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Monotone continuous multiple priors*

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Summary. In a multiple priors model à la Gilboa and Schmeidler (1989), we provide necessary and sufficient behavioral conditions ensuring the countable additivity and non-atomicity of all priors.

Keywords and Phrases: Multiple priors, Countable additivity, Non-atomicity.

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1 Introduction

Decision theorists have often debated whether to use countably or finitely additive probabilities to model decision makers' subjective beliefs. The two most notable advocates of finite additivity were de Finetti and Savage, who argued that countable additivity is a purely technical property devoid of a clear behavioral content and whose assumption prevents the analysis of significant phenomena (see Savage, 1954; Finetti 1931, 1970).

On the other hand, countable additivity is a very convenient property, which leads to many important results in probability theory like, for example, the classic limit laws. As a result, its use is pervasive in mathematical economics and finance.

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For a decision theorist the problem is to understand whether the added analytic power of countable additivity offsets its supposed shaky behavioral underpinning.

Arrow (1970) provided an important contribution to this issue by identifying the precise behavioral conditions under which subjective beliefs can be represented by a countably additive probability. Building on Villegas (1964), Arrow (1970) obtained a subjective expected utility representation with a countably additive probability by adding the following monotone continuity axiom to a set of standard Savage-type axioms.

Axiom 1 (Monotone Continuity) *Given any acts $f \succ g$ in L, consequence x in X, and sequence of events $\{E_n\}_{n \geq 1}$ in Σ with $E_1 \supseteq E_2 \supseteq ...$ and $\bigcap_{n \geq 1} E_n = \emptyset$, there exists $\bar{n} \geq 1$ such that*

$$\begin{bmatrix} x & \text{if } s \in E_{\bar{n}} \\ f(s) & \text{if } s \notin E_{\bar{n}} \end{bmatrix} \succ g \quad \text{and} \quad f \succ \begin{bmatrix} x & \text{if } s \in E_{\bar{n}} \\ g(s) & \text{if } s \notin E_{\bar{n}} \end{bmatrix}.$$

Arrow showed that monotone continuity is the behavioral condition which underlies the use of countably additive probabilities in subjective expected utility theory. The question is, therefore, whether or not monotone continuity is also a sensible behavioral property. It is not, however, our purpose to expatiate on this, ultimately subjective, issue.[1]

In contrast, our aim is to study the implications of monotone continuity for the *multiple priors model*, a popular generalization of subjective expected utility theory axiomatized by Gilboa and Schmeidler (1989). In this model the decision makers' beliefs are represented by a set C of priors in order to capture the vagueness of beliefs (also called *ambiguity*), and acts are ranked according to the minimum expected utilities with respect to C or, more generally, according to a weighted average of the minimum and the maximum expected utilities with respect to C. Conventional subjective expected utility theory is the special case in which the set of priors C is a singleton, modelling in this way a situation where there is no vagueness.

Not surprisingly, countable additivity turns out to be a very convenient property in applications of the multiple priors model. For example, the recent applications in economics and finance of Epstein and Wang (1994, 1995), Billot, Chateauneuf, Gilboa, and Tallon (2000), Delbaen (2002), and Chen and Epstein (2002) critically depend on the countable additivity of the probabilities forming the decision makers' set of priors, and on some compactness properties of such a set.

It is natural to wonder whether such a convenient property has its behavioral counterpart in the monotone continuity of preferences. Theorem 1 below shows that, fortunately, this is indeed the case. In particular, a preference relation having a multiple priors representation is monotone continuous if and only if the set of priors is a relatively weak compact subset of countably additive probabilities.

As well-known, the subjective probability derived in Savage (1954) is convex ranged. This is another convenient property, which recently has been used in a

[1] The reader may find interesting this quotation from Arrow (1970) "the assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem."

multiple priors setting (see Nehring, 2001; Amarante, 2002, 2003). Theorem 2 below extends another classic result of Villegas (1964) by showing that a simple atomlessness property of preferences is a necessary and sufficient condition for the range convexity of all the priors in C. All proofs are relegated to the Appendix.

2 Set-up

2.1 Mathematical preliminaries

Throughout the paper, Σ is a σ-algebra of subsets of a space S. Subsets of S are understood to be in Σ even where not stated explicitly.

We denote by $ba(\Sigma)$ and $ca(\Sigma)$, respectively, the vector spaces of finitely additive and countably additive bounded real-valued set functions on Σ; we call *charges* the elements of $ba(\Sigma)$ and *measures* the elements of $ca(\Sigma)$. Clearly, $ca(\Sigma)$ is a vector subspace of $ba(\Sigma)$. In particular, both $ca(\Sigma)$ and $ba(\Sigma)$ become Banach spaces when equipped with the variation norm. An element μ of $ca(\Sigma)$ is *non-atomic* if, for all $A \in \Sigma$ with $\mu(A) \neq 0$, there exists $B \in \Sigma$ such that $B \subseteq A$ and $\mu(A) \neq \mu(B) \neq 0$; non-atomic elements of $ca(\Sigma)$ form a closed subspace of $ca(\Sigma)$.

We denote by $B(\Sigma)$ the set of all bounded and Σ-measurable functions $\varphi : S \to \mathbb{R}$. The vector space $B(\Sigma)$ is a Banach space with respect to the supnorm $\|\cdot\|_s$. The standard duality between $ba(\Sigma)$ and $B(\Sigma)$ endows $ba(\Sigma)$ and its subsets of a weak* topology.[2]

Finally, $ba^1(\Sigma)$ and $ca^1(\Sigma)$ denote, respectively, the sets of probabilities in $ba(\Sigma)$ and $ca(\Sigma)$; we reserve the letter P for elements of $ba^1(\Sigma)$ and $ca^1(\Sigma)$.

2.2 Decision-theoretic preliminaries

States of nature and *events* are represented by the pair (S, Σ), while X is the space of *consequences*. An *act* is a map $f : S \to X$ and it is *simple* when it is finite valued; L_0 denotes the set of all simple Σ-measurable acts. The decision maker has a preference relation $\succsim$ on L_0, which in turn induces a preference over X, obtained in the standard way by identifying consequences with constant acts.

A binary relation $\succsim$ on L_0 is an *α-maximin expected utility* (α-MEU) preference relation if there exist a utility index $u : X \to \mathbb{R}$, a non-empty set $C \subseteq ba^1(\Sigma)$ and a constant $\alpha \in [0, 1]$ such that $\succsim$ is represented by the preference functional $V : L_0 \to \mathbb{R}$ defined by

$$V(f) = \alpha \inf_{P \in C} \int u(f(s))\, dP(s) + (1-\alpha) \sup_{P \in C} \int u(f(s))\, dP(s) \qquad (2.1)$$

for all $f \in L_0$. When $C = \{P\}$ is a singleton, α-MEU preferences collapse to the *Subjective Expected Utility* (SEU) case $V(f) = \int u(f(s))\, dP(s)$.

[2] See, e.g., Aliprantis and Border (1999), p. 457.

We assume that the range $u(X)$ of u is not a nowhere dense subset of $\mathbb{R}$, that is, the interior of the closure $\overline{u(X)}$ is non-empty. This is obviously the case when $u(X)$ is an interval of $\mathbb{R}$; for instance, when X is a convex set and u is non-constant and affine, or when X is a connected topological space and u is non-constant and continuous. This assumption implies that X has to be at least countably infinite.

Notice that, given any set of priors $C \subseteq ba^1(\Sigma)$ and denoted by $\overline{co}^{w^*}(C)$ its weak* closed convex hull, we have

$$V(f) = \alpha \min_{P \in \overline{co}^{w^*}(C)} \int u(f(s))\, dP(s) + (1-\alpha) \max_{P \in \overline{co}^{w^*}(C)} \int u(f(s))\, dP(s). \tag{2.2}$$

For this reason the set C itself is often assumed to be convex and weak* closed.

Axiomatic characterizations of this kind of preferences for $\alpha = 1$ (MEU) can be found in Gilboa and Schmeidler (1989), Casadesus-Masanell, Klibanoff, and Ozdenoren (2000), and Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003), while the general case of $\alpha \in [0, 1]$ is considered in Ghirardato, Maccheroni, and Marinacci (2003) and Kopylov (2002).

Let L be the set of all acts $f : S \to X$ that are both *preference measurable*, i.e., $\{s \in S : f(s) \succsim x\}$ and $\{s \in S : f(s) \succ x\}$ belong to Σ for all x in X, and *preference bounded*, i.e., there exist $\underline{x}$ and $\overline{x}$ in X such that $\overline{x} \succsim f(s) \succsim \underline{x}$ for all $s \in S$. Since for all $f \in L$ we have $u \circ f \in B(\Sigma)$, the natural extension of the functional V defined by (2.1) from L_0 to L allows to extend $\succsim$ from L_0 to L too; the extensions of V and $\succsim$ to L are still denoted by V and $\succsim$.

3 Monotone continuity

We can now state our main result.

Theorem 1 *Let $\succsim$ be an α-MEU preference relation on L, with a set C of priors. Then, the following conditions are equivalent:*

(i) $\succsim$ is monotone continuous.
(ii) C is a relatively weak compact subset of $ca(\Sigma)$.

If, in addition, C is weak closed, then (i) is equivalent to:*

(iii) C is a subset of $ca(\Sigma)$.

This theorem generalizes the aforementioned results of Arrow and Villegas, who dealt with singleton sets of priors. It is also related to some other results in the literature. Schmeidler (1972) p. 220 noticed that the core of a continuous exact game is a weak sequentially compact subset of $ca^1(\Sigma)$, while Epstein and Wang (1995) p. 44 showed that the set of priors is a weak sequentially compact subset of $ca(\Sigma)$ when the MEU functional $\min_{P \in C} \int u(f)\, dP$ is continuous at certainty. Finally, Marinacci (2002) showed that C is included in $ca(\Sigma)$ whenever $\succsim$ is monotone continuous.

The Eberlein-Smulian Theorem, Theorems IV.9.1 and IV.9.2 of Dunford and Schwartz (1958), and Lemmas A.3 and A.4 provide several topological conditions equivalent to (ii).

4 Range convexity

By using bets, the preference $\succsim$ on L induces in a well-known way a likelihood ordering $\succsim_l$ on the event σ-algebra Σ, which takes the form $A \succsim_l B$ if and only if

$$\alpha \inf_{P \in C} P(A) + (1-\alpha) \sup_{P \in C} P(A) \geq \alpha \inf_{P \in C} P(B) + (1-\alpha) \sup_{P \in C} P(B).$$

Villegas (1964)'s results imply that for a standard monotone continuous SEU ordering $\succsim$, the single probability measure P that represents $\succsim_l$ is non-atomic if and only if $\succsim_l$ satisfies the following condition.

Axiom 2 (Downward Atomlessness) *If $A \succ_l \emptyset$, there exists $B \subseteq A$ such that $A \succ_l B \succ_l \emptyset$.*

In the standard SEU case, in which C is a singleton, downward atomlessness is equivalent to:

Axiom 3 (Upward Atomlessness) *If $A \prec_l S$, there exists $B \supseteq A$ such that $A \prec_l B \prec_l S$.*

For $\alpha \in (0,1)$, downward and upward atomlessness always coincide; for $\alpha \in \{0,1\}$, some further conditions are needed (see Lemma A.7). The next result shows that downward atomlessness is the appropriate non-atomicity requirement for 0-MEU preferences, upward atomlessness is the appropriate one for 1-MEU preferences, and either one works for α-MEU preferences when $\alpha \in (0,1)$.

Theorem 2 *Let $\succsim$ be a monotone continuous α-MEU preference relation on L, with a set C of priors. If $\alpha \neq 1$ ($\alpha \neq 0$, resp.), the following conditions are equivalent:*

(i) $\succsim$ is downward atomless (upward, resp.),
(ii) all priors P in C are non-atomic.

A Proofs and related material

A.1 Compactness

The following result - essentially due to Bartle, Dunford and Schwartz (1955) - shows a noteworthy relation existing between compactness in the weak and weak* topologies of $ca(\Sigma)$. It can be proved by standard Banach lattice techniques.[3]

Lemma 3 *Let C be a subset of $ca(\Sigma)$. Then, the following statements are equivalent:*

(i) C is weak closed and relatively weak compact.*
(ii) C is weak closed and norm bounded.*
(iii) C is weak compact.*

[3] See, e.g., Aliprantis and Burkinshaw, 1985, Chapter 4, and especially Section 13.

(iv) *C is weak compact.*

Moreover, if $C \subseteq ca^1(\Sigma)$ *is convex and (i) holds, then there exists* $P_0 \in C$ *such that for all* $P \in C$ *we have* $P(A) = 0$ *whenever* $P_0(A) = 0$.

Another useful lemma, which is essentially Theorem IV.9.1 of Dunford and Schwartz (1958).

Lemma 4 *Let C be a subset of* $ca(\Sigma)$*. The following facts are equivalent:*

(i) *C is relatively weak compact.*
(ii) *C is bounded and* $\sup_{\mu \in C} |\mu(A_n)| \to 0$ *whenever* $A_n \downarrow \emptyset$.

A.2 Monotone continuity

If $f, g \in L$ and $A \in \Sigma$, we set

$$fAg(s) = \begin{cases} f(s) & s \in A \\ g(s) & s \in A^c \end{cases}.$$

Clearly, $fAg \in L$.

Lemma 5 *Let* $\succsim$ *be an* α*-MEU preference relation on L, with a set C of priors. If* $\succsim$ *is monotone continuous, then C is a relatively weak compact subset of* $ca(\Sigma)$.

Proof. Let $\alpha \neq 1$. Choose $y, z \in X$ such that $y \succ z$ and there exists a sequence $\{z_k\}_{k \geq 1}$ of consequences such that $z_k \succ z_{k+1} \succ z$ for all $k \geq 1$, and $\lim_{k \to \infty} u(z_k) = u(z)$. W.l.o.g., set $u(y) = 1$ and $u(z) = 0$. If $E_n \downarrow \emptyset$, by monotone continuity, for all $k \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ such that $yE_{\bar{n}}z \prec z_k$. That is, $\alpha \inf_{P \in C} P(E_{\bar{n}}) + (1-\alpha)\sup_{P \in C} P(E_{\bar{n}}) < u(z_k)$. As the sequence $\alpha \inf_{P \in C} P(E_n) + (1-\alpha)\sup_{P \in C} P(E_n)$ is decreasing, this implies

$$\lim_{n \to \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1-\alpha) \sup_{P \in C} P(E_n) \right) < u(z_k).$$

Passing to the limit for $k \to \infty$, we get

$$\lim_{n \to \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1-\alpha) \sup_{P \in C} P(E_n) \right) = 0. \tag{A.3}$$

As $0 \leq \inf_{P \in C} P(E_n) \leq \alpha \inf_{P \in C} P(E_n) + (1-\alpha)\sup_{P \in C} P(E_n)$, by (A.3) $\lim_{n \to \infty} (\inf_{P \in C} P(E_n)) = 0$. Therefore,

$$\begin{aligned} 0 &= \lim_{n \to \infty} \left(\alpha \inf_{P \in C} P(E_n) + (1-\alpha) \sup_{P \in C} P(E_n) \right) \\ &= \alpha \lim_{n \to \infty} \inf_{P \in C} P(E_n) + (1-\alpha) \lim_{n \to \infty} \sup_{P \in C} P(E_n) \\ &= (1-\alpha) \lim_{n \to \infty} \sup_{P \in C} P(E_n), \end{aligned}$$

and we can conclude

$$\lim_{n\to\infty} \sup_{P\in C} P(E_n) = 0. \tag{A.4}$$

Hence, for all $E_n \downarrow \emptyset$ and all $Q \in C$, $0 \le Q(E_n) \le \sup_{P\in C} P(E_n)$ implies $Q(E_n) \downarrow 0$, so that $C \subseteq ca(\Sigma)$. Eq. (A.4) yields relative weak compactness by Lemma 4.

If $\alpha = 1$, define $f \succsim' g$ iff $g \succsim f$ to obtain a monotone continuous 0-MEU preference on L with set of priors C (and utility index $-u$). □

Lemma 6 *Let $\succsim$ be an α-MEU preference relation on L, with a set C of priors. If C is a relatively weak compact subset of $ca(\Sigma)$, then $\succsim$ is monotone continuous.*

Proof. Let $f, g \in L$ with $f \succ g$, $x \in X$, and $\Sigma \ni E_n \downarrow \emptyset$. For all $\varepsilon > 0$,

$$\begin{aligned} A_n &= \{s \in S : |u(f(s)) - u(xE_n f(s))| > \varepsilon\} \\ &= E_n \cap \{s \in S : |u(f(s)) - u(x)| > \varepsilon\} \downarrow \emptyset. \end{aligned}$$

Then, by Lemma 4, $\lim_n (\sup_{P\in C} P(A_n)) = 0$. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that $P(A_n) < \varepsilon$ for all $n \ge n_\varepsilon$ and all $P \in C$. Let $M = \|u \circ f - u(x)\|_s$. It holds:

$$\begin{aligned} \int |u\circ f - u\circ(xE_n f)|\, dP &= \int_{A_n} |u\circ f - u\circ(xE_n f)|\, dP \\ &\quad + \int_{A_n^c} |u\circ f - u\circ(xE_n f)|\, dP \le M\varepsilon + \varepsilon \end{aligned}$$

for all $n \ge n_\varepsilon$ and all $P \in C$. Then $\int u\circ(xE_n f)\, dP \to \int u\circ f dP$ uniformly with respect to $P \in C$. Whence

$$\begin{aligned} &\inf_{P\in C} \int u\circ(xE_n f)\, dP \to \inf_{P\in C} \int u\circ f dP \quad \text{and} \\ &\sup_{P\in C} \int u\circ(xE_n f)\, dP \to \sup_{P\in C} \int u\circ f dP, \end{aligned}$$

and so

$$V(xE_n f) \to V(f) > V(g).$$

Analogously, $V(xE_n g) \to V(g) < V(f)$, as desired. □

Lemmas 3, 5, and 6 immediately yield Theorem 1.

A.3 Range convexity

Proof of Theorem 2. By Lemma 3, there exists $P_0 \in \overline{co}^{w^*}(C)$ such that $P \ll P_0$ for all $P \in \overline{co}^{w^*}(C)$.

Let $\alpha \neq 1$ and let $\succsim_l$ be downward atomless. We show that P_0 is non-atomic. Suppose, *per contra*, that A is an atom for P_0. Then $P_0(A) > 0$, $A \succ_l \emptyset$,[4] and for all $B \subseteq A$, either $P_0(B) = 0$ or $P_0(B) = P_0(A)$. In the former case, $P(B) = 0$ for all $P \in C$, so that $B \sim_l \emptyset$; in the latter case, $P_0(A - B) = 0$, so that $P(A) = P(B)$ for all $P \in C$, and so $B \sim_l A$. This is a contradiction, since $\succsim_l$ is downward atomless. Therefore, P_0 is non-atomic. As a consequence any $P \in C$ is non-atomic since $P \ll P_0$ (see, e.g., Marinacci, 1999, p. 360).

Conversely, let $A \in \Sigma$ be such that $A \succ_l \emptyset$. Then $\sup_{P \in C} P(A) > 0$, so that $P_0(A) > 0$. Since P_0 is non-atomic, there exists a decreasing sequence $B'_n \downarrow B'$ such that $B'_n \subseteq A$ for all $n \in \mathbb{N}$ and $P_0(B'_n) = \frac{1}{2^n} P_0(A)$. Then, the sequence $B_n = B'_n - B'$ decreases to $\emptyset$, with $B_n \subseteq A$ for all $n \in \mathbb{N}$, and $P_0(B_n) = \frac{1}{2^n} P_0(A)$. Thus $\sup_{P \in C} P(B_n) \downarrow 0$, which implies

$$\alpha \inf_{P \in C} P(B_n) + (1-\alpha) \sup_{P \in C} P(B_n) \downarrow 0,$$

and

$$\begin{aligned} &\alpha \inf_{P \in C} P(B_n) + (1-\alpha) \sup_{P \in C} P(B_n) \geq (1-\alpha) \sup_{P \in C} P(B_n) \\ &= (1-\alpha) \max_{P \in \overline{co}^{w^*}(C)} P(B_n) \geq \frac{1-\alpha}{2^n} P_0(A) > 0. \end{aligned}$$

For n large enough,

$$\alpha \inf_{P \in C} P(A) + (1-\alpha) \sup_{P \in C} P(A) > \alpha \inf_{P \in C} P(B_n) + (1-\alpha) \sup_{P \in C} P(B_n) > 0,$$

that is, $A \succ_l B_n \succ_l \emptyset$.

Let $\alpha \neq 0$, and consider the dual likelihood relation $A \succsim^l B$ iff $B^c \succsim_l A^c$. Set $\beta = 1 - \alpha \in [0, 1)$ and notice that $\succsim_l$ is upward atomless iff $\succsim^l$ is downward atomless, and that

$$\begin{aligned} A \succsim^l B &\Leftrightarrow \alpha \inf_{P \in C} P(A^c) + (1-\alpha) \sup_{P \in C} P(A^c) \\ &\leq \alpha \inf_{P \in C} P(B^c) + (1-\alpha) \sup_{P \in C} P(B^c) \\ &\Leftrightarrow \beta \inf_{P \in C} P(A) + (1-\beta) \sup_{P \in C} P(A) \\ &\geq \beta \inf_{P \in C} P(B) + (1-\beta) \sup_{P \in C} P(B). \end{aligned}$$

As a result, if $\succsim_l$ is upward atomless, then the argument used in the case $\alpha \neq 1$, when applied to $\succsim^l$, shows that C consists of non-atomic measures. Conversely, if C consists of non-atomic measures, the argument used in the case $\alpha \neq 1$, shows that $\succsim^l$ is downward atomless and $\succsim_l$ is upward atomless. □

[4] $\alpha \inf_{P \in C} P(A) + (1-\alpha) \sup_{P \in C} P(A) = \alpha \min_{P \in \overline{co}^{w^*}(C)} P(A) + (1-\alpha) \max_{P \in \overline{co}^{w^*}(C)} P(A)$

Lemma 7 *Let $\succsim$ be a monotone continuous α-MEU preference relation on L, with a set C of priors.*

(a) If $\alpha \in (0,1)$, then $\succsim_l$ is downward atomless iff it is upward atomless.
(b) If $\alpha \in \{0,1\}$, then, downward and upward atomlessness coincide provided that, for any $A \in \Sigma$, we have $A^c \sim_l \emptyset$ if and only if $A \sim_l S$.

Proof. A direct proof of (a), not building on monotone continuity is possible. But, under monotone continuity the result immediately follows from Theorem 2. Next we prove (b).

Suppose $\alpha = 1$. We first show that downward atomlessness implies upward atomlessness. Let $\inf_{P\in C} P(A) < 1$. Then $\inf_{P\in C} P(A^c) > 0$. In fact, $\inf_{P\in C} P(A^c) = 0$ implies $A^c \sim_l \emptyset$. Then there exists $B^c \subseteq A^c$ such that $0 < \inf_{P\in C} P(B^c) < \inf_{P\in C} P(A^c)$. But, $A \subseteq B$ implies $\inf_{P\in C} P(A) \leq \inf_{P\in C} P(B) < 1$. If $\inf_{P\in C} P(A) = \inf_{P\in C} P(B)$, then

$$0 \leq \inf_{P\in C} P(B-A) = \inf_{P\in C} (P(B) - P(A)) \leq$$
$$\leq \inf_{P\in C} P(B) - \inf_{P\in C} P(A) = 0,$$

so that $\inf_{P\in C} P((B-A)^c) = 1$. In turn, this implies $1 - P(B) + P(A) = 1$ for all $P \in C$, i.e., $P(A) = P(B)$ for all $P \in C$. Hence, $P(A^c) = P(B^c)$ for all $P \in C$, and so $\inf_{P\in C} P(A^c) = \inf_{P\in C} P(B^c)$, a contradiction.

As to the other implication, let $\inf_{P\in C} P(A) > 0$. Then $\inf_{P\in C} P(A^c) < 1$, so that there exists $B^c \supseteq A^c$ such that $\inf_{P\in C} P(A^c) < \inf_{P\in C} P(B^c) < 1$. This implies $0 < \inf_{P\in C} P(B) \leq \inf_{P\in C} P(A)$. If $\inf_{P\in C} P(A) = \inf_{P\in C} P(B)$, we can proceed as before (exchanging the roles of B and A) to reach a contradiction.

Suppose $\alpha = 0$. Consider the dual likelihood relation $A \succsim^l B$ iff $B^c \succsim_l A^c$. Notice that,

1. $A \succsim^l B \Leftrightarrow \inf_{P\in C} P(A) \geq \inf_{P\in C} P(B)$.
2. $\succsim_l$ is upward atomless iff $\succsim^l$ is downward atomless.
3. $\succsim_l$ is downward atomless iff $\succsim^l$ is upward atomless.
4. $A^c \sim^l \emptyset$ if and only if $A \sim_l S$ if and only if $A^c \sim_l \emptyset$ if and only if $A \sim^l S$.

Hence, the argument used for $\alpha = 1$, when applied to $\succsim^l$ shows that $\succsim^l$ is downward atomless iff $\succsim^l$ is upward atomless at S, and the same is true for $\succsim_l$. □

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