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Unawareness and bankruptcy: A general equilibrium model★

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Summary. We present a consistent pure-exchange general equilibrium model where agents may not be able to foresee all possible future contingencies. In this context, even with nominal assets and complete asset markets, an equilibrium may not exist without appropriate assumptions. Specific examples are provided.

An existence result is proved under the main assumption that there are sufficiently many states that all the agents foresee. An intrinsic feature of the model is bankruptcy, which agents may involuntarily experience in the unforeseen states.

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1 Introduction

In existing general equilibrium models all agents are assumed to perceive uncertainty as being represented by the same all-inclusive 'objective' state space, say S. If one imagines occurrence of each $s \in S$ as being determined by the truth value of a set of facts – the sources of uncertainty – the above assumption amounts to supposing that all agents take into account all

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relevant facts in making their plans. This paper presents a pure exchange, two-period model in which this assumption is dropped, so that an agent h perceives uncertainty as represented by the subjective space which he can construct on the basis of the facts which he can think of, i.e. of which he is aware [cfr. Modica-Rustichini (1994a, b) for a discussion of a modal logic analysis of the concept; a different version, with set theoretic methods, is in Dekel, Lipman, Rustichini (1997)].

For example, suppose there are two sources of uncertainty, W = "there is war" and E = "a new source of energy is discovered", so that the objective S is the four-state space $S = \{(\mathsf{W},\mathsf{E}), (\mathsf{W}, \text{ not}\mathsf{E}), (\text{not}\mathsf{W},\mathsf{E}), (\text{not}\mathsf{W}, \text{ not}\mathsf{E})\}$. If the possibility of the discovery of a new energy source is out of agent h's mind, he will only think in $\mathsf{W}/\text{not}\mathsf{W}$ terms, and his subjective space will be $\{\mathsf{W}, \text{not}\mathsf{W}\}$. To relate this space to S, we suppose that the agent unconsciously attributes a truth value to the facts of which he is not aware and the states he perceives correspond to the real states under those values –a subset $S^h \subseteq S$–, so that he will perceive any function defined on S (like endowments, prices or assets) as its restriction to S^h. In the above example, we may imagine agent h perceiving what in fact are the two real states under "notE". Then in one-to-one correspondence with h's subjective space $\{\mathsf{W}, \text{not}\mathsf{W}\}$ there is $S^h = \{(\mathsf{W}, \text{not}\mathsf{E}), (\text{not}\mathsf{W}, \text{not}\mathsf{E})\}$; and if f is a map defined on S, he will perceive it as the map $\tilde{f}$ defined on $\{\mathsf{W}, \text{not}\mathsf{W}\}$ by $\tilde{f}(\mathsf{W}) = f(\mathsf{W}, \text{not}\mathsf{E})$, $\tilde{f}(\text{not}\mathsf{W}) = f(\text{not}\mathsf{W}, \text{not}\mathsf{E})$. In other words, he will perceive the vector $(f(s))_{s \in S}$ as the vector $(f(s))_{s \in S^h}$.

This is in particular true of assets. The J nominal assets of this paper, the j-th being $a^j : S \longrightarrow \mathbb{R}$, are meant to mimick dividends of (unmodelled) firms. They are not traded as contingent contracts, but simply as random variables; traders sign no contract. The distinction is immaterial when $S^h = S$ all h, but here it is important. When agent h sells to h' asset j in exchange for asset j', h thinks he has sold the income stream $(a^j(s))_{s \in S^h}$ in exchange for the stream $(a^{j'}(s))_{s \in S^h}$, and h' thinks he has bought $(a^j(s))_{s \in S^{h'}}$ for $(a^{j'}(s))_{s \in S^{h'}}$. In this way they implement income transfers across subjective states, and the 'role of securities' in this context is to economize not only on contingent goods' markets (Arrow, 1953; Kreps, 1982) but also on not-easy-to-write financial contingent contracts among people aware of different things. In such setting of course there is no guarantee that things will go as the agents expect; and it must be said that although they may know this, in the present model agents are assumed to do nothing about it (that is, we do not consider subjective space revision).

On his subjective space agent h will perceive $\#S^h$ (not S) budget constraints, one for each $s \in S^h$; and the assumption on h's 'conditionally correct' perception amounts to perfect foresight on S^h, which is effectively (i.e. in one-to-one correspondence with) the space on which h makes plans. So if $s \in S^h$ occurs, agent h will simply carry out his plans. If on the other hand an unforeseen scenario $s \notin S^h$ materializes he will have to re-optimise, taking as given his period-zero financial trades (incidentally, we may contrast the position of this agent h with that of a fully aware agent assigning probability zero to $s \in S \backslash S^h$, who unlike h makes complete plans for all $s \in S$ in advance; cfr. section 2.1).

In particular, at $s \in S^h$ agent h's debts and credits (together with endowments and prices) are exactly as anticipated in period zero. In such states h is assumed to always honour his debts (for example because a court would impose large penalties if it discovered that the agent is bankrupt). But by definition he does not take into account the consequences of his period-zero actions in the unforeseen scenarios $s \notin S^h$. Therefore he may involuntarily be bankrupt in those states. By the same token, at any $s \in S$ there may be agents who go bankrupt; so agent h must anticipate that in $s \in S^h$ his credits may not be entirely repaid; and this must be part of the equilibrium. In comparing the present model with the bankruptcy models of Dubey-Geanakoplos-Shubik (1988) and Zame (1993), one may notice that in the latter bankruptcy is voluntary and with penalty, and it must be so (it cannot be involuntary with full awareness, and once voluntary there must be a penalty); in the present model it is only involuntary, and the 'only' part is for simplicity.

The equilibrium concept retains as much flavour of the traditional rational expectations concept as it can; and the proof follows the main lines of Werner (1985), with the main complication that while in his case there is a sequence of truncations of a fixed price space, converging to the latter, in our case the sequence is one of truncations of a sequence of price spaces, converging to the limit price space. For existence it is critical to assume that there be enough states which all agents foresee (precisely no less than J), assumption C below, which reminds of that of agreement among agents' expectations in temporary equilibrium theory (Grandmont, 1982). For existence, stating that there be a in the intersection of the supports of all agents' assumption C is critical because there may not exist equilibrium in economies satisfying all the assumptions of our existence theorem but that one; and this implies in particular that if there are no more than J states the model need not be consistent except under full awareness, and hence that in the presence of unawareness the 'domain of consistency' of the model is a set of economies where markets are incomplete.

Section 2 contains the formal description of the model and definition of equilibrium; a proof of existence of an equilibrium is in sect.3. In section 4 we sketch an alternative existence proof, which also demonstrates a degree of indeterminacy of equilibrium asset prices; a last section collects comments and examples.

Miscellaneous notation

For $x, y \in \mathbb{R}^N$ we write $x \geq y$ if $x_i \geq y_i, i \in \{1, \ldots, N\} = N$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$, $i \in N$. $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x \geq 0\}$; $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x \gg 0\}$. $|x| = \sum_{i=1}^N |x_i|$. $u = \mathbf{1} = (1, \ldots, 1)$, in each occurrence of the appropriate dimension.

A matrix $A \in \mathbb{R}^{M \times N}$ has element $A^n(m)$ in row m, column $n, m \in \{1, \ldots, M\} = M$, $n \in N$; A^n (resp. $A(m)$) denotes column n (row m). For any $T \subseteq M$, A_T is the $T \times N$-matrix with rows $A_T(m) = A(m)$ for $m \in T$.

We define the operation $\otimes : \mathbb{R}^{M\times N} \times \mathbb{R}^{M\times N} \longrightarrow \mathbb{R}^{M\times N}$ as elementwise multiplication, i.e. taking the matrices A, B to the matrix $A \otimes B$ defined by $(A \otimes B)^n(m) = A^n(m)B^n(m),\ m \in M,\ n \in N$.

Standard notation is used from topology and convex analysis. For $X \subseteq \mathbb{R}^N$, cl X, int X, ri X, conv X and aff X are resp. X's closure, interior, relative interior, convex hull and affine hull, and cone X is the cone generated by X, i.e. the set of points of the form $\sum_{i=1}^{k} \lambda_i x_i$ with k a positive integer, the x_i's in X and the λ_i's nonnegative. Recall that if X is convex, cone $X = \mathbb{R}_+ X = \{tx : t \in \mathbb{R}_+,\ x \in X\}$ (Rockafellar 1970, p. 14). Another result we use is that for any finite set $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^N$, $\text{ri conv } \{x_1, \ldots, x_k\} = \{\sum_{i=1}^{N} \lambda_i x_i : \sum_{i=1}^{N} \lambda_i = 1, \lambda_i > 0\, \forall\, i\}$ (easily proved using Rockafellar 6.4). Finally, from p. 50 of Rockafellar we use the fact that if X is convex, $\text{ri cone } X = \mathbb{R}_{++} \text{ri} X = \{tx : t \in \mathbb{R}_{++},\ x \in \text{ri} X\}$.

2 The model

The model is built on the standard two-period pure exchange general equilibrium setup, with uncertainty in the second period (e.g. Magill-Shafer 1991). There are two periods, 0 and 1, and in the second period one state $s \in \{1, \ldots, S\} = S$ is determined. We also view the first period as state zero and write $s \in \{0, 1, \ldots, S\}$. There are L consumption goods indexed by $\ell \in \{1, \ldots, L\} = L$ in the first period and in each state in the second; and J assets, indexed by $j \in \{1, \ldots, J\} = J$, denominated in units of accounts, with return matrix $a = (a^j(s)) \in \mathbb{R}^{S\times J}$. In the first period there is trade in goods and assets and goods' consumption; in the second there is collection of assets' (net) returns and goods' consumption.

Consumers are indexed by $h \in \{1, \ldots, H\} = H$. $S^h \subseteq S$ is the set of states which reflects h's awareness. Consumer h's consumption set as seen from period zero is $\mathbb{R}_+^{L(1+S^h)}$, and he gets utility $u^h(x(0), (x(s))_{s\in S^h})$ from the plan $(x(0), (x(s))_{s\in S^h}) \in \mathbb{R}_+^{L(1+S^h)}$. In $s \notin S^h$, he has consumption set $\mathbb{R}_+^L$ and utility $u_s^h(x(s))$ from the bundle $x(s) \in \mathbb{R}_+^L$. There are endowments of goods, denoted by $e^h(s) \in \mathbb{R}_+^L$, $s \in \{0, 1, \ldots, S\}$. We let $z^h(s) = x^h(s) - e^h(s),\ s \in \{0, 1, \ldots, S\}$. To ease notation it is assumed that assets are in zero initial endowment.

As we mentioned, the model could also be cast in the numéraire asset framework, in which assets in state s pay off in a fixed commodity bundle. The proof that we construct is flexible enough to accomodate this case as well. Details are presented in a special section.

The portfolio of agent h is composed by asset purchases $\phi^h = (\phi^{hj})_{j\in J} \in \mathbb{R}_+^J$ and asset sales $\psi^h = (\psi^{hj})_{j\in J} \in \mathbb{R}_+^J$. We use $\theta^h = \phi^h - \psi^h$ for net holdings.

Prices are denoted as follows: for $s \in \{0, 1, \ldots, S\}$, $p(s) \in \mathbb{R}_+^L$ is the price vector of the goods in state s, and $p = (p(0), (p(s))_{s\in S})$. Assets' prices are $q \in \mathbb{R}_+^J$.

Agents will be assumed to be forced to honour their debts in the states they foresee, but they may be unable to do so in the other ones; hence in any $s \in S$ only a fraction of the total debts will be repaid. The way losses are distributed among the creditors may vary in different institutional arrangements. To prove existence of an equilibrium only two things are needed: that the total amount of default equals the amount of losses of creditors; and that the allocation of this amount is done in a continuous way. The rule which we adopt reflects anonymous trade as in Dubey-Geanakoplos-Shubik (1988), and is such that in any given state, each debtor repays in the same proportion the debts he has in the various assets (the proportion being equal to the ratio of his total receipts to his total debts); and all creditors of any one asset obtain the same fraction of their credits (the fraction being the ratio of total repayments to total debts on the asset).

Thus in any given state, the exigible fraction of credits on a given asset depends on how much the debtors can pay; but of course this in turn depends on the repayments they get on their own credits. Formally, portfolios $(\phi^h, \psi^h)_{h\in H}$, prices $(p(s))_{s\in S}$ and an $S \times J$ matrix of repayment fractions $K = (K^j(s)) \in [0,1]^{S\times J}$ generate an $S \times J$ matrix $\kappa = \beta((p(s))_{s\in S}, K, (\phi^h, \psi^h)_{h\in H})$ (β standing for book-keeping map) defined as follows. Let

$$M^h(s)\big(p(s), K(s), \phi^h, \psi^h\big) = \min\big\{p(s)e^h(s) + (a \otimes K)(s)\phi^h,\ a(s)\psi^h\big\}\ ,$$

$M^h(s)$ for short (it is the minimum between h's receipts and his debts in state s); and let

$$\iota(h,s) = \begin{cases} \dfrac{M^h(s)}{a(s)\psi^h} & \text{if } a(s)\psi^h > 0 \\ 1 & \text{otherwise} \end{cases} .$$

Then we define

$$\kappa^j(s) = \begin{cases} \dfrac{\sum\limits_h \iota(h,s)a^j(s)\psi^{hj}}{\sum\limits_h a^j(s)\psi^{hj}} & \text{if } \sum\limits_h a^j(s)\psi^{hj} > 0 \\ [0,1] & \text{otherwise} \end{cases} . \tag{BK}$$

Notice that since repayment rate on asset j in state s is a fraction of the total debts on that asset, it is not well defined if total debts are zero. We have for convenience defined it to be the whole interval $[0,1]$ in that case. This makes β a correspondence. In equilibrium when total debts are zero so are total credits, so which number we define as repayment rate is irrelevant.

Actual repayment rates must be a fixed point of (BK), i.e. must satisfy $\kappa^j(s) \in \beta^j(s)(p(s), \kappa(s), (\phi^h, \psi^h)_{h\in H})$ for all j, s. We note in passing that in (BK) the $a^j(s)$'s simplify; they are left there for clarity.

Next the budget sets which agent h faces. In period zero

$$\begin{aligned} B^h(p,q,\kappa) = \Big\{ & \big(x(0), (x(s))_{s\in S^h}, \phi, \psi\big) \in \mathbb{R}_+^{L(1+S^h)} \times \mathbb{R}_+^J \times \mathbb{R}_+^J : \\ & p(0)\big(x(0) - e^h(0)\big) + q(\phi - \psi) \le 0; \\ & p(s)\big(x(s) - e^h(s)\big) \le (a \otimes \kappa)(s)\phi - a(s)\psi, s \in S^h; \phi\psi = 0 \Big\} \ . \end{aligned}$$

So agent h cannot go bankrupt in $s \in S^h$. Notice that if $s \in \cap_{h\in H} S^h$, for any $((p(s))_{s\in S}, \kappa, (\phi^h, \psi^h)_{h\in H})$ such that $(\phi^h, \psi^h)_{h\in H}$ satisfy the budget constraints at (p, q, κ) one has $\beta^j(s)(p(s), \kappa, (\phi^h, \psi^h)_{h\in H}) = 1$ or [0,1] for all $j \in J$.

Note also that we are imposing that the agents cannot at the same time buy and sell the same asset (condition $\phi\psi = 0$, which given nonnegativity is equivalent to $\phi^j\psi^j = 0$ all j). This extra constraint never hurts any agent and helps in the existence proof (since agents' asset purchases and sales appear separately in the book-keeping map, in the sequence of the nth-stage fixed points which will converge to equilibrium – section 3.5 – we need to ensure that both sales and purchases, not just their difference, converge). In period 1, if $s \notin S^h$ materializes, agent h may be bankrupt, in which case he gets zero income:

$$B^h_s(p(s), \kappa(s), \phi, \psi) = \{x(s) \in \mathbb{R}^L_+ : \\ p(s)(x(s) - e^h(s)) \le (a \otimes \kappa)(s)\phi - M^h(s)\} \quad s \notin S^h \ .$$

The constraint may be written as: $p(s)x(s) \le \max\{0, p(s)e^h(s) + (a \otimes \kappa)(s)\phi - a(s)\psi\}$. Thus h's problems are:

$$CP^h(p, q, \kappa) : \max u^h\big(x(0), (x(s))_{s\in S^h}\big) \\ s.to\big(x(0), (x(s))_{s\in S^h}, \phi, \psi\big) \in B^h(p, q, \kappa)$$

$$CP^h_s(p(s), \kappa(s), \phi, \psi) : \max u^h_s(x(s)) \\ s.to\ x(s) \in B^h_s(p(s), \kappa(s), \phi, \psi)$$

2.1 Definition *An equilibrium is an array of prices* (p, q), *repayment rates* κ *and actions* $(x^h(0), (x^h(s))_{s\in S}, \phi^h, \psi^h)_{h\in H}$ *such that*

(i) $(x^h(0), (x^h(s))_{s\in S^h}, \phi^h, \psi^h)$ *solves* $CP^h(p, q, \kappa), h \in H$
(ii) $x^h(s)$ *solves* $CP^h_s(p(s), \kappa(s), \phi^h, \psi^h),\ s \notin S^h,\ h \in H$
(iii) $\sum_{h\in H} x^h(s) = \sum_{h\in H} e^h(s), h \in H, s \in \{0, 1, \ldots, S\}$ *and* $\sum_{h\in H}(\phi^h - \psi^h) = 0$
(iv) κ *is a fixed point of* (BK).

Notice that at equilibrium each agent h correctly anticipates prices and repayment rates in S^h. If $S^h = S$ for all $h \in H$ there is no bankruptcy and the model and equilibrium concept reduce to the standard ones.

We assume that there are no less states than assets, $J \le S$. It is checked that this entails no loss of generality in section 3.7. We denote by $C \subseteq S$ the set of the first J states. Thus $\{1, \ldots, J\} = C$ as a set of states, even though the set in brackets is called J when seen as the set of assets. Assumption C below implies $J \le \#S^h \le S$.

2.2. Theorem *An equilibrium exists under the following assumptions*:

P. $a \gg 0$
U. u^h *and* u^h_s *are continuous, strictly increasing and strictly concave for all* $h \in H$ *and* $s \notin S^h$
E. $e^h(s) \gg 0$ *for all* $h \in H, s \in \{0, 1, \ldots, S\}$
C. $C \subseteq S^h$ *for all* $h \in H$
FR. a_C *is nonsingular.*

An equilibrium may not exist in an economy satisfying all the stated assumptions except C.

2.1 Two remarks

We discuss two issues related to the above theorem and model in the context of a simple (complete market) structure with two agents, two states, $S = \{s_1, s_2\}$ and two Arrow-securities:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Unawareness with complete markets. If we let $S^h = S,\ h = 1, 2$ we obtain an Arrow-Debreu economy in which equilibrium exists under U, E. If we introduce unawareness by taking for example $S^1 = \{s_1\}$ and $S^2 = \{s_2\}$ (both agents perceive no uncertainty), assumption C is violated. Agent 1 sees an asset giving 1 for sure (a^1) and one giving zero for sure (a^2); symmetrically for agent 2. As long as endowments are positive, no equilibrium can exist in this economy, no matter how one completes its specification. For: in equilibrium portfolios would be finite, so since endowments are positive it should be $\kappa \gg 0$; but for such κ at any (q_1, q_2) at least one agent sees arbitrage (at $(q_1, q_2) \gg 0$ both do), hence his problem has no solution.

Thus with unawareness equilibrium may not exist even with complete markets.

Unawareness versus zero-probability. It will be clear from the formulation of the agents' optimization problems that the behaviour of an agent who does not foresee some of the states is different from that of an agent who gives zero probability to those states. Something stronger is true: *it is not possible to interpret a model where some agents do not foresee some states as a model in which they give zero probability to those states.* To see this we show how unawareness and zero probability have different equilibrium implications, by contrasting the last non-existence example with the case where agents are fully aware, have von Neumann-Morgenstern utilities, and h's belief has support $\{s_h\},\ h = 1, 2$. We obtain again a standard Arrow-Debreu economy (with non strictly increasing utilities due to zero-probability events), in which again an equilibrium exists under classical assumptions.

In concrete terms, let us assume that there is one good in each state, so we may let $p(0) = p(1) = p(2) = 1$. If we look at agent 1's position in the zero-probability case, since his utility is independent of $x^1(2)$, clearly he will want to give away his endowment in state 2 by selling asset 2, and buy asset 1 to increase consumption in state 1. Hence his budget constraints become:

$$x^1(0) + q_1\phi^{11} - q_2\psi^{12} = e^1(0)$$
$$x^1(1) = e^1(1) + \phi^{11}$$
$$x^1(2) = e^1(2) - \psi^{12}$$

and he will set $\psi^{12} = e^1(2),\ x^1(2) = 0$. Agent 2 will be quite willing to accept such trade at the 'right' prices, for he is interested in exactly the opposite operations.

With unawareness agent 1 sees asset 1 as giving 1 (times repayment rate) for sure while asset 2 as giving zero for sure, so he too will want to sell asset 2 and buy 1. But his budget constraints are

$$x^1(0) + q_1\phi^{11} - q_2\psi^{12} = e^1(0)$$
$$x^1(1) = e^1(1) + \kappa^1(1)\phi^{11}$$

so by selling asset 2 and buying asset 1 with the sale's revenue he can make $x^1(1)$ as large as he wants as long as $\kappa^1(1) > 0$ (he will go bankrupt in state 2, but he is not aware of that). Hence no solution to his problem exists if $\kappa^1(1) > 0$ (but in equilibrium it should be $\kappa^1(1) > 0$, so no equilibrium exists). Similar is agent 2's position.

We emphasize that the different form of the budget constraint in the two cases is more than a convention. Two of us have provided (see Modica and Rustichini, 1996) a decision theoretic model, which generalizes the classical Anscombe Aumann theory, where bankruptcy plays the role of a *very bad prize* (in the formal sense that all other prizes are preferred to it), and where the maximization problem of the agent has a formulation with budget constraint equal to the first presented in this section. In light of this model, we can summarize the situation as follows.

It is clear that in a standard general equilibrium model (Arrow-Debreu-McKenzie) there must be a penalty for people who violate the budget constraints. More than that, this penalty must be infinite: because if it is finite, there will be a tradeoff between the penalty and the utility of consuming additional goods; so typically the budget constraint will be violated. The situation becomes more complicated when we want to model agents who, under uncertainty, give zero probability to some event; or (alternatively) are not aware of it.

What do we do when an agent gives zero probability to an event? Here we have a situation of zero (probability) times infinity (the disutility of the penalty). What is the product of the two terms? Clearly in standard analysis the question itself is not very clear. In non-standard analysis it is a very clear question, and has a very simple answer (see Modica and Rustichini, 1996). Here we report, informally, the main result. We will think of the zero probability as infinitesimally small, and of the large penalty as infinitely large. Two cases are now possible.

If the product of the two terms in the expected utility evaluation is infinite, then the concern about the bankruptcy prevails, and the agent who gives infinitesimally small probability to an event will not go bankrupt in it. But then his behavior will be different from the person who is not aware of an event, because this second person *will* go bankrupt in that event, simply because he "did not think" of it.

In the second case the product is infinitesimally small. Then the utility of additional consumption prevails, and the person who gives infinitesimally small probability will go bankrupt, by *arbitrarily large amount* (since he does not care about the penalty). In this case of course an equilibrium typically will not exist. Note that *again*, his behavior will be different from the person who is not aware, because this second will go bankrupt only by a finite amount, determined by how much he spends in the states that he can think of. Also, equilibrium exists in this case, under two technical assumptions. This is the main existence result of this paper.

We finally report that partial awareness and zero probability may have different equilibrium implications when the assumption of strictly positive endowments is relaxed. Indeed, in the structure under discussion, with non strictly positive endowments there are cases where equilibrium exists with partial awareness (namely, $S^h = \{s_h\},\ h = 1, 2,\ e^1(0) = e^2(0) = e^1(2) = e^2(1) = 0$) and cases where equilibrium does not exist with full awareness and zero probability ($e^1(0) = e^2(0) = 0$ and logarithmic utility).

3 Model consistency

The proof of existence of an equilibrium goes through the following steps: truncate the price spaces along a sequence, indexed by n; get a fixed point for each n; note that prices and repayment rates converge (along a subsequence); show that a subsequence of excess demand for goods converges, and that this implies that a subsequence of demand for assets converges; show that the limit is an equilibrium. The rest of this section presents the details.

The proof of the last assertion is contained in point 2 of the concluding section.

3.1 The set of no-arbitrage prices

Take $\kappa \in [0,1]^{S\times J}$ and $h \in H$. We denote by $Q_{S^h}(\kappa)$ the set of no-arbitrage (asset) prices for agent h when he faces repayment rates κ. Formally for any $T \subseteq S$ one has

$$Q_T(\kappa) = \left\{ q \in \mathbb{R}^J : \nexists\ (\phi,\psi) \in \mathbb{R}^{2J}_+ \ \text{ s.t. } \begin{pmatrix} -q(\phi-\psi) \\ (a\otimes\kappa)_T\phi - a_T\psi \end{pmatrix} > 0 \right\} .$$

3.1.1 Lemma $Q_T(\kappa) = \{q \in \mathbb{R}^J_{++} : \exists\ \lambda \in \mathbb{R}^T_{++} \ \text{ s.t. } \ \lambda(a\otimes\kappa)_T \le q \le \lambda a_T\}$.

Proof. Letting

$$A = \begin{pmatrix} -q & q \\ (a\otimes\kappa)_T & -a_T \end{pmatrix}, \qquad y = (\phi,\psi)$$

q permits arbitrage iff $\exists\, y \ge 0$ s.t. $Ay > 0$, that is iff $\exists\, 0 < y \in \mathbb{R}^{2J}$, $0 < v \in \mathbb{R}^{T+1}$ s.t. $Ay - Iv = 0$ where I is the unit $(T+1)\times(T+1)$ matrix; that is, iff

$$\exists\ 0 \le (y,v) \in \mathbb{R}^{2J+T+1} \ \text{ s.t. } \begin{pmatrix} A & -I \\ 0_{1\times 2J} & \mathbf{1}_{1\times(T+1)} \end{pmatrix} (y,v) = \begin{pmatrix} 0_{(T+1)\times 1} \\ 1 \end{pmatrix} ,$$

where subscripts denote dimensions. The alternative, by Gale (1960) theorem 2.6 is that $\exists\, x \in \mathbb{R}^{T+1}, \eta \in \mathbb{R}$ which solve

$$\begin{cases} xA + \eta 0 \geq 0 \\ -Ix + \eta \mathbf{1}_{1\times(T+1)} \geq 0 \\ \eta < 0 \end{cases}$$

that is $\exists\, x \in \mathbb{R}^{T+1}$ s.t. $xA \leq 0$ and $x \geq -\eta \mathbf{1} \gg 0$, equivalently $\exists\, 0 \ll \lambda \in \mathbb{R}^T$ s.t. $(1, \lambda)A \leq 0$. This is just what is asserted. □

As is clear from the above expression, $Q_{S^h}(\kappa)$ is a cone. Next we wish to have conditions under which $Q_{S^h}(\kappa)$ is open. For $A, B \in \mathbb{R}^{T\times J}$ with $0 \leq A \leq B$, we let

$$\mathrm{Co}(A,B) = \left\{ q \in \mathbb{R}^J_+ : \exists\, \lambda \in \mathbb{R}^T_{++} \text{ s.t. } \lambda A \leq q \leq \lambda B \right\} .$$

By the previous lemma, $Q_{S^h}(\kappa) = \mathrm{Co}((a \otimes \kappa)_{S^h}, a_{S^h})$. We now state two useful characterizations of $\mathrm{Co}(A,B)$. Given $A, B \in \mathbb{R}^{T\times J}$ with $0 \leq A \leq B$, let

$$[A,B] = \left\{ D \in \mathbb{R}^{T\times J} : A^j(s) \leq D^j(s) \leq B^j(s),\ s \in T,\ j \in J \right\} .$$

Notice that if A and B are 1-row matrices, i.e. vectors in $\mathbb{R}^J$, the symbol $[A,B]$ does not denote a segment but a rectangle.

3.1.2 Lemma (i) $\mathrm{Co}(A,B) = \{ q \in \mathbb{R}^J_+ : \exists\, (\lambda, D) \in \mathbb{R}^T_{++} \times [A,B] \text{ s.t. } q = \lambda D \}$

(ii) *Let* $A_i, B_i \in \mathbb{R}^{T_i \times J}$ *with* $A_i \leq B_i$, $i = 1, 2$, *and let* $A = \binom{A_1}{A_2}$, $B = \binom{B_1}{B_2}$, *both in* $\mathbb{R}^{(T_1+T_2)\times J}$. *Then* $\mathrm{Co}(A,B) = \mathrm{Co}(A_1,B_1) + \mathrm{Co}(A_2,B_2)$.

Proof. (i) That the set on the right is contained in $\mathrm{Co}(A,B)$ is clear; conversely, suppose q is s.t. for a $\lambda \in \mathbb{R}^T_{++}$, $\lambda A \leq q \leq \lambda B$. For each j we have $\lambda A^j \leq q^j \leq \lambda B^j$; then denote for any $\alpha \in [0,1]$, $D(\alpha)^j = A^j + \alpha(B^j - A^j)$. We have $\{\lambda D(\alpha)^j : \alpha \in [0,1]\} = [\lambda A^j, \lambda B^j]$, so that for some $\alpha_j \in [0,1]$, $\lambda D(\alpha_j)^j = q^j$. Now let $D = (D(\alpha_1)^1, \ldots, D(\alpha_J)^J)$.

(ii) $q \in \mathrm{Co}(A,B)$ iff $\exists \lambda \in \mathbb{R}^{T_1+T_2}_{++}$, $D = \binom{D_1}{D_2} \in [A,B]$ s.t. $q = \lambda D$; iff $\exists\, (\lambda_1, \lambda_2) \in \mathbb{R}^{T_1}_{++} \times \mathbb{R}^{T_2}_{++}$, $D_i \in [A_i, B_i]$ s.t. $q = \lambda_1 D_1 + \lambda_2 D_2$; iff $q \in \mathrm{Co}(A_1, B_1) + \mathrm{Co}(A_2, B_2)$. □

The set of asset prices at which no agent sees arbitrage opportunities given κ is denoted by $Q(\kappa)$:

$$Q(\kappa) = \bigcap_{h\in H} Q_{S^h}(\kappa)$$

3.1.3 Lemma *If C and FR hold and* $\kappa_C = \mathbf{1}$, *then* $Q_{S^h}(\kappa)$ *is open for all* $h \in H$, *and* $Q(\kappa)$ *is open.*

Proof.

$$Q_{S^h}(\kappa) = \mathrm{Co}\big((a\otimes\kappa)_{S^h}, a_{S^h}\big) = \mathrm{Co}\left(\begin{pmatrix} a_C \\ (a\otimes\kappa)_{S^h\setminus C} \end{pmatrix}, \begin{matrix} a_C \\ a_{S^h\setminus C} \end{matrix} \right)$$

$$= \mathrm{Co}(a_C, a_C) + \mathrm{Co}\Big((a\otimes\kappa)_{S^h\setminus C}, a_{S^h\setminus C}\Big) .$$

From assumption *FR* $\mathrm{Co}(a_C, a_C)$ is open, hence so is $Q_{S^h}(\kappa)$. □

The map $(T,\kappa) \longrightarrow Q_T(\kappa)$ has the following monotonicity properties.

3.1.4 Lemma *Let* $T, T_1, T_2 \in S, \kappa, \kappa_1, \kappa_2 \in [0,1]^{S\times J}$.
(i) *if* $\kappa_1 \le \kappa_2$, *then* $Q_T(\kappa_2) \subseteq Q_T(\kappa_1)$
(ii) *if* $T_1 \subseteq T_2$ *and* $Q_{T_1}(\kappa)$ *is open, then* $Q_{T_1}(\kappa) \subseteq Q_{T_2}(\kappa)$.

Proof.

$$\text{(i) } Q_T(\kappa_2) = \cup\left\{\text{Co}(D,D) : D \in \left[(a\otimes\kappa_2)_T, a_T\right]\right\}$$
$$\subseteq \cup\left\{\text{Co}(D,D) : D \in \left[(a\otimes\kappa_1)_T, a_T\right]\right\} = Q_T(\kappa_1) \ .$$

$$\text{(ii) } Q_{T_2}(\kappa) = \text{Co}((a\otimes\kappa)_{T_2}, a_{T_2}) = \text{Co}\left(\begin{pmatrix} (a\otimes\kappa)_{T_1} \\ (a\otimes\kappa)_{T_2\setminus T_1}\end{pmatrix}, \begin{pmatrix} a_{T_1} \\ a_{T_2\setminus T_1}\end{pmatrix}\right)$$
$$= \text{Co}\left((a\otimes\kappa)_{T_1}, a_{T_1}\right) + \text{Co}\left((a\otimes\kappa)_{T_2\setminus T_1}, a_{T_2\setminus T_1}\right)$$
$$= Q_{T_1}(\kappa) + Q_{T_2\setminus T_1}(\kappa)$$
$$\supseteq Q_{T_1}(\kappa) \text{ if } Q_{T_1}(\kappa) \text{ is open (easy to see)} \ . \quad \square$$

3.2 Truncation of the set of first period prices

A main step in the proof of Werner is to construct a sequence of truncations of the price set, converging to the full set, such that at each step the truncated price set is compact; we will do the same, with the complication that the (full) set of no-arbitrage asset prices varies along the sequence, since repayment rates vary. Our n-th stage truncated asset price set will be a compact subset of the n-th stage no-arbitrage price set, converging to the limit set of no-arbitrage prices. We start with asset prices, then consider the totality of first period prices (goods and assets). Second period spot prices are no problem, the truncation will be as in Werner (1985).

The idea of the truncation is the following. For each κ and each awareness set S^h take the intersection of the closure of $Q_{S^h}(\kappa)$ with the simplex Δ^J, and get a convex set. In fact, this convex set is generated by a finite set of points (which correspond to the finitely many generators of $Q_{S^h}(\kappa)$). Now shrink this set, making sure that the set which is obtained is contained in the relative interior of the original set. Then take the cone generated by this smaller set: its closure stays in the interior of the set $Q_{S^h}(\kappa)$. On this smaller cone the consumers' problems are well defined, and we can derive a demand function defined on the set. We then have the basic element to get the fixed point at the n-th stage of the sequence.

We present here the shrinking by a $\delta \in [0,1]$; in the n-th stage of the sequence it will be $\delta = 1 - 1/n$.

To go on with the details, fix first $\kappa \in [0,1]^{S\times J}$ and $s \in S$, and consider the rectangle $[(a\otimes\kappa)(s), a(s)]$ in $\mathbb{R}^J$. It has 2^J vertices $v_i,\ i = 1,\dots,2^J$ (actually $v_i = v_i(s,\kappa)$; we are suppressing dependence on (s,κ) momentarily), not necessarily distinct (in fact if e.g. $\kappa(s) = \mathbf{1}$ so that $(a\otimes\kappa)(s) = a(s)$, all vertices coincide). Each vertex v_i has j-th coordinate equal to $(a\otimes\kappa)^j(s)$ or to

$a^j(s)$; formally the 2^J vertices are the elements of the set $\times_{j=1}^{J}\{(a\otimes\kappa)^j(s), a^j(s)\}$. Now list them in any order such that $v_1 = (a\otimes\kappa)(s)$ (the closest to the origin). For any $i \geq 2$, for at least one j, $v_i^j = a^j(s) > 0$, so $|v_i| > 0$ and thus $v_i/|v_i| \in \Delta^J$ is well defined. Then let

$$G_s(\kappa) = \left\{\frac{v_i(s,\kappa)}{|v_i(s,\kappa)|} : i = 2, \ldots 2^J\right\}$$

(in the example with $\kappa(s) = \mathbf{1}$, $G_s(\kappa) = \{a(s)/|a(s)|\}$).

Now for $T \subseteq S$, define $G_T(\kappa) = \cup\{G_s(\kappa) : s \in T\}$. $G_T(\kappa)$ is a finite subset of Δ^J, which (since properly ordered) can also be viewed as a vector in $\mathbb{R}^{J(2^J-1)T}$. The context should make clear which interpretation is appropriate. For example, viewing $G_T(\kappa)$ as a vector one clearly has that

3.2.1 Lemma *For every $T \subseteq S$, the function $\kappa \mapsto G_T(\kappa)$ is continuous on $[0,1]^{S\times J}$.*

The next result is that cone $G_T(\kappa) = \text{cl}\, Q_T(\kappa)$. First we write the latter set in a convenient way, then prove the equality.

3.2.2 Lemma *For $T \subseteq S$,*

$$\text{cl}\, Q_T(\kappa) = \left\{q \in \mathbb{R}_+^J : \exists\, \lambda \in \mathbb{R}_+^T \ \text{ s.t. } \ \lambda(a\otimes\kappa)_T \leq q \leq \lambda a_T\right\} \ .$$

Proof. Exactly as in Lemma 3.1.2(i) one proves that the set on the right is equal to $\{q \in \mathbb{R}_+^J : \exists\,(\lambda, D) \in \mathbb{R}_+^T \times [(a\otimes\kappa)_T, a_T]$ s.t. $q = \lambda D\}$, which is clearly equal to $\text{cl}\{q \in \mathbb{R}_+^J : \exists\,(\lambda, D) \in \mathbb{R}_{++}^T \times [(a\otimes\kappa)_T, a_T]$ s.t. $q = \lambda D\} = \text{cl}\, Q_T(\kappa)$, the last equality from Lemmas 3.1.1 and 3.1.1(i). □

3.2.3 Lemma *For any $T \subseteq S$ and $\kappa \in [0,1]^{S\times J}$,* cl $Q_T(\kappa) =$ cone $G_T(\kappa)$.

Proof. First we prove the inclusion $\subseteq$. From the proof of the last lemma we know that $q \in$ cl $Q_T(\kappa)$ iff there is $(\lambda, D) \in \mathbb{R}_+^T \times [(a\otimes\kappa)_T, a_T]$ such that $q = \lambda D$. Also, if $D(s) = 0$ for some $s \in T$ we can ignore it in the sum giving q, so we may assume that $D(s) \neq 0\,\forall s \in T$. Take an $s \in T$ and define $\bar\alpha = \max\{\alpha \in \mathbb{R} : \alpha D(s) \leq a(s)\}$. Note that $\bar\alpha \geq 1$, and that for some $j \in J$ it is $\bar\alpha D^j(s) = a^j(s)$. Since $\bar\alpha D(s) \in [(a\otimes\kappa)(s), a(s)]$ we can write

$$\bar\alpha D(s) = \sum_{i=1}^{2^J} \mu_i v_i(s,\kappa), \quad \text{for some } \mu_i \geq 0, \ \sum_{i=1}^{2^J} \mu_i = 1, \ \ \mu_1 = 0 \ .$$

The last restriction can be imposed because $\bar\alpha D^j(s) = a^j(s)$ for a $j \in J$; in fact one could impose $\mu_i = 0$ whenever $v_i^j(s,\kappa) = (a\otimes\kappa)^j(s)$. To see this observe that $\bar\alpha D(s)$ is in the cube

$$\left[(a\otimes\kappa)^1(s), a^1(s)\right] \times \cdots \times \{a^j(s)\} \times \cdots \times \left[(a\otimes\kappa)^J(s), a^J(s)\right] \ ,$$

whose extreme points are contained in the set of vertices $v_i(s,\kappa)$ with $v_i^j(s,\kappa) = a^j(s)$.

For each $s \in T$ find $\bar\alpha(s)$ and the corresponding $\mu_i(s)$; then since $D(s) = \bar\alpha(s)^{-1}\sum_{i=2}^{2^J}\mu_i(s)v_i(s,\kappa)$, we may write

$$\begin{aligned} q &= \sum_{s\in T} \lambda_s D(s) \\ &= \sum_{s\in T} \sum_{i=2}^{2^J} \left(\lambda_s \frac{\mu_i(s)}{\bar{\alpha}(s)} |v_i(s,\kappa)| \right) \frac{v_i(s,\kappa)}{|v_i(s,\kappa)|} \\ &\in \text{ cone } G_T(\kappa) \ . \end{aligned}$$

For the reverse inclusion, let $q \in$ cone $G_T(\kappa)$. Then

$$\begin{aligned} q &= \sum_{s\in T} \sum_{i=2}^{2^J} v(s,i) \frac{v_i(s,\kappa)}{|v_i(s,\kappa)|}, \quad v(s,i) \geq 0 \\ &= \sum_{s\in T} \lambda_s \sum_{i=2}^{2^J} \left(\frac{v(s,i)}{|v_i(s,\kappa)|} \lambda_s^{-1} v_i(s,\kappa) \right) \end{aligned}$$

where $\lambda_s = \sum_{i=2}^{2^J} \frac{v(s,i)}{|v_i(s,\kappa)|} \geq 0,\ s \in T$. Since, for each $s \in T$, λ_s multiplies a point in $[(a \otimes \kappa)(s), a(s)]$, the result is established. □

Now we shall shrink the set conv $G_T(\kappa)$ generating cl $Q_T(\kappa) =$ cone $G_T(\kappa) =$ cone conv $G_T(\kappa)$ and take as truncated asset price space the cone generated by the shrunk set. With assumption C and the budget sets in mind, we assume in the rest of the present subsection that

$$T \supseteq C, \ \ \kappa_C = \mathbf{1} \quad \text{and} \quad FR \text{ holds} \ .$$

With these assumptions, for all $s \in C,\ G_s(\kappa) = \{a(s)/|a(s)|\}$, so, recalling that $C = \{1, \ldots, J\}$,

$$\begin{aligned} \text{conv } G_T(\kappa) &= \text{conv}\left\{ \frac{a(1)}{z|a(1)|}, \ldots, \frac{a(J)}{|a(J)|}, \bigcup_{s\in T\setminus C} \left\{ \frac{v_i(s,\kappa)}{|v_i(s,\kappa)|} : i = 2, \ldots, 2^J \right\} \right\} \\ &= \text{conv}\{g(1), \ldots, g(J), g(J+1), \ldots, g(N)\} \end{aligned}$$

say, where $g(s) = a(s)/|a(s)|$ for $s \in C$. To do the shrinking we take the barycenter $\bar{g}$ of the first $C = J$ points, $\bar{g} = J^{-1} \sum_{s=1}^{J} g(s)$, and for $\delta \in [0,1]$ define

$$g^\delta(i) = \delta g(i) + (1-\delta)\bar{g},\ i = 1, \ldots, N$$

(we have taken the barycenter of only the first J points to have it independent of T and κ and ease notation; nothing substantial is involved).

3.2.4 Lemma *Let* $g(1), \ldots, g(J), \ldots, g(N) \in \Delta^J$ *be* s.t. $g(1), \ldots, g(J)$ *are linearly independent. Then* $\bar{g} \in$ ri conv$\{g(1), \ldots, g(J), \ldots, g(N)\}$.

Proof. $\bar{g} \in$ ri conv$\{g(1), \ldots, g(J)\} \subseteq$ ri conv$\{g(1), \ldots, g(J), \ldots, g(N)\}$ since aff conv $\{g(1), \ldots, g(J)\} =$ aff conv$\{g(1), \ldots, g(J), \ldots, g(N)\} = \Delta^J$. □

3.2.5 Lemma *For* $g(1), \ldots, g(J), \ldots, g(N) \in \Delta^J$ *as in the previous lemma and any* $\delta \in [0,1)$, conv $\{g^\delta(i) : i = 1, \ldots, N\} \subseteq$ ri conv$\{g(i) : i = 1, \ldots, N\}$.

Proof. Take $x \in$ conv$\{g^\delta(i) : i = 1, \ldots, N\}$; then for some $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$,

$$x = \sum_{i=1}^{N} \lambda_i g^\delta(i) = \sum_{i=1}^{N} \lambda_i(\delta g(i) + (1-\delta)\bar{g})$$
$$= \delta \sum_{i=1}^{N} \lambda_i g(i) + (1-\delta)\bar{g} \ .$$

But $\sum_{i=1}^{N} \lambda_i g(i) \in$ conv$\{g(i) : i = 1, \dots, N\}$ and by the previous lemma $\bar{g} \in$ ri conv $\{g(1), \dots, g(N)\}$; then apply Rockafellar 6.1. □

Now define $G_T^\delta(\kappa) = \{g^\delta(i) : i = 1, \dots, N\}$, $\bar{Q}_T^\delta(\kappa) = \text{cone } G_T^\delta(\kappa)$ and let the truncated cone of no-arbitrage prices be

$$\bar{Q}^\delta(\kappa) = \bigcap_{h \in H} \bar{Q}_{S^h}^\delta(\kappa) \ .$$

Note that $\bar{Q}_T^1(\kappa) = \text{cl } Q_T(\kappa)$, and (since the interiors have nonempty intersection) $\bar{Q}^1(\kappa) = \text{cl } Q(\kappa)$. The wanted inclusion is the following:

3.2.6 Lemma *For any* $\delta \in [0,1)$, $\bar{Q}_T^\delta(\kappa) \subseteq Q_T(\kappa)$.

Proof. Take $q \in \bar{Q}_T^\delta(\kappa)$ and note that $q/|q| \in$ conv $G_T^\delta(\kappa)$. By the previous lemma $q/|q| \in$ ri conv $G_T(\kappa)$, hence (Rockafellar p. 50) $q \in$ ricone conv $G_T(\kappa) = \text{ri cone } G_T(\kappa) = \text{ri cl } Q_T(\kappa) = \text{ri } Q_T(\kappa) = Q_T(\kappa)$, using lemmas 3.2.3 and 3.1.3. □

Next we have to ensure that the correspondences $\kappa \mapsto \bar{Q}_T^\delta(\kappa)$ and $\kappa \mapsto \bar{Q}^\delta(\kappa)$ are well behaved for $\delta \leq 1$.

3.2.7 Lemma *Let the function* $\kappa \mapsto \gamma(\kappa) = (\gamma_1(\kappa), \dots, \gamma_N(\kappa)) \in (\Delta^J)^N$ *be continuous on* $[0,1]^{S \times J}$. *Then*

(i) *the correspondence* $\kappa \mapsto \text{conv } \{\gamma_1(\kappa), \dots, \gamma_N(\kappa)\}$ *is compact-, convex-valued and continuous*

(ii) *the correspondence* $\kappa \mapsto \text{cone}\{\gamma_1(\kappa), \dots, \gamma_N(\kappa)\}$ *is convex-valued, closed and lower hemicontinuous* (*lhc for short*).

Proof. (i) denote by $\mathscr{H}$ the given correspondence. It is clear that $\mathscr{H}$ is compact- and convex-valued. To prove that it is continuous it then suffices (lemma p. 33 in Hildenbrand) that it is closed and lhc. $\mathscr{H}$ is closed: let $(q_n, \kappa_n) \to (q, \kappa)$ with $q_n \in \mathscr{H}(\kappa_n)$; then $q_n = \sum_{i=1}^{N} \mu_{n,i}\gamma_i(\kappa_n)$; the sequence $\{\mu_n\}$ is in the simplex Δ^N, so it has a convergent subsequence $\mu_{n_k} \to \mu$; then $q_{n_k} \to \sum_{i=1}^{N} \mu_i \gamma_i(\kappa) = q$, whence the claim. $\mathscr{H}$ is lhc: let $\kappa_n \to \kappa$ and $q = \sum_{i=1}^{N} \mu_i \gamma_i(\kappa) \in \mathscr{H}(\kappa)$; then $\mathscr{H}(\kappa_n) \ni q_n = \sum_{i=1}^{N} \mu_i \gamma_i(\kappa_n) \to q$.

(ii) denote by $\mathscr{F}$ the given correspondence. $\mathscr{F}$ is closed: let $(q_n, \kappa_n) \to (q, \kappa)$ and $q_n = \sum_{i=1}^{N} \mu_{n,i}\gamma_i(\kappa_n) \in \mathscr{F}(\kappa_n)$ (with $\mu_{n,i} \geq 0$ for all n, i). For each $i = 1, \dots, N$ it is $\sup_n \mu_{n,i} < \infty$ (if not there would be an i_0 and a sequence $\mu_{n_k,i_0} \to \infty$ as $k \to \infty$; but then $|q_{n_k}| \geq |\mu_{n_k,i_0}\gamma_i(\kappa_{n_k})| = \mu_{n_k,i_0} \to \infty$, contradicting $|q_{n_k}| \to |q|$). Then on a convergent subsequence $\mu_{n_k} \to \mu$ one has

$$q_{n_k} = \sum_{i=1}^{N} \mu_{n_k,i}\gamma_i(\kappa_{n_k}) \to \sum_{i=1}^{N} \mu_i \gamma_i(\kappa) \in \mathscr{F}(\kappa) \ .$$

$\mathscr{F}$ is lhc: let $\kappa_n \to \kappa$ and $q = \sum_{i=1}^N \mu_i \gamma_i(\kappa) \in \mathscr{F}(\kappa)$. Then $q_n \equiv \sum_{i=1}^N \mu_i \gamma_i(\kappa_n) \to q$, and $q_n \in \mathscr{F}(\kappa_n)$ for all n. □

3.2.8 Corollary *For every* $\delta \in [0,1]$,

(i) *The correspondence* $\kappa \mapsto \bar{Q}^\delta$ *is closed and lhc*

(ii) $\bar{Q}^\delta(\kappa)$ *is a closed cone for every* $\kappa \in [0,1]^{S\times J}$.

Proof. (i) By the previous lemma, for each $h \in H$ the correspondence $\bar{Q}^\delta_{S^h}$ is convex-valued, closed and lhc. Moreover for any κ the $\bar{Q}^\delta_{S^h}(\kappa)$'s have interiors with nonempty intersection. Then the result follows from Hildenbrand (1974) problem 6, (2) p. 35. (ii) the set is an intersection of closed cones. □

We now pass to first period assets and goods prices. For $\delta \in [0,1]$ let

$$P^\delta = \left\{(p,q) \in \Delta^{L+J} :\ p_\ell \geq 1-\delta,\ \ell = 1,\dots,L\right\}$$
$$P^\delta(\kappa) = \left\{(p,q) \in P^\delta :\ q \in \bar{Q}^\delta(\kappa)\right\}$$

3.2.9 Lemma *For each* $\delta \in [0,1]$, *the correspondence* $\kappa \mapsto P^\delta(\kappa)$ *is compact-convex-valued and continuous.*

Proof. Each $P^\delta(\kappa)$ is the intersection of two convex sets; one compact (P^δ), the other closed ($\mathbb{R}^L_+ \times \bar{Q}^\delta(\kappa)$). To prove continuity we show that P^δ is closed and lhc, then invoke the lemma on p. 33 of Hildenbrand.

Closedness: let $(p_n, q_n, \kappa_n) \to (p, q, \kappa)$ with $(p_n, q_n) \in P^\delta(\kappa_n)$. It is $p_{n,\ell} \geq 1-\delta$ for all n, ℓ, so $p_\ell \geq 1-\delta$ for all $\ell \in L$; $(p,q) \in \Delta^{L+J}$ since each $(p_n, q_n) \in \Delta^{L+J}$; and $q \in \bar{Q}^\delta(\kappa)$ by corollary 3.2.8.

Lower hemicontinuity: let $\kappa_n \to \kappa$ and $(p,q) \in P^\delta(\kappa)$. From corollary 3.2.8 there is a sequence $\tilde{q}_n \to q$ with $\tilde{q}_n \in \bar{Q}^\delta(\kappa_n)$ for all n; also $q_n \equiv (\tilde{q}_n/|\tilde{q}_n|)|q| \in \bar{Q}^\delta(\kappa_n)$ (because $\bar{Q}^\delta(\kappa_n)$ is a cone), and $q_n \to q$. Now let $(p_n, q_n) = (p, q_n)$, and notice that for every n, $|(p_n, q_n)| = 1$ and $p_{n,\ell} \geq 1-\delta$ for all $\ell \in L$. □

3.3 n-th stage correspondence and fixed point

Following is the list of price spaces which will be used. As usual, $T \subseteq S$.

$$P = \left\{(p,q) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ :\ \sum_{\ell=1}^L p_\ell + \sum_{j=1}^J q_j = 1\right\} = \Delta^{L+J}$$
$$R = \left\{p \in \mathbb{R}^L_+ :\ \sum_{\ell=1}^L p_\ell = 1\right\} = \Delta^L$$
$$Q_T(\kappa) = \left\{q \in \mathbb{R}^J_{++} : \exists\, \lambda \in \mathbb{R}^T_{++}\ \text{ s.t. }\ \lambda(a \otimes \kappa)^T \leq q \leq \lambda a^T\right\} \text{(as before)}$$
$$Q(\kappa) = \bigcap_{h\in H} Q_{S^h}(\kappa) \text{(as before)}$$
$$P(\kappa) = \{(p,q) \in P :\ q \in Q(\kappa)\}$$
$$P^n = \{(p,q) \in P :\ p_\ell \geq 1/n, \ell = 1,\dots,L\}$$
$$R^n = \{p \in R :\ p_\ell \geq 1/n, \ell = 1,\dots,L\}$$

$$Q_T^n(\kappa) = \bar{Q}_T^{1-\frac{1}{n}}(\kappa)$$
$$Q^n(\kappa) = \bigcap_{h\in H} Q_{S^h}^n(\kappa)$$
$$P^n(\kappa) = \{(p,q) \in P^n :\ q \in Q^n(\kappa)\}$$

Now fix n. We look for a fixed point in the space

$$P^n \times (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} \times \left(\mathbb{R}_+^{L(1+S)} \times \mathbb{R}^J \times \mathbb{R}^J\right)^H$$

of the correspondence defined as the product of three correspondences:

$$\mu^n :\ \left(\mathbb{R}_+^{L(1+S)} \times \mathbb{R}^J \times \mathbb{R}^J\right)^H \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} \longrightarrow P^n \times (R^n)^S$$
$$(\tilde{z}^{h,n})_{h\in H} :\ P^n \times (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} \times \mathbb{R}^J \times \mathbb{R}^J \longrightarrow \left(\mathbb{R}_+^{L(1+S)} \times \mathbb{R}^J \times \mathbb{R}^J\right)^H$$
$$\tilde{\beta}^n : (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} \times \left(\mathbb{R}^J \times \mathbb{R}^J\right)^H \longrightarrow \{1\}^{J\times J} \times [0,1]^{(S-J)\times J}$$

Here μ^n is the market maker correspondence; $(\tilde{z}^{h,n})_{h\in H}$ is the vector of excess demands; and $\tilde{\beta}^n$ is (part of) the book-keeping map. The price spaces are of course truncated. Moreover, the domain of repayment rates is restricted to 1 instead of $[0,1]$ on C; this will make excess demands well behaved, and at the end we will check that 1 is in fact the value of the full book-keeping map on C. Of course $(\mathbb{R}_+^{L(1+S)} \times \mathbb{R}^J \times \mathbb{R}^J)^H$ is not compact; but it will be reduced to a compact set in the usual way. We now describe the three correspondences in detail.

The market maker correspondence

Let $z(s) = \sum_{h\in H} z^h(s),\ s \in \{0,\dots,S\}$ and $\theta = \sum_{h\in H}(\phi^h - \psi^h)$. An array $((z^h(s))_{s\in\{0\}\cup S}, \phi^h, \psi^h)_{h\in H} \times \kappa$ is first mapped into $(z,\theta,\kappa) \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J}$, and then into

$$\Big\{ \left(p(0), q, (p(s))_{s\in S}\right) \in P^n(\kappa) \times (R^n)^S \text{ such that}$$
$$p(0)z(0) + q\theta = \max_{(p',q')\in P^n(\kappa)} p'z(0) + q'\theta$$
$$p(s)z(s) = \max_{p'\in R^n} p'z(s),\ s \in S \Big\}\ .$$

The correspondence P^n is continuous and compact-valued by lemma 3.2.9 (the truncation was designed with the main purpose of making that lemma hold). Hence the market maker problem is well defined and we can state that:

3.3.1 Lemma *The correspondence* μ^n *is nonempty- compact- convex-valued and upper hemicontinuous* (*uhc*).

The book-keeping map

We define $\tilde{\beta}^n = \begin{pmatrix} \mathbf{1}_{J\times J} \\ \beta_{(S-J)\times J} \end{pmatrix}$, where $\beta = \beta\big((p(s))_{s\in S}, K, (\phi^h, \psi^h)_{h\in H}\big)$ is the book-keeping map defined in section 2. That is, $(\tilde{\beta}^n)^j(s) = 1$ for all $j \in J,\ s \in C$ for any point in $(R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} \times (\mathbb{R}^J \times \mathbb{R}^J)^H$, and equal to the 'true' book-keeping map for $s \in S\backslash C$.

3.3.2 Lemma *$\tilde{\beta}^n$ is convex-valued and uhc on its domain.*

Proof. Convex-valuedness follows from the definition. For upper hemicontinuity, it suffices to prove that β is uhc on all of $(R^n)^S \times [0,1]^{S\times J} \times (\mathbb{R}^J \times \mathbb{R}^J)^H$.

First, $\imath(h,s)$ is continuous. It is clearly so when $a(s)\psi^h > 0$, i.e. when $\psi^h > 0$. Points such that $\psi^h = 0$ have neighbourhoods in which $M^h(s) = a(s)\psi^h$ (since $p(s)e^h(s) > 0$), i.e. in which $\imath(h,s) = 1$, the value it takes if $\psi^h = 0$.

So $\beta^j(s)$ is a continuous function when $\sum_{h\in H} a^j(s)\psi^{hj} > 0$, i.e. when $\sum_{h\in H} \psi^{hj} > 0$. Otherwise $\psi^{hj} = 0$, in which case $\beta^j(s) = [0,1]$ hence trivially uhc (for this purpose it was defined thus). □

The excess demand correspondence

To verify that $(\tilde{z}^{h,n})_{h\in H}$ is well behaved requires more work. We must establish the analogues of Werner's lemmas 1 and 2, with three additional difficulties: one, the consumers' problems do not have solution for all prices in the domain (which typically contains arbitrage prices) so we have to ensure that demand can be continuously extended from the domain where it is well defined to the whole price space; two, first period's budget sets are not convex due to the constraint $\phi^h\psi^h = 0$; three, we have to keep track of the states $s \notin S^h$ for $h \in H$, and be careful about the non-minimum income condition in states $s \notin \cup_{h\in H} S^h$.

We start with period zero. For each $h \in H$, the set

$$\left\{ \big(p(0), q, (p(s))_{s\in S}, \kappa\big) \in P^n \times (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J} : \, q \in Q^n_{S^h}(\kappa) \right\} \ ,$$

on which h's problem surely has solution, is closed (easy consequence of lemma 3.2.7). On this set define the correspondence $(\xi^{h,n}, \phi^{h,n}, \psi^{h,n})$ taking $(p(0), q, (p(s))_{s\in S^h}, \kappa)$ to the set of solutions of $\mathcal{CP}^h((p(0), q, (p(s))_{s\in S^h}, \kappa)$. We prove that, under our assumptions, on a superset of the above closed set the solution is unique and the resulting maximum-value function is continuous. Thus it will be possible to extend $(\xi^{h,n}, \phi^{h,n}, \psi^{h,n})$ (continuous on a closed subset of a compact set) continuously to all of $P^n \times (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J}$. The extension will be the first-period part of the $\tilde{z}^{h,n}$ map, denoted by $(\tilde{\xi}^{h,n}, \tilde{\phi}^{h,n}, \tilde{\psi}^{h,n})$. Next we consider the second period.

An $h \in H$ is fixed. We write (p,q) for $(p(0), q, (p(s))_{s\in S^h})$. Actually in period zero everything concerning agent h only depends on $s \in S^h$. Sometimes we write the full vectors, to simplify notation. Preliminarily we study the budget correspondence.

3.3.3 Lemma (i) The correspondence $(p,q,\kappa) \mapsto B^h(p,q,\kappa)$ *is closed on* $P \times (R)^{S^h} \times [0,1]^{S\times J}$.

(ii) Under *C and FR*, $B^h(p,q,\kappa)$ *is compact for all* $(p(0),(p(s))_{s\in S^h}) \gg 0$, $q \in Q_{S^h}(\kappa)$ *and* $\kappa \in [0,1]^{S\times J}$.

(iii) Under assumption E, the correspondence $(p,q,\kappa) \mapsto B^h(p,q,\kappa)$ *is lhc on* $P \times (R)^{S^h} \times [0,1]^{S\times J}$.

Proof. (i) is clear.

(ii) $B^h(p,q,\kappa)$ is clearly closed, so it suffices to show that it is bounded. The proof is a minor modification of the argument in Werner (lemma 1(ii)). Since $q \in Q_{S^h}(\kappa)$, there exists a λ such that:

$$\lambda(a\otimes\kappa)_{S^h} \leq q \leq \lambda a_{S^h} \quad .$$

Now multiply the budget constraint in each state $s \in S^h$ by the corresponding element of the vector λ, and add over the states in the second period and over the two periods. From the previous equation we derive that

$$\lambda(a\otimes\kappa)_{S^h}\phi - \lambda a_{S^h}\psi \leq q(\phi-\psi) \;\; ;$$

and from this inequality it follows that the set of feasible allocations and portfolios is bounded, by the assumptions *C* and *FR*.

(iii) Let $\breve{B}^h(p,q,\kappa)$ be the set obtained from $B^h(p,q,\kappa)$ by substituting strict inequalities for weak ones. This set is nonempty. In fact recall that from assumption *E*, $e^h(s) \gg 0$ so for any $p(s) \in R = \Delta^L$, $p(s)e^h(s) > 0$. Now if $p(0) > 0$, then $x(0) = 0, \phi = \psi = 0, x(s) = 0 \; \forall\, s \in S^h$ is in $\breve{B}^h$. If $p(0) = 0$, then $q > 0$ so $x(0) = 0, \phi = 0, x(s) = 0 \; \forall\, s \in S^h$ and a $\psi \gg 0$ (so that $q\psi > 0$) but so small that $\forall\, s \in S^h \; a(s)\psi > -p(s)e^h(s)$ is a plan in $\breve{B}^h$.

The conclusion now follows as in Werner (lemma 1(iii)): take $(p_i,q_i,\kappa_i) \longrightarrow (p,q,\kappa)$ and $(x(0),(x(s))_{s\in S^h},\phi,\psi) \in \breve{B}^h(p,q,\kappa)$. The constant sequence of actions equal to the given one is eventually in $\breve{B}^h(p_i,q_i,\kappa_i)$ which is thus lhc. $B^h = \text{cl}\,\breve{B}^h$ is then lhc too. □

3.3.4 Corollary *Assume* C, E and *FR*. *Then* $(p,q,\kappa) \mapsto B^h(p,q,\kappa)$ *is a continuous correspondence for* $(p(0),(p(s))_{s\in S^h}) \gg 0$, $q \in Q_{S^h}(\kappa)$ *and any* κ.

Proof. The lemma p. 33 in Hildenbrand cannot be invoked directly because B^h is not convex-valued: we refine that argument. At any (p,q,κ) as in the statement B^h is lhc, so it remains to show upper hemicontinuity. By part (ii) of the previous lemma B^h is compact-valued at these (p,q,κ), so the characterization of upper hemicontinuity in Hildenbrand's Theorem 1 p. 24 applies. B^h is (nonempty and) closed, thus as in Hildenbrand p. 33 we have to show that: if $(p_i,q_i,\kappa_i) \longrightarrow (p,q,\kappa)$, any sequence $\chi_i = (x_i(0),(x_i(s))_{s\in S^h}, \phi_i,\psi_i) \in B^h(p_i,q_i,\kappa_i)$ is bounded. The proof is by contradiction along standard lines. Suppose in fact that the sequence χ_i is not bounded: dividing each element in the sequence by the norm, and taking limits we find a limit vector of unit norm. Using the joint continuity (in the two variables) of the finite dimensional inner product, it is immediate that the no arbitrage condition is violated by q at κ. But $q \in Q_{S^h}(\kappa)$, a contradiction. Again, we are using here the two assumptions *C* and *FR*. □

We now turn to the demand correspondence.

3.3.5 Lemma *Let* (ξ^h, ϕ^h, ψ^h) *denote the correspondence from* $P \times (R)^{S^h} \times [0,1]^{S\times J}$ *to* $\mathbb{R}_+^{L(1+S^h)} \times \mathbb{R}_+^{2J}$ *taking* (p, q, κ) *to the set of solutions of* $\mathcal{CP}^h\,(p, q, \kappa)$.

Assume E, C, U, FR. *Then for any* $(p(0), (p(s))_{s\in S^h}) \gg 0$, $q \in Q_{S^h}(\kappa)$ *and any* κ, *the correspondence* (ξ^h, ϕ^h, ψ^h) *is nonempty- compact-valued and uhc. If moreover* $\kappa_C = \mathbf{1}$, *it is also single-valued.*

Proof. From the previous corollary and the maximum theorem (Hildenbrand p. 30) we get all except convex-valuedness. To conclude we show that if $\kappa_C = \mathbf{1}$ the solution to the consumer's problem is unique.

Because of the constraint $\phi\psi = 0$ –which, given $\phi, \psi \in \mathbb{R}_+^J$, is equivalent to $\phi^j\psi^j = 0\ \ \forall i \in J$–, $B^h(p, q, \kappa)$ is not convex. However, given $(x(0), (x(s))_{s\in S^h}, \phi, \psi)$ satisfying all the restrictions in $B^h(p, q, \kappa)$ except $\phi\psi = 0$, there is an element $(x'(0), (x'(s))_{s\in S^h}, \phi', \psi') \in B^h$ with $u^h(x'(0),\ (x'(s))_{s\in S^h}) \geq u^h(x(0), (x(s))_{s\in S^h})$, for: suppose j is such that $\phi^j\psi^j > 0$, say $\phi^j > \psi^j$. Then $\phi'^j = \phi^j - \psi^j, \psi'^j = 0$ has the same cost and guarantees the same income or higher in every $s \in S^h$. In fact $q^j(\phi^j - \psi^j) = q^j(\phi'^j - \psi'^j)$; and $\forall s \in S^h$,

$$\begin{aligned}&(a\otimes\kappa)^j(s)\phi^j - a^j(s)\psi^j = (a\otimes\kappa)^j(s)(\psi^j + \phi'^j) - a^j(s)\psi^j\\ &\leq (a\otimes\kappa)^j(s)\phi'^j = (a\otimes\kappa)^j x(s)\phi'^j - a^j(s)\psi'^j \quad (\text{for } \kappa^j(s) \leq 1)\ .\end{aligned}$$

Similarly if $\psi^j > \phi^j > 0$, the trade $\psi'^j = \psi^j - \phi^j,\ \phi'^j = 0$ would be at least as good.

Now suppose $\chi = (x(0), (x(s))_{s\in S^h}, \phi, \psi)$ and $\chi' = (x'(0), (x'(s))_{s\in S^h}, \phi', \psi')$ were two solutions to $\mathcal{CP}^h(p, q, \kappa)$. If $(x(0), (x(s))_{s\in S^h}) \neq ((x'(0), (x'(s))_{s\in S^h})$, then by assumption U, $\hat\chi = \frac{1}{2}(\chi + \chi')$ is such that $u^h(\hat x(0), (\hat x(s))_{s\in S^h}) > u^h(x(0), (x(s))_{s\in S^h})$; but $\hat\chi$ may fail to satisfy $\hat\phi\hat\psi = 0$. However, taking $\chi'' \in B^h(p, q, \kappa)$ such that $u^h(x''(0), (x''(s))_{s\in S^h}) \geq u^h(\hat x(0), (\hat x(s))_{s\in S^h})$ (we have seen above how this can be done) enables to conclude that goods' demand is unique. By monotonicity of u^h, then also optimal asset returns in the states in S^h are unique. But then assumptions C, FR, and $\kappa_C = \mathbf{1}$ imply that (since $(a\otimes\kappa)_C = a_C$ is full rank and $C \subseteq S^h$) net asset demand $\phi - \psi$ is unique. This and $\phi^j\psi^j = 0\ \forall j \in J$ then imply that also ϕ and ψ are unique (e.g. if $(\phi-\psi)^j = (\phi'-\psi')^j > 0$, $(\phi-\psi)^j = \phi^j,\ (\phi'-\psi')^j = \phi'^j$). □

The set of prices and κ's in the statement of this lemma contains the set on which $(\xi^{h,n}, \phi^{h,n}, \psi^{h,n})$ is defined, for goods' prices are all positive in P^n and R^n and by lemma 3.2.6 $Q^n_{S^h}(\kappa) \subseteq Q_{S^h}(\kappa)$. Thus $(\xi^{h,n}, \phi^{h,n}, \psi^{h,n})$ can be extended to the 'pseudo-demand' $(\tilde\xi^{h,n}, \tilde\phi^{h,n}, \tilde\psi^{h,n})$, continuous on $P^n \times (R^n)^S \times \{1\}^{J\times J} \times [0,1]^{(S-J)\times J}$.

Now turn to the second period, $h \in H,\ s \notin S^h$. First the budget correspondence again. We can write the budget set as

$$\begin{aligned}&B^h_s(p(s), \kappa(s), \phi, \psi)\\ &= \left\{x(s) \in \mathbb{R}^L_+ :\quad p(s)x(s) \leq \max\{0, p(s)e^h(s) + (a\otimes\kappa)(s)\phi - a(s)\psi\right\}\end{aligned}$$

3.3.6 Lemma (i) *The correspondence* $(p(s), \kappa(s), \phi, \psi) \mapsto B_s^h(p(s), \kappa(s), \phi, \psi)$ *is closed on* $R \times [0,1]^J \times \mathbb{R}_+^{2J}$

(ii) $B_s^h(p(s), \kappa(s), \phi, \psi)$ *is compact for any* $p(s) \gg 0$

(iii) *The correspondence in (i) is lhc for any* $p(s) \gg 0$.

Proof. (i) Let $(p_i(s), \kappa_i(s), \phi_i, \psi_i) \longrightarrow (p(s), \kappa(s), \phi, \psi)$, $x_i(s) \in B_s^h((\cdot)_i)$ and $x_i(s) \longrightarrow x(s)$. To show $x(s) \in B_s^h((\cdot))$, with self-evident notation. Let 'net income' $NI = p(s)e^h(s) + (a \otimes \kappa)(s)\phi - a(s)\psi$, NI_i along the sequence. If $NI < 0$, for i large enough $NI_i < 0$ so $p_i(s)x_i(s) = 0$ whence $p(s)x(s) = 0$ i.e. $x(s) \in B_s^h((\cdot))$. If $NI > 0$ eventually $p_i(s)x_i(s) = NI_i$ whence again from $p(s)x(s) = NI$, $x(s) \in B_s^h((\cdot))$. If $NI = 0$, $p_i(s)x_i(s) \longrightarrow 0$ so again $p(s)x(s) = \lim p_i(s)x_i(s) = NI$ and $x(s) \in B_s^h((\cdot))$.

(ii) This is clear, since we are working with $\mathbb{R}_+^L$ as consumption set.

(iii) If in the definition of the budget set $\max\{0, \cdot\} > 0$, do it the standard way. If $\max\{0, \cdot\} = 0$, then $p(s) \gg 0$ implies that $B_s^h(\cdot) = \{0\} \in \mathbb{R}^L$. Since $0 \in B_s^h(\cdot)$ for any $p(s), \kappa(s), \phi, \psi)$, the argument here is trivial (take the constant 0 sequence). □

3.3.7 Corollary *The correspondence* $(p(s), \kappa(s), \phi, \psi) \mapsto B_s^h(p(s), \kappa(s), \phi, \psi)$ *is continuous and compact-valued whenever* $p(s) \gg 0$.

Proof. This correspondence is clearly convex-valued, so continuity follows from the previous lemma and Hildenbrand p. 33. Compact-valuedness is also clear. □

3.3.8 Lemma *Let* ξ_s^h $(s \notin S^h)$ *be the correspondence from* $R \times [0,1]^J \times \mathbb{R}_+^{2J}$ *to* $\mathbb{R}_+^L$ taking $(p(s), \kappa(s), \phi, \psi)$ *to the set of solutions of* $\mathcal{CP}^h(p(s), \kappa(s), \phi, \psi)$.

Assume U. Then ξ_s^h *is a continuous function at any* $(p(s), \kappa(s), \phi, \psi)$ *such that* $p(s) \gg 0$.

Proof. From the previous corollary, the maximum theorem (Hildenbrand p. 30) and U (which gives single-valuedness) the conclusion is direct. □

We let $\xi_s^{h,n}$ be the restriction of the ξ_s^h of the previous lemma to $R^n \times [0,1]^J \times \mathbb{R}_+^{2J}$. We can now define the $(\tilde{z}^{h,n})_{h \in H}$-correspondence as having components

$$\tilde{z}^{h,n} = \left(\left(\tilde{\xi}^{h,n}(s) - e^h(s) \right)_{s \in \{0\} \cup S^h}, \left(\xi_s^{h,n} - e^h(s) \right)_{s \in S \setminus S^h}, \tilde{\phi}^{h,n}, \tilde{\psi}^{h,n} \right) \quad h \in H \ .$$

The n-th stage fixed point

3.3.9 Lemma *For every* $n \geq 1$, *the correspondence defined as the product of* μ^n, $(\tilde{z}^{h,n})_{h \in H}$ *and* $\tilde{\beta}^n$ *has a fixed point, denoted by*

$$\left(p_n, q_n, \kappa_n, \left((z_n^h(s))_{s \in \{0\} \cup S}, \phi_n^h, \psi_n^h \right)_{h \in H} \right)$$

(*where* $(p_n = (p_n(0), (p_n(s))_{s \in S})$). *Period-zero excess demand is equal to 'true' demand at the fixed-point prices* (p_n, q_n, κ_n), *i.e. for all* $h \in H$

$$\left(\left(z_n^h(s) + e^h(s) \right)_{s \in \{0\} \cup S^h}, \phi_n^h, \psi_n^h \right) = \left(\left(\xi^{h,n}(s) \right)_{s \in \{0\} \cup S^h}, \phi^{h,n}, \psi^{h,n} \right) (p_n, q_n, \kappa_n) \ .$$

And finally, the fixed point is such that, letting $z_n(s) = \sum_{h \in H} z_n^h(s)$, $s \in \{0\} \cup S$ *and* $\theta_n = \sum_{h \in H}(\phi_n^h - \psi_n^h)$, *one has for all* $(p(0), q) \in P^n\ (\kappa_n), p \in R^n$,

$$p(0)z_n(0) + q\theta_n \leq 0 \tag{1}$$

$$pz_n(s) \leq (a \otimes \kappa_n)(s)\theta_n \quad s \in S \tag{2}$$

(*analogue of* (1) *and* (2) *of Werner*).

Proof. Let $\Xi^h \subseteq \mathbb{R}^{L(1+S^h)_+} \times \mathbb{R}_+^{2J}$ be a compact convex superset of the compact range of the function $(\tilde{\xi}^{h,n}, \tilde{\phi}^{h,n}, \tilde{\psi}^{h,n})$, continuous on its compact domain $P^n \times (R^n)^S \times \{1\}^{J \times J} \times [0,1]^{(S-J) \times J}$.

Let Ξ_{2J}^h be the compact projection of Ξ^h into $\mathbb{R}_+^{2J}$; restrict $\xi_s^{h,n}$ to $R^n \times [0,1]^J \times \Xi_{2J}^h$ and denote by $\Xi_s^h \subseteq \mathbb{R}_+^L$ a compact superset of the range of this restriction (continuous function on a now compact domain).

Now for each $h \in H$ take the product of Ξ^h and $\times_{s \notin S^h} \Xi_s^h$ to get a compact subset $\overline{\Xi}^h$ of $\mathbb{R}_+^{L(1+S)} \times \mathbb{R}_+^{2J}$; let $\overline{\Xi} = \times_{h \in H} \overline{\Xi}^h$; and finally, define the restricted product-correspondence on the convex compact set

$$P^n \times (R^n)^S \times \{1\}^{J \times J} \times [0,1]^{(S-J) \times J} \times \overline{\Xi}\ .$$

By lemmas 3,3.1, 3.3.2, 3.3.5 and 3.3.8 this correspondence is convex-, compact-valued and uhc, hence Kakutani's theorem gives existence of a fixed point.

The assertion about excess demand follows from the definition of the μ-map, which puts prices in $P^n(\kappa_n) \times (R^n)^S$ where the consumers' problems have solution.

Validity of the two equations in the final part of the lemma follows from Walras' laws, which we next state, and the market maker problem, in the standard way. We return on this at the end of the next subsection, after stating and proving Walras' laws. □

3.4 Walras' laws

3.4.1 Lemma *Let* $(p, q, \kappa) \in P \times R^S \times [0,1]^{SJ}$. *For period zero, one has*

$$p(0) \sum_{h \in H} z^h(0) + q \sum_{h \in H} \theta^h = 0. \quad WL(0)$$

Period 1, *state s*: *assume that* $\kappa(s)$ *is a fixed point of* (*BK*), *or that* $\kappa(s) = \mathbf{1}$ *and* $s \in \cap_{h \in H} S^h$. *Then*

$$p(s) \sum_{h \in H} z^h(s) = (a \otimes \kappa)(s) \sum_{h \in H} \theta^h \quad WL(s)$$

Proof. Period zero: Summing over $h \in H$ the individual restrictions in the budget constraints gives $(WL)(0)$.

Period 1, state s: summing over $h \in H$ the individual constraints (in B^h if $s \in S^h$, in B_s^h otherwise) one gets

$$p(s) \sum_{h \in H} z^h(s) = (a \otimes \kappa)(s) \sum_{h \in H} \phi^h - \sum_{h \in H} M^h(s)\ . \tag{1}$$

Observe that (since $a \gg 0$) $\sum_{h\in H} a^j(s)\psi^{hj} = 0$ iff $\psi^{hj} = 0\ \forall\, h \in H$, and that $a(s)\psi^h = 0$ iff $\psi^{hj} = 0\ \forall\, j \in J$. Then from (BK) one has for any $j \in J$

$$\kappa^j(s)\sum_{h\in H} a^j(s)\psi^{hj} = \sum_{\{h:a(s)\psi^h>0\}} \frac{M^h(s)}{a(s)\psi^h}\ a^j(s)\psi^{hj}$$

(summation over an empty set is defined to be zero). The same equality holds if $\kappa(s) = \mathbf{1}$ and $s \in \cap_{h\in H} S^h$, for in that case $M^h(s) = a(s)\psi^h$ all $h \in H$. Summing over j one obtains $LHS = \sum_{j\in J}\sum_{h\in H} \kappa^j(s)a^j(s)\psi^{hj} = \sum_{h\in H}(a\otimes\kappa)(s)\psi^h$, and

$$RHS = \sum_{j\in J}\sum_{\{h:a(s)\psi^h>0\}} \frac{M^h(s)}{a(s)\psi^h} a^j(s)\psi^{hj} = \sum_{\{h:a(s)\psi^h>0\}} M^h(s) = \sum_{h\in H} M^h(s)$$

(the last equality because $a(s)\psi^h = 0$ implies $M^h(s) = 0$), i.e.

$$\sum_{h\in H} M^h(s) = (a\otimes\kappa)(s)\sum_{h\in H}\psi^h \tag{2}$$

Substituting into (1) gives $(WL)(s)$. □

Observe that by definition of the map $\tilde{\beta}^n$, $(WL)(0)$ and $(WL)(s)$ hold at the fixed point of last subsection, for every n. Then equations (1) and (2) in lemma 3.3.9 are direct consequences of the maximization on the part of the market maker.

3.5 Convergence of fixed points

At this step the objective is to find a convergent subsequence of the n-th stage fixed points. The next and last step will be to prove that limit actions and limit prices form an equilibrium.

Since the sequence of fixed-point prices (p_n, q_n, κ_n) is bounded, it converges (along a subsequence) to a limit $(\bar{p}, \bar{q}, \bar{\kappa})$. Note that $(\bar{p}(0), \bar{q}) \in P$ and $\bar{p}(s) \in R$ for all $s \in S$. And $\bar{\kappa}_C = \mathbf{1}$ (for $(\kappa_n)_C = \mathbf{1}\ \forall\, n$).

3.5.1 Lemma *The sequence $((z^h_n(s))_{s\in\{0,\dots,S\}}, \phi^h_n, \psi^h_n)_{h\in H})_{n\geq 1}$ converges along a subsequence, to a limit $((\bar{z}^h(s))_{s\in\{0,\dots,S\}}, \bar{\phi}^h, \bar{\psi}^h)_{h\in H})$. Along this subsequence also aggregate demands converge: $z_n(s) \to \bar{z}(s) = \sum_{h\in H}\bar{z}^h(s),\ s \in \{0,\dots,S\}$ and $\theta_n \to \bar{\theta} = \sum_{h\in H}(\bar{\phi}^h - \bar{\psi}^h)$.*

Proof. We start by proving that for $s \in \{0,\dots,S\}$ the sequence $(z_n(s))_{n\geq 1}$ is bounded. From this, convergence of a subsequence of $((z^h_n(s))_{h\in H})_{n\geq 1}$ follows.

Recall that equations $(1), (2)$ of p. 21 hold for all $(p(0), q) \in P^n(\kappa_n), p \in R^n$, where $P^n(\kappa_n) = \{(p,q) \in P:\ p_\ell \geq 1/n, q \in \cap_{h\in H} Q^n_{S^h}(\kappa_n)\}$; and observe that since for any δ, κ, T, $G^\delta_T(\mathbf{1}) \subseteq G^\delta_T(\kappa)$, then for every κ_n, $\cap_{h\in H} Q^n_{S^h}(\mathbf{1}) \subseteq \cap_{h\in H} Q^n_{S^h}(\kappa_n)$.

Fix $\eta \in (0,1)$. We claim that: there exist a $\tilde{q} \in \mathbb{R}^J_+$ with $|\tilde{q}| = 1-\eta$, a vector v and a sequence $(v^n)_{n\geq 1}$ in $\mathbb{R}^S_{++}$ with $v^n \geq v$ for all $n \geq 1$, such that

$$\tilde{q} = v^n(a \otimes \kappa_n) \in Q^n(\kappa_n) \quad \text{for all } n \ .$$

Assuming the claim true, let $\tilde{p}_\ell(0) = \eta/L,\ \ell \in L,\ \tilde{p}_\ell(s) = 1/L,\ \ell \in L, s \in S$ and $\tilde{q}$ as in the claim. Then this vector of prices is in $P^n(\kappa_n) \times (R^n)^S$ for n sufficiently large; hence from (1), (2) of p. 21 we get:

$$\tilde{p}(0)z_n(0) + v^n(a \otimes \kappa_n)\theta_n \leq 0$$
$$v_s^n\tilde{p}(s)z_n(s) \leq v^n(a \otimes \kappa_n)(s)\theta_n$$

and therefore $\tilde{p}(0)z_n(0) + \sum_{s\in S} v_s^n\tilde{p}(s)z_n(s) \leq 0$ for all n. This implies boundedness of $(z_n(s))_{n\geq 1}$, since the v_s^n are bounded away from zero uniformly in n.

Now we prove the claim: recall first that $\kappa_n(s) = \mathbf{1}$ for $s = 1, \ldots, C$. Define

$$v_s^n = v_s = \frac{\epsilon}{\#(S\backslash C)|a|_\infty}, \quad s \in S\backslash C$$

where ϵ is to be determined and $|a|_\infty = \max_{s\in S, j\in J} a^j(s)$. Then $\sum_{s\in S\backslash C} v_s (a \otimes \kappa_n)(s) \leq \epsilon\mathbf{1}$ for every n.

Define $\tilde{q} = \alpha \sum_{s\in C} a(s)$ where α is such that $\alpha \sum_{j\in J}\sum_{s\in C} a^j(s) = 1 - \eta$, so that $|\tilde{q}| = 1 - \eta$. We claim that for n large enough $\tilde{q} \in Q^n(\kappa_n)$. To see this: $\tilde{q} = \lambda a_C$ for a $\lambda \gg 0$, hence $\tilde{q} \in Q_C(\mathbf{1}) \subseteq Q_T(\mathbf{1})\ \forall\ T \supseteq C$ (by lemma 3.1.3), so $\tilde{q} \in Q(\mathbf{1})$ and therefore $\tilde{q} \in Q^n(\mathbf{1}) \subseteq Q^n(\kappa_n)$ for n large.

Now choose ϵ so that the closed ball $B(\tilde{q}, \epsilon) \subseteq \text{int Co}\ (a_C, a_C)$. This is clearly possible from the definition of $\tilde{q}$ (and *FR*). Let $q_n^\epsilon = \sum_{s\in S\backslash C} v_s(a \otimes \kappa_n)(s)$ ($\leq \epsilon\mathbf{1}$ as we have seen) and write $\tilde{q} = \tilde{q} - q_n^\epsilon + q_n^\epsilon$. For ϵ small enough the vector $\tilde{q} - q_n^\epsilon$ is in $B(\tilde{q}, \epsilon)$ so there exists $v_C = (v_1, \ldots, v_J) \in \mathbb{R}_{++}^J$ such that $v_C^n \equiv (\tilde{q} - q_n^\epsilon)a_C^{-1} \geq v_C$ for all n, and the claim is proved.

To finish the proof we show that for each $h \in H$ the sequence $(\phi_n^h, \psi_n^h)_{n\geq 1}$ has a convergent subsequence. Given $\kappa_n = \mathbf{1}$ all n, one has for all $s \in C,\ n \geq 1$, $p_n(s)z_n^h(s) = a(s)(\phi_n^h - \psi_n^h)$, so the sequence $\phi_n^h - \psi_n^h = a_C^{-1}(p_n\ (s) \times z_n^h(s))_{s\in C}$ converges along the same subsequence where $p_n(s)z_n^h(s)$ does. But $\phi_n^h, \psi_n^h \geq 0$ and $\phi_n^h\psi_n^h = 0$ for all $h \in H, n \geq 1$ imply that $\phi_n^{hj} = \max\{0, \phi_n^{hj} - \psi_n^{hj}\}$ and $\psi_n^{hj} = -\min\{0, \phi_n^{hj} - \psi_n^{hj}\}$; hence ϕ_n^h and ψ_n^h converge too. □

3.6 The limit is an equilibrium

We start with prices.

3.6.1 Lemma *As in lemma* 3.3.5, *let* $\chi^h = (\xi^h, \phi^h, \psi^h)$ *be the correspondence taking* (p, q, κ) *to the set of maximizers of* $\mathcal{CP}^h(p, q, \kappa)$.

(i) Under assumption E, χ^h *is closed on* $P \times (R)^S \times [0, 1]^{J\times S}$.

(ii) *Assume* E, C, U and *FR*. *Let* $(p_i, q_i, \kappa_i) \to (p, q, \kappa)$ *with* $(\kappa_i)_C = \kappa_C = \mathbf{1}$, $p_i \gg 0$, $q_i \in Q_{S^h}(\kappa_i)\ \forall\ i \geq 1$ *and* $(p(0), (p(s))_{s\in S^h}) \not\gg 0$ or $q \notin Q_{S^h}\ (\kappa)$, *and let* $(x_i(0), (x_i(s))_{s\in S^h}, \phi_i, \psi_i) \in \chi^h(p_i, q_i, \kappa_i)$. *Then* $|(x_i(0), (x_i(s))_{s\in S^h})| \longrightarrow \infty$.

Proof. (i) Take $(p_i, q_i, \kappa_i) \longrightarrow (p, q, \kappa)$, $(x_i(0), (x_i(s))_{s\in S^h}, \phi_i, \psi_i) \in \chi^h(p_i, q_i, \kappa_i)$ converging to $(x(0), (x(s))_{s\in S^h}, \phi, \psi)$; to show that the latter is in $\chi^h(p, q, \kappa)$.

First, by closedness of B^h (lemma 3.3.3) it is in $B^h(p,q,\kappa)$. Take now an arbitrary point $(x'(0),(x'(s))_{s\in S^h},\phi',\psi') \in B^h(p,q,\kappa)$. By lower hemicontinuity of B^h (lemma 3.3.3) there exists a sequence $(x_i'(0),(x_i'(s))_{s\in S^h},\phi_i',\psi_i') \in B^h(p_i,q_i,\kappa_i)$ which converges to $(x'(0),(x'(s))_{s\in S^h},\phi',\psi')$. Since $u^h(x_i(0), (x_i(s))_{s\in S^h}) \geq u^h(x_i'(0),(x_i'(s))_{s\in S^h})$, by continuity of u^h then $u^h(x(0),(x(s))_{s\in S^h}) \geq u^h(x'(0),(x'(s))_{s\in S^h})$.

(ii) Assume not. Then $(x_i(0),(x_i(s))_{s\in S^h}) \longrightarrow (x(0),(x(s))_{s\in S^h})$ along a subsequence. For all $s \in C$ one has $p_i(s)(x_i(s) - e^h(s)) = a(s)\theta_i$ (where $\theta_i = \phi_i - \psi_i$), so $a_C\theta_i$ converges, and by *FR* θ_i converges. Since $\phi_i\psi_i = 0\ \forall i$, then also (ϕ_i^j,ψ_i^j) converge, say to (ϕ,ψ). By part (i) then $(x(0), (x(s))_{s\in S^h},\phi,\psi) \in \chi^h(p,q,\kappa)$, which is impossible because $\chi^h(p,q,\kappa) = \emptyset$. □

3.6.2 Lemma *The limit to which the (sub)sequence of n-th stage fixed-points converges (see the lemma of section* 3.5) *is such that*: $\bar{q} \in Q(\kappa)$, $\bar{p}(0) \gg 0$ *and* $\bar{p}(s) \gg 0$ for all $s \in \cup_{h\in H} S^h$; *and for each* $h \in H$ $((\bar{x}^h(s))_{s\in\{0\}\cup S^h},\bar{\phi}^h,\bar{\psi}^h)$ *is solution of* $CP^h(\bar{p},\bar{q},\bar{\kappa})$.

Proof. Since excess demands converge, the assertion about prices is a direct consequence of the previous lemma and lemma 3.2.6. The other follows from continuity of the excess demand functions at those prices (lemma 3.3.5). □

Next we derive the analogue of Werner's equations (6) and (7), which we too will label (6) and (7), from our equations (1), (2) of p. 21.

Equation (7) follows as in Werner, for ri $R = \cup_{n\geq 1} R^n$:

$$p(s)\bar{z}(s) \leq (a \otimes \bar{\kappa})(s)\bar{\theta}(s) \quad \forall\, p(s) \in \text{ri } R\ . \tag{7}$$

For equation (6) we have the usual complication given by the fact that the n-th stage price space depends on κ_n. First express ri $P(\kappa)$ conveniently:

3.6.3 Lemma $\text{ri}\, P(\kappa) = \{(p,q) \in \text{ ri } P:\ q \in \text{ri}\, Q(\kappa)\}$

Proof. We can write $P(\kappa) = P \cap (\mathbb{R}^L \times Q(\kappa))$, two sets with ri's with non-empty intersection. Then from Rockafellar 6.5 the ri of the intersection is equal to the intersection of the ri's, which is what is claimed. □

3.6.4 Lemma If $p(0)z_n(0) + q\theta_n \leq 0$ for all $(p(0),q) \in P^n(\kappa_n)$, and $z_n(0) \rightarrow \bar{z}(0)$, $\theta_n \rightarrow \bar{\theta}$, $\kappa_n \rightarrow \bar{\kappa}$, then

$$p(0)\bar{z}(0) + q\bar{\theta} \leq 0 \quad \forall\ (p(0),q) \in \text{ ri}\, P(\bar{\kappa}) \tag{6}$$

Proof. It clearly suffices to prove that if $(p(0),q) \in \text{ri } P(\bar{\kappa})$ then for n large enough $(p(0),q) \in P^n(\kappa_n)$. Recall that $Q(\bar{\kappa})$ is open. Then (see the expression of ri $P(\bar{\kappa})$ above and the definition of $P^n(\kappa_n)$) it is enough to show that if $q \in Q(\bar{\kappa})$ then $q \in Q^n(\kappa_n)$ for n large.

Take first any $T \supseteq C$. If $q \in Q_T(\bar{\kappa})$ and $\kappa_n \longrightarrow \bar{\kappa}$, then $q \in Q_T^n(\kappa_n)$ for n large. To see this: from lemma 3.2.3 and Rockafellar p. 50, $q \in Q_T(\bar{\kappa})$ (open) is equivalent to $q/|q| \in \text{ri conv}\, G_T(\bar{\kappa})$. Since as $n \rightarrow \infty$ conv $G_T^{1-1/n}(\kappa_n) \longrightarrow$ conv $G_T(\bar{\kappa})$, for n large $q/|q| \in \text{conv} G_T^{1-1/n}(\kappa_n)$, hence $q \in$ cone conv $G_T^{1-1/n}(\kappa_n) = Q_T^n(\kappa_n)$.

To conclude, if $q \in Q(\bar{\kappa})$, then $q \in Q_{S^h}(\bar{\kappa})$ for each $h \in H$, hence for n large $q \in Q^n_{S^h}(\kappa_n)$, so $q \in \cap_{h \in H} Q^n_{S^h}(\kappa_n) = Q^n(\kappa_n)$. □

Next we show that (6) and (7) imply $((\bar{z}(s))_{s \in \{0\} \cup_{h \in H} S^h}, \bar{\theta}) = 0$. First observe that

3.6.5 Claim *$\bar{\kappa}$ is a fixed point of (BK) at the limit prices and asset demands, i.e.*

$$\bar{\kappa}^j(s) \in \beta^j(s)\big(\bar{p}(s), \bar{\kappa}(s), (\bar{\phi}^h, \bar{\psi}^h)_{h \in H}\big) \quad j \in J, \ \ s \in S \ .$$

Proof. For $s \in C$, this holds because $s \in \cap_{h \in H} S^h$ and for each $h \in H$ asset demands satisfy budget constraints at $\bar{p}(s)$ so for j such that $\sum_{h \in H} \bar{\psi}^{hj} > 0$ one has $\beta^j(s)(\cdot) = 1 = \bar{\kappa}^j(s)$, while for the other j's (if any) $\beta^j(s)(\cdot)$ is arbitrary hence still contains $\bar{\kappa}^j(s) = 1$.

For $s \in S \backslash C$, $\kappa_n(s)$ is a fixed point of (BK) at fixed-point prices and asset demands $(p_n, q_n, (\phi^h_n, \psi^h_n)_{h \in H})$ for all n, by definition of the correspondence $\tilde{\beta}^n$ $(= \beta$ for such $s)$. Then the conclusion follows from convergence of n-th stage fixed points and closedness of β. □

Therefore, also Walras laws $(WL)(0)$ and $(WL)(s)$ hold in the limit. In period zero, this means

$$\bar{p}(0)\bar{z}(0) + \bar{q}\bar{\theta} = 0 \ . \tag{8}$$

3.6.6 Claim (*As in Werner*) (6) *and* (8) imply $\bar{z}(0) = \bar{\theta} = 0$.

Proof. We spell out the argument.

1. (6) implies $p\bar{z}(0) \leq 0$ and $q\bar{\theta} \leq 0$ for all $(p, q) \in \mathrm{ri}\ \ P(\kappa)$.

Proof of this: say there is $(\tilde{p}, \tilde{q}) \in \mathrm{ri}\, P(\kappa)$ such that $\tilde{p}\bar{z}(0) > 0$. Chose any $q \in Q(\kappa)$ such that $|q| = 1 - |\tilde{p}|$. Then for ϵ small enough $(\frac{1-\epsilon|q|}{|\tilde{p}|}\tilde{p}, \epsilon q)$ $\in \mathrm{ri}\, P(\kappa)$, and as $\epsilon \ \rightarrow \ 0$

$$\frac{1 - \epsilon|q|}{|\tilde{p}|}\tilde{p}\bar{z}(0) + \epsilon q\bar{\theta} \longrightarrow \frac{\tilde{p}}{|\tilde{p}|}\bar{z}(0) > 0 \ \ ,$$

contradicting (6). Say now $\tilde{q}\theta > 0$ for a $(\tilde{p}, \tilde{q}) \in \mathrm{ri}\, P(\kappa)$. Similarly to the previous case, we reach a contradiction by fixing $p \in \mathrm{ri}\, P$ such that $|p| = 1 - |\tilde{q}|$ and observing that as $\epsilon \longrightarrow 0$, $\epsilon p\bar{z}(0) + ((1 - \epsilon|p|)/|\tilde{q}|)\tilde{q}\bar{\theta}$ $\longrightarrow (\tilde{q}/|\tilde{q}|)\bar{\theta} > 0$.

2. Then $\bar{p}(0)\bar{z}(0) = \bar{q}\bar{\theta} = 0$ (this is clear from 1 and equation (8)).

3. $\bar{z}(0) = 0$

Proof: $p\bar{z}(0) \leq 0$ for all $(p, q) \in \ \mathrm{ri}\, P(\kappa)$ (true from 1) implies $\bar{z}(0) \leq 0$ (in fact if for some ℓ, $\bar{z}_\ell(0) > 0$, choose ϵ small enough, $q \in Q(\kappa)$ with $|q| = \epsilon$, $p_{\ell'}$ close to zero for $\ell' \neq \ell$, p_ℓ close to $1 - \epsilon$ with $|p| + |q| = 1$ to reach a contradiction). Then use $\bar{p}(0) \gg 0$ and $\bar{p}(0)\bar{z}(0) = 0$.

4. $\bar{\theta} = 0$

Proof: we know that $q\bar{\theta} \leq 0$ for all $(p, q) \in \mathrm{ri}\, P(\kappa)$, and $\bar{q}\bar{\theta} = 0$, with $|\bar{q}| = 1 - |\bar{p}(0)| \in (0, 1)$. If $\bar{\theta} \neq 0$, choose $q \in Q(\bar{\kappa})$ close to $\bar{q}$ such that $q\bar{\theta} > 0$ and $0 < |q| < 1$ (which we can do since $Q(\bar{\kappa})$ is open), and complete to a $(p, q) \in \mathrm{ri}\ \ P(\bar{\kappa})$ such that $q\bar{\theta} > 0$: contradiction. □

3.6.7 Claim $((\bar{z}(s)))_{s\in\cup_{h\in H}S^h} = 0$.

Proof. This follows easily from the limit $(WL)(s)$, which given $\bar{\theta} = 0$ is

$$\bar{p}(s)\bar{z}(s) = 0\ , \tag{9}$$

equation (7) and the fact that $\bar{p}(s) \gg 0\ \forall\ s \in \cup_{h\in H}S^h$. □

Lastly we consider $s \notin \cup_{h\in H}S^h$, of which no one is aware. These have to be treated separately because since nobody's actions in period zero depend on $p(s)$ for such s, as $p_n(s)$ goes to the boundary of R we cannot rely on first period planned demands' explosions. We will check that in the second period there must be at least one agent for whom the non-minimum income condition is satisfied, and *his* demand would then explode as $p_n(s)$ goes to the boundary.

3.6.8 Lemma *For $h \in H$ and $s \notin S^h$, let ξ^h_s denote as in lemma* 3.3.8 *the correspondence taking $(p(s), \kappa(s), \phi, \psi)$ to the set of solutions to $CP^h(p(s),\ \kappa(s), \phi, \psi)$, and assume U.*

(i) *Suppose $(p_i(s), \kappa_i(s), \phi_i, \psi_i) \longrightarrow (p(s), \kappa(s), \phi, \psi)$ with $p_i(s) \gg 0\ \forall\ i$, $p(s) \not\gg 0$ and $p(s)e^h(s) + (a \otimes \kappa)(s)\phi - a(s)\psi > 0$. Let $x_i(s) \in \xi^h_s(p_i(s), \kappa_i(s), \phi_i, \psi_i)$. Then $|x_i(s)| \longrightarrow \infty$.*

(ii) *Take $s \notin \cup_{h\in H}S^h$. Suppose $(p_i(s), \kappa_i(s), \phi_i, \psi_i) \longrightarrow (p(s), \kappa(s), \phi, \psi)$* with *$p_i(s) \gg 0\ \forall\ i$, $p(s) \not\gg 0$, $\kappa(s)$ fixed point of (BK), and $\sum_{h\in H}(\phi^h - \psi^h) = 0$. Let $x_i(s) \in \xi^h_s(p_i(s), \kappa_i(s), \phi_i, \psi_i)$* for all $h \in H,\ i \geq 1$. *Then $|\sum_{h\in H} x^h_i(s)| \longrightarrow \infty$.*

Proof. (i) is standard, e.g. Hildenbrand p. 103. For (ii), from part (i) it suffices to prove that under the stated hypotheses there is an $h \in H$ such that $p(s)e^h(s) + (a \otimes \kappa)(s)\phi^h - a(s)\psi^h > 0$. To see this observe that $\max\{0, p(s)e^h(s) + (a \otimes \kappa)(s)\phi^h - a(s)\psi^h\} = p(s)e^h(s) + (a \otimes \kappa)(s)\phi^h\ - M^h(s)$. Add over $h \in H$, plug in equation (2) of section 3.4 and use $\sum_{h\in H}(\phi^h - \psi^h) = 0$. The result is $\sum_{h\in H} \max\{0, \cdot\} = p(s)\sum_{h\in H} e^h(s) > 0$. □

The use of this lemma is clear. The sequence of n-th stage fixed points satisfies the conditions of part (ii), hence from boundedness of $(z^h_n(s))_{h\in H}$ we conclude that $\bar{p}(s) \gg 0,\ s \notin \cup_{h\in H}S^h$. This implies first that $\bar{x}^h(s)$ solves $CP^h_s(\bar{p}(s), \bar{\kappa}(s), \bar{\phi}^h, \bar{\psi}^h)$, by continuity of ξ^h_s (lemma 3.3.8); and second, that $\bar{z}(s) = 0$ also for $s \notin \cup_{h\in H}S^h$, via Walras law (9), equation (7) and $\bar{p}(s) \gg 0$.

This, the last three claims and the last assertion of lemma 3.6.2 above show that $(\bar{p}, \bar{q}, \bar{\kappa})$ and $((\bar{x}^h(s))_{s\in\{0,\dots,S\}}, \bar{\phi}^h, \bar{\psi}^h)_{h\in H}$ constitute an equilibrium.

3.7 Redundant assets

We check here that there is no loss of generality in eliminating redundant assets. To this end we show that to each equilibrium of the reduced economy with $a \in \mathbb{R}^{S\times J}$ there corresponds an equilibrium of the original one.

Suppose $(p, q, \kappa, (x^h, \phi^h, \psi^h)_{h\in H})$ is an equilibrium of the reduced economy, obtained by removing an asset $a^{J+1} = a\mu,\ \mu \in \mathbb{R}^J$. We then verify that the tuple $(\tilde{p}, \tilde{q}, \tilde{\kappa}, (\tilde{x}^h, \tilde{\phi}^h, \tilde{\psi}^h)_{h\in H})$ we now define is an equilibrium of the original economy. Let $\tilde{p} = p$, $\tilde{x}^h = x^h, h \in H, \tilde{\kappa}^j(s) = \kappa^j(s), j \in J$, $\tilde{\kappa}^{J+1}(s) = \kappa(s)\mu$, $\tilde{\phi}^{hj} = \phi^{hj}$ and $\tilde{\psi}^{hj} = \psi^{hj}, h \in H, j \in J$, $\tilde{\phi}^{h,J+1} = \tilde{\psi}^{h,J+1} = 0$ for $h \in H$. Finally, since q is an equilibrium (no-arbitrage) price, we may invoke lemmas 3.1.4 (ii) and 3.1.3 to assert that for some $\lambda \in \mathbb{R}^S_{++}$, $D \in [a \otimes \kappa, a]$ it is $q = \lambda D$. And we define $\tilde{q} = (\lambda D, \lambda D\mu)$.

To see that this is an equilibrium it suffices to show (since markets for goods and assets obviously clear, and $\tilde{\kappa}^{J+1}(s) \in [0,1]$ for all $s \in S$) that each income allocation that an agent can achieve in the original economy can also be achieved at the same cost in the reduced economy. In fact any income allocation

$$\left(a \otimes \kappa, a^{J+1} \otimes \kappa^{J+1}\right)\left(\phi^h, \phi^{h,J+1}\right) - \left(a, a^{J+1}\right)\left(\psi^h, \psi^{h,J+1}\right) \ ,$$

which costs $\lambda D(\phi^h - \psi^h) + \lambda D\mu(\phi^{h,J+1} - \psi^{h,J+1})$, can be obtained at the same cost by the portfolio $\phi^h + \mu\phi^{h,J+1}$, $\psi^h + \mu\psi^{h,J+1}$. It is immediate to check that the cost is the same; as for the returns, one uses the equalities $\tilde{\kappa}^{J+1} = \kappa\mu$ and $(a \otimes \kappa)\mu = a\mu \otimes \kappa\mu$.

3.8 Numéraire assets

The proof we have proposed for nominal assets can easily be adapted to prove existence for the case of numéraire assets. More precisely, suppose that assets pay off in state s in units of a fixed commodity bundle. Let $d(s)$, an L dimensional non negative vector be that bundle. Asset j hence yields a payoff in state s equal to $a^j(s)k^j(s)d(s)p(s)$. Now observe that if assets pay off in each state in terms of a byndle consisting of one unit of each commodity, so that $d(s) = (1, \ldots, 1)$ for all s, then asset j yields $a^j(s)k^j(s)\sum_l p^l(s)$. Similarly, if one holds asset j, one has to pay $a^j(s)\sum_l p^l(s)$ in state s. Given the normalization we adopted, $p(s)$ lies in the simplex. Therefore, an agent's budget constraint in state s can be written as

$$p(s)\left(x^h(s) - e^h(s)\right) \leq (a \otimes k)(s)\Phi^h - a(s)\Psi^h$$

which is equivalent to the one with nominal assets.

Thus, our theorem also yields existence for the case of assets paying a bundle of one unit of each good. An alternative, more general model is possible: in this case it is possible to adapt the proof of Chae (1988) of existence of an equilibrium in the numéraire asset case to our framework; the adaptation required is of the same nature as the one we had to do to adapt Werner's existence proof for nominal assets.

4 Indeterminacy of asset prices

In models of equilibria with incomplete financial markets where agents are fully aware of all possible events it is well known (Cass 1984) that for any vector of asset prices which prevents arbitrage there is an equilibrium with that vector as the equilibrium price for the assets.

A similar result holds in our framework, with two natural modifications: first, the asset prices must prevent arbitrage for all agents, and, second, since the return of assets depends on the matrix of repayment rates κ, the asset price vector must prevent arbitrage at κ.

The characterization of the maximal cone of asset prices that can be imbedded in an equilibrium might not be easy. A subset of it can be easily determined, and this is what we do in this section. Take any vector q in the interior of the cone generated by the rows of the matrix a_C, that we denote by cone a_C. Certainly for every agent, and for every matrix κ this is a no-arbitrage price. Any such vector may be the component of an equilibrium:

4.1 Proposition *For any asset price vector* $q \in \text{int cone}\, a_C$ *there exists an equilibrium with q as equilibrium price vector for the assets.*

Sketch of Proof. The proof is of course an alternative proof of the existence of the equilibrium, this time along the lines of Cass (1984). Since most of the basic steps are unchanged with respect to the previous proof, we shall only provide the main lines.

The first step is to express the fixed vector of asset prices as an appropriate linear combination of the asset returns, with coefficients that may depend on κ but in such a way to remain strictly positive for any value of the repayment rates. This has already been done as a step in the proof of lemma 3.5.1, but here is the idea again. Take the arbitrary vector $q \in \text{int cone}\, a_C$, so that $q = v_C a_C$ for a vector $v_C \in \mathbb{R}^J_{++}$.

Now consider the function of ϵ and κ defined by:

$$\Phi(\epsilon, \kappa) = \left[q - \sum_{s \in S \setminus C} \epsilon (a \otimes \kappa)(s) \right] a_C^{-1} \ .$$

This is clearly an uniformly continuous function and $\Phi(0, \kappa) = v_C$. Hence for some ϵ_0 small enough the function $\Phi(\epsilon_0, \cdot)$ is strictly positive. Now define the function v from κ to $\mathbb{R}^S_{++}$ by $\kappa \mapsto v(\kappa) = (\Phi(\epsilon_0, \kappa), \epsilon_0, \ldots, \epsilon_0)$; then $q = v(\kappa)(a \otimes \kappa)$ for every κ.

Notice that here again the assumption *FR* plays a critical role. In fact in the construction that we have just seen it is crucial that small perturbations of the fixed vector q remain in int cone a_C.

Since asset prices are fixed, we may reduce the price space to the component of the goods prices:

$$P = \left\{ \left(p(0), (p(s))_{s \in S} \right) \in \Delta^{(S+1)L} \right\} \ .$$

Now the proof follows again two steps. First restrict the goods price space to the subset

$$P^n = \left\{ p \in P : p(s) \geq \frac{1}{n}, s \in \{0\} \cup S \right\} \ .$$

Consumers face the same maximisation problems as defined previously (with the fix asset price vector q), which yield goods and asset demands, the latter again expressed as two separate components of positive and negative holdings. The book-keeping map is unchanged, while the market maker now solves the problem:

$$\max_{p \in P^n} p(0)z(0) + \sum_{s \in S} v(\kappa, s)p(s)z(s) \ .$$

The correspondence from the restricted set of goods prices, repayment rates, and goods and assets demands into itself has fixed point. The corresponding sequence of fixed points has a limit, and this part is identical to the one presented above. We have now only to prove that the limit is an equilibrium.

First note that at the limit values $(\bar{p}(0), (\bar{p}(s))_{s \in S}, \bar{z}(0), (\bar{z}(s))_{s \in S})$ we have

$$\bar{p}(0)\bar{z}(0) + \sum_{s \in S} v(\bar{\kappa}, s)\bar{p}(s)\bar{z}(s) = 0 \ .$$

This follows by adding each equality of the Walras' Law, in the special form which holds for this model, after having multiplied each of the terms for the second period by the corresponding coefficient of the vector $v(\bar{\kappa})$, and then cancelling terms. It is now easy to show that the excess demand of goods are non positive. Our assumption P that endowments of all agents are strictly positive in all states now gives a positive income for at least one agent in at least one state, and hence limit prices are strictly positive. It now follows that goods market clear. Now the assumption of full rank of the matrix a_C and the previous result give that asset markets also clear (this is, obviously, one of the main ideas in Cass (1984), and does not require any special role played by some of the consumers (the "Mister 1" of the Cass trick): see for instance Duffie 1987). □

5 Comments and examples

5.1 Assumptions E, U, FR and P

Of none of them can we say it is made 'for simplicity'. Assumption E on positive endowments can be weakened but not much, as Werner (1985) has shown.

Assumption U gives single-valuedness of the solution of the consumers' problems. Without that the constraint $\phi\psi = 0$ (of not buying and selling one asset at the same time) poses a problem. To see this suppose there is only one asset and that for agent h the two choices of buying it – say $(\phi, \psi) = (1, 0)$ – and selling it – say $(\phi, \psi) = (0, 1)$ – are both optimal. The points of the segment between $(1, 0)$ and $(0, 1)$ have coordinates both positive, so they violate the above constraint, hence are not feasible. So in this example, where the solution is not single-valued, the optimal-choice correspondence is not convex-valued. Moreover, the set $\{(1, 0), (0, 1)\}$ is not even contractible. One might believe that if $(1, 0)$ and $(0, 1)$ are both optimal so are $(\alpha, 0)$ and $(0, \beta)$ for any $0 \leq \alpha, \beta \leq 1$, in which case one would recover the contractible optimal set $\{(\alpha, 0) \mid \alpha \in [0, 1]\} \cup \{(0, \beta) \mid \beta \in [0, 1]\}$. We have not pursued this point. It may be mentioned that since we have strictly positive endowments, equilibrium exists also under weak monotonicity of utility functions. The argument is standard (an instance of it appears in point 2 below).

Assumption *FR* has force because of C, for essentially by C one gets $\kappa_C = \mathbf{1}$ so $(a \otimes \kappa)_C = a_C$. Without C the last equality may fail, and then non-singularity of a_C would be irrelevant (what matters is always a and $a \otimes \kappa$). We will see that C cannot be dispensed with; on the other hand in the presence of C, we would not know how to proceed without *FR* (other than perturb a).

Assumption P too seems hard to dispense with, at least this is the impression we have by looking at the first non-existence example of section 2.1 (in which besides P, also C and *FR* are violated).

5.2 Existence may fail without assumption C

In the aforementioned example it is natural to think that non-existence is due to the fact that repayment rates cannot do their job because of the zeros in a. In the next example we see that this is not quite so: the burden placed on repayment rates as equilibrating force may be excessive even with $a \gg 0$. Since the example satisfies all the assumptions of the theorem except C, it shows that the latter is to some extent 'necessary' for existence. The basic structure is as in that example, now with

$$a = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

and $0 < \alpha < 1$. There is one good per state, $e^1 = e^2 = (1, 1, 1)$, and

$$u^1(x^1(0), x^1(1)) = \ln(1 + x^1(1)) \quad u^1_2(x^1(2)) = \ln(1 + x^1(2))$$
$$u^2(x^2(0), x^2(2)) = \ln(1 + x^2(2)) \quad u^2_1(x^2(1)) = \ln(1 + x^2(1)) \ .$$

Notice that these utilities are not strictly increasing; we will make them satisfy this condition by perturbing them. Obviously it cannot be $\kappa = \mathbf{1}$ in equilibrium, otherwise at any q some agent would see arbitrage; so financial markets must be active. As long as $p(0) > 0$, all agents would sell $e^h(0)$ and buy assets to transfer income to period 1: but then from $p(0)0 + q(\phi^h - \psi^h) = p(0)e^h(0) > 0$, summing over $h \in H$ one would get $\sum_{h \in H}(\phi^h - \psi^h) > 0$, which cannot be part of an equilibrium. Hence at equilibrium it should be $p(0) = 0$. Similarly, since at equilibrium it should be $\kappa \gg 0$, it must also be $q(\phi^h - \psi^h) = 0$ for all h ($q(\phi^h - \psi^h) < 0$ would be improved upon by buying assets and transferring income to the second period). Thus both agents must make some financial trade (not null, for κ must move away from $\mathbf{1}$) which costs zero and then yields zero (no-arbitrage). There are only two possible situations in which this can happen:

A) Agent 1 buys asset 1 from agent 2 and sells him asset 2. Notice that in this case $\kappa^2(1) = 1$ for the only debtor on asset 2 is agent 1 who sees state 1, hence there can be no default on asset 2 in state 1. Similarly $\kappa^1(2) = 1$. Zero cost and yield of agent 1's trade are:

$$q_1\phi^{11} - q_2\psi^{12} = 0 \quad \text{and} \quad \kappa^1(1)\phi^{11} - \alpha\psi^{12} = 0$$

from which one gets $\kappa^1(1) = \alpha q_1/q_2$. Similarly, for agent 2 from $q_2\phi^{22} - q_1\psi^{21} = 0$ and $-\alpha\psi^{21} + \kappa^2(2)\phi^{22} = 0$ one has $\kappa^2(2) = \alpha q_2/q_1$. On the other hand, the book-keeping condition on $\kappa^1(1)$ is

$$\kappa^1(1) = \imath(2,1) = \frac{1 + \alpha\phi^{22}}{\psi^{21}} = \frac{1}{\psi^{21}} + \frac{\alpha q_1}{q_2} \quad .$$

9 > Thus the restrictions imposed on $\kappa^1(1)$ by no-arbitrage and book-keeping are inconsistent (it would take an infinite amount of asset 2 sold by agent 1 to agent 2 to make them compatible); hence no equilibrium exists in which agent 1 buys asset 1 and sell asset 2. For the sake of curiosity, it is immediately checked that book-keeping forces $\kappa^2(2) = \imath(1,2) = 1/\psi^{12} + \alpha q_2/q_1 \neq \alpha q_2/q_1$.

What happens here is that any given q puts restrictions on optimal financial trades, and these generate, via book-keeping, matrices κ such that $q \notin Q_{S^h}(\kappa)$.

B) It is entirely analogous to check that the same phenomenon occurs in the other possible case, where agent 1 sells asset 1 and buys asset 2. In this case $\kappa^1(1) = \kappa^2(2) = 1$. Cost and revenue for agent 1 are $-q_1\psi^{11} + q_2\phi^{12} = 0$ and $-\psi^{11} + \kappa^2(1)\alpha\phi^{12} = 0$ so $\kappa^2(1) = q_2/\alpha q_1$. For agent 2 we have $q_1\phi^{21} - q_2\psi^{22} = 0$ and $-\psi 22 + \kappa^1(2)\alpha\phi^{21} = 0$, whence $\kappa^1(2) = q_1/\alpha q_2$. On the other hand from book-keeping it should be

$$\kappa^2(1) = \imath(2,1) = \frac{1}{\alpha\psi^{22}} + \frac{q_2}{\alpha q_1} \qquad \kappa^1(2) = \imath(1,2) = \frac{1}{\alpha\psi^{11}} + \frac{q_1}{\alpha q_2} \quad ,$$

again inconsistent with the other restrictions. Conclusion: no equilibrium exists in this economy.

In this example the utility functions of the two agents are not strictly increasing, so the assumption U of the theorem is not satisfied. Adding a dependence on first period consumption would complicate computations, so we prefer to resort to an indirect argument to prove that existence may fail in an economy where all the assumptions of the theorem except C are satisfied.

Consider the example we have just described, but now set

$$u^1(x^1(0), x^1(1)) = \epsilon x^1(0) + \ln(1 + x^1(1))$$
$$u^2(x^2(0), x^2(2)) = \epsilon x^2(0) + \ln(1 + x^2(2)) \quad .$$

We claim that equilibrium does not exist for some $\epsilon > 0$. Suppose to the contrary that it does for all $\epsilon > 0$. Take a convergent subsequence of equilibrium prices, repayment rates, allocations and portfolios as $\epsilon \longrightarrow 0$. Since the utility functions specified above converge uniformly to the function with $\epsilon = 0$ on the cube between zero and the aggregate endowments, $[0,2]^3$, it is easy to show that the limit point would itself be an equilibrium; a contradiction.

5.3 A discontinuity and a bad-equilibrium example

In the previous example, modify endowments in the unforeseen states:

$$e^1 = (1, 1, \epsilon_1) \qquad e^2 = (1, \epsilon_2, 1) \ .$$

Everything is as in the previous example except that from book-keeping one now has, respectively in cases A and B:

$$\kappa^1(1) = \frac{\epsilon_2}{\psi^{21}} + \frac{\alpha q_1}{q_2} \qquad \kappa^2(2) = \frac{\epsilon_1}{\psi^{12}} + \frac{\alpha q_2}{q_1}$$

$$\kappa^2(1) = \frac{\epsilon_2}{\alpha\psi^{22}} + \frac{q_2}{\alpha q_1} \qquad \kappa^1(2) = \frac{\epsilon_1}{\alpha\psi^{11}} + \frac{q_1}{\alpha q_2}$$

so there is no equilibrium for any $\epsilon_1, \epsilon_2 > 0$. Let $\epsilon_1, \epsilon_2 \longrightarrow 0$ and consider the limit economy:

$$e^1 = (1, 1, 0) \qquad e^2 = (1, 0, 1) \ .$$

Here a continuum of equilibria appear. Essentially anything will do, provided the κ's go down to eliminate arbitrage.

As before it must be $p(0) = 0$ and the agents must engage in 'useless' financial trade which costs zero and gives zero revenue (useful only to establish equilibrium). Any such trade is optimal and leaves everybody with their endowments in the foreseen states (and bankrupt in the others). Set $q_1 = q_2 = 1$ and let agent buy asset 1 and sell asset 2, as in case A above. Pick any positive volume of trade $v > 0$ and set $\phi^{11} = \psi^{21} = \psi^{12} = \phi^{22} = v$, which are feasible (from $q_1 = q_2$) market clearing plans. From the same two equations as before we now get $\kappa^1(1) = \alpha$; and now from book-keeping it must be $\kappa^1(1) = (0 + \alpha\phi^{22})/\psi^{21} = \alpha$. Similarly the two restrictions for $\kappa^2(2)$ are consistent, and we have an equilibrium (for each $v > 0$).

A weakness of the model is revealed by considering the agents' positions in the last equilibrium. Take agent 1 for example. He is getting a certain number of units of asset 1 – which he originally sees as yielding 1 unit of income for sure – by giving away the same number of units of asset 2 which he sees as yielding only α for sure. Nonetheless he does not think he is making any profit, for he correctly believes that agent 2 is making such a bad deal that he is not going to be able to repay his debts, so in the end asset 1 is only worth α (just as much asset 2). The model assumes that agent 1's reasoning goes as far as here. On the other hand, such reasoning could continue, for agent 1 has considered only the 'dark side' of agent 2's position. Indeed he knows that agent 2 does not see state 1, for he anticipates 2 being bankrupt there. But agent 2 must see some other states in which he is better off than he is in state 1 (otherwise why would he trade?). Therefore agent 1 could go on suspecting that symmetrically in such states (unforeseen by himself) he might end up being worse off than he is in state 1, perhaps bankrupt. But then he should think twice before engaging in the (useless) financial trade which he accepts in the above equilibrium; more precisely he should first consider revising his state space somehow. This is beyond of the present model's reach.

5.4 Space revision

Let us go back to the war-energy example we started with. If all agent h can think of is whether or not there will be war, he is by definition incapable of enlarging his $\{\mathsf{W}, \mathsf{notW}\}$ space by adding more relevant facts. However, these facts are relevant insofar as the relevant economic variables depend on them; and the latter – endowments, prices and asset returns – are certainly well present to h's mind. To focus on asset returns, let us suppose that endowments are constant and price expectations in one-to-one correspondence with asset returns, so that the relevant state space becomes $\mathbb{R}^J_{++}$, the space of asset returns. Agent h starts with what may be interpreted as his support, namely the set $a(S^h) = \{a(s) \mid s \in S^h\} \subseteq \mathbb{R}^J_{++}$. On $\mathbb{R}^J_{++}$ it is meaningful to talk about space extension; for h does not know which other sources of uncertainty may be at play; but he knows their possible effect, that is, return vectors not in $a(S^h)$. The problem is now that while h had nothing to add to $\{\mathsf{W}, \mathsf{notW}\}$, on $\mathbb{R}^J_{++}$ he has too large a choice. Indeed, it is no solution to suggest to expand $a(S^h)$ to all of $\mathbb{R}^J_{++}$: for the set of possible asset returns would become unbounded, so any non-null trade would drive him bankrupt in some states. How to expand $a(S^h)$ to a smaller, say bounded or finite, subset of $\mathbb{R}^J_{++}$ determined by the initial $a(S^h)$, by what (prices) the agent observes in the markets and by his presumption that the others are rational is object of current research.

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