligned}$$

and therefore $f \succeq f_\ell$ for all ℓ. □

Proof of Proposition 2

The following implications are straightforward, [$(iv) \Rightarrow (i)$], [$(i) \Rightarrow (ii)$], and [$(i) \Rightarrow (iii)$]. [$(iii) \Rightarrow (iv)$] follows from Theorem 3.

What remains to be proved is [$(ii) \Rightarrow (iv)$]. To that effect, suppose u is not concave on $\mathbb{R}_+$. Hence, there exists $x_0 \in \mathbb{R}_{++}$ such that $u''(x_0) > 0$, and therefore there exist $a,b \in \mathbb{R}_{++}$, $a < b$, such that $u''(x) > 0$ on $[a,b]$. u is hence strictly convex on $[a,b]$.

Let A and A^c be events with probability π and $1-\pi$ such that $0 < \pi \leq 1-\pi$.

Now, since u is strictly increasing, continuous and $\pi \leq 1/2$, there exists $a' \in \mathbb{R}_+$, $b > a' \geq a$ such that

$$\pi u(a) + (1-\pi)u(b) = \pi u(b) + (1-\pi)u(a')$$

Consider now the following two acts $C_1 = a\mathbf{1}_A + b\mathbf{1}_{A^c}$ and $C_2 = b\mathbf{1}_A + a'\mathbf{1}_{A^c}$. Notice that $C_1 \sim C_2$.

Let $\alpha = \frac{b-a'}{b-a'+b-a} \in (0,1)$. A straightforward computation gives:

$$\alpha C_1 + (1-\alpha)C_2 = k\mathbf{1}_S \quad \text{with} \quad k = \frac{b^2 - aa'}{2b - a - a'} \in \mathbb{R}_{++}$$

But $u(k) = E(u(\alpha C_1 + (1-\alpha)C_2)) < \alpha E(u(C_1)) + (1-\alpha)E(u(C_2))$ by strict convexity of u on $[a, b]$. $E(u(C_1)) = E(u(C_2))$ then implies $C_1 \succ k\mathbf{1}_S$, a contradiction. □

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