Coping with Imprecise Information: a Decision Theoretic Approach *

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Abstract

We provide a model of decision making under uncertainty in which the decision maker reacts to imprecision of the available data. Data is represented by a set of probability distributions. We axiomatize a decision criterion of the maxmin expected utility type, in which the set of priors explicitly depends on the available data. The central axiom is one of aversion towards imprecision of the information. We then characterize notions of comparative aversion to imprecision of the data as well as traditional notions of risk aversion. Comparative aversion to imprecision can be studied independently of the utility function, which embeds risk attitudes. We also give a more specific result, in which the functional representing the decision maker's preferences is the convex combination of the minimum expected utility with respect to the available data and the expected utility with respect to a subjective probability distribution, interpreted as a reference prior. This particular form is shown to be equivalent to some form of constant aversion to imprecision. We finally provide an example of application to portfolio choice.

Keywords: Imprecision, Ambiguity, Uncertainty, Decision, Multiple Priors.
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1 Introduction

In many problems of choice under uncertainty, some information is available to the decision maker. Yet, this information is often far from being sufficiently precise to allow the decision maker to come up with an estimate of a probability distribution over the relevant states of nature. The archetypical example of such a situation is the so-called Ellsberg paradox (Ellsberg (1961)), in which subjects are given some imprecise information concerning the composition of

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an urn and are then asked to choose among various bets on the color of a ball drawn from that urn.

In this paper, we model a decision maker who reacts to imprecision of the available data in a given choice problem. We do so assuming that data can be represented by sets of probability distributions. Thus, we define preferences as a binary relationship on the cross product of acts (mappings from states of the world to outcomes) and available information (sets of probability distributions over the state space). Denoting $\mathcal{P}$ the set of probability distributions over the state space that represents the information available to the decision maker (hereafter *information sets*), preferences bear on couples $(f, \mathcal{P})$ where f is an act in the usual sense. This means that, at least conceptually, we allow decision makers to compare the same acts in different informational settings. Our general representation theorem axiomatizes a class of functionals of the maxmin expected utility type à la Gilboa and Schmeidler (1989), where the set of priors is a subset of the available information. Hence, compared to Gilboa and Schmeidler (1989) we enrich the space on which preferences are defined: in their setting, the (un-modelled) prior information that the decision maker has is fixed. For each set of probability distributions representing available information, we get a transformed set. More precisely, the general decision criterion we axiomatize takes the following form: for two sets of probability distributions $\mathcal{P}$ and $\mathcal{Q}$ and two acts f and g, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp.$$

In this expression $\mathcal{F}(.)$ is a function, the *transformation function*, that maps sets of probability distributions to sets of probability distributions. It associates to each information set $\mathcal{P}$ a transformed set $\mathcal{F}(\mathcal{P})$. Our representation result imposes some consistency conditions on $\mathcal{F}(.)$. These conditions (e.g., $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$) are obtained with an economy of axioms. The main axiom in this construction is one of aversion to imprecision which, loosely speaking, states that the decision maker always prefers to act in a setting in which he possesses more information. An advantage of having defined preferences on pairs (act,information) is to be able to simply capture imprecision aversion as aversion towards a "garbling" of the information at hand. At this stage, we simply remark that the notion we adopt of what it means for a set of probability distributions to be more imprecise than another one is rather weak and partial in the sense that it does not enable one to compare many sets (this will be discussed at length and illustrated via an example when we introduce our axiom.)

Based on this representation theorem, one can characterize a comparative notion of aversion towards imprecision, with the feature that it can be completely separated from risk attitudes. We say that a decision maker b is more imprecision averse than a decision maker a if whenever a prefers to *bet* on an event when the information is given by a (precise) probability distribution rather than some imprecise information, b prefers the bet with the precise information as well. This notion captures in rather natural terms a preference for precise information, which does not require the two decision makers that are compared to have the same risk attitudes, the latter being captured, as we show, by the concavity of the utility function.[1] Our result states that two decision makers can be compared according to that notion if and only if the transformed set of one of them is included in the other's. A Bayesian decision maker (that is, a subjective expected utility maximizer) will have a transformation function whose range consists of singleton sets only, while an extremely imprecision averse decision maker's transformation function will be equal to the identity function. Whenever the information sets have some underlying symmetric structure (which we'll define precisely), it is possible to define absolute and relative imprecision premia that characterize this notion of aversion towards imprecision.

The representation theorem described above does not pin down a functional form but rather a class of functional forms compatible with aversion towards imprecision. If one is willing to assume extra properties of the preference relation, one can come up with more precise functional forms. For instance, a convenient one, already suggested in Ellsberg (1961), consists of taking the convex combination of the minimum expected utility with respect to all the probability distributions compatible with prior information, with the expected utility with respect to a particular probability distribution in this set. The coefficient in the convex combination has then a direct interpretation in terms of attitude towards imprecision. As it turns out, this functional form can be axiomatized in a natural way for a large class of sets of probability distributions (including notably cores of beliefs functions), the extra axiom being one of constant relative imprecision premium. This gives rise to the following representation: $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\theta \min_{p \in \mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}} \geq \theta \min_{p \in \mathcal{Q}} \int u \circ g dp + (1-\theta) \int u \circ f dc_{\mathcal{Q}}.$$

where $c_{\mathcal{P}}$ is the, suitably defined, center of the family $\mathcal{P}$ and θ is the value of the constant relative imprecision aversion premium that allows all the attitudes towards imprecision from paranoic pessimism ($\theta = 1$) to Bayesianism ($\theta = 0$).

Our approach, we believe, might be of interest for economic applications where imprecision about the data is the rule. First, our theory can provide orderings that permit to develop results of comparative statics in the spirit of comparative statics for risk based on second order stochastic dominance. Second, the separation of attitude towards risk and attitude towards imprecision that we achieve, paves the way for exploring the respective consequences of these two features in economic situations. Although we do not develop these ideas in full generality in this paper, we provide a simple example of a portfolio choice problem when three types of assets (certain, risky, and uncertain) are available and illustrate how imprecision of the information, risk attitude, and imprecision attitude affect assets holding. This suggests that our model might have the potential to help explaining some usual financial puzzles.

[1] Whenever information is precise in the sense that it is compatible with only one probability distribution, our axioms imply that the decision maker is maximizing expected utility.

Organization of the paper

The paper is organized as follows. The next subsection discusses some related literature. The following section describes the setup and establishes the notation. It also presents a few important definitions on how to use our framework to represent "Anscombe-Aumann acts". Section 3 is divided into two. In the first subsection we introduce and discuss our axioms. In particular, we provide a lengthy discussion of our axiom of aversion to imprecision. In the second subsection we state our representation theorem and discuss some of its implications. Section 4 defines and characterizes a notion of comparative imprecision aversion and subsequently the usual notion of comparative risk aversion transposed to our model. Section 5 contains an exploration of our model when the information is restricted to symmetrically decomposable sets of probability distributions. In particular, we define a notion of imprecision premium and characterize a more specific functional form based on constant relative imprecision premium. In Section 6, we provide an application of our model to a portfolio choice problem. All proofs are gathered in the Appendix.

Comparison with the related literature

We conclude this introduction by mentioning some related literature, whose precise relationship with our model and results will be discussed further in the text. We also make clear what are the main conceptual differences between our approach and much of the recent literature.

Our model incorporates explicitly information as an object on which the decision maker has well defined preferences. To the best of our knowledge, Jaffray (1989) is the first to axiomatize a decision criterion that takes into account "objective information" in a setting that is more general than risk. In his model, preferences are defined over belief functions. The criterion he axiomatizes is a weighted sum of the minimum and of the maximum expected utility. This criterion prevents a decision maker from behaving as an expected utility maximizer, contrary to ours, which obtains as a limit case the expected utility criterion. Interest in this approach has been renewed recently, in which object of choices are sets of lotteries (Ahn (2003), Olszewski (2002), Stinchcombe (2003)). We will explain in Section 3 why our model does not reduce to one of choice over sets of lotteries. More closely related to our analysis, and actually a point of inspiration of this paper, is Wang (2003). In his approach the available information is explicitly incorporated in the decision model. That information takes the form of a set of probability distributions together with an anchor, i.e., a probability distribution that has particular salience. As in our analysis, he assumes that decision makers have preferences over couples (act, information). However, his axiom of ambiguity aversion is much stronger than ours and forces the decision maker to be a maximizer of the minimum expected utility taken over the entire information set. There is no scope in his model for less extreme attitude towards ambiguity. Following Wang's approach, we proposed in Gajdos, Tallon, and Vergnaud (2004) a weaker version of aversion towards imprecision still assuming that information was coming as a set of priors together with an anchor.

The notion of aversion towards imprecision that we develop here is based on the one analyzed in our previous work and is different from the one defined in Gilboa and Schmeidler (1989) and Schmeidler (1989) and the subsequent literature. There, aversion towards ambiguity is defined *via* a preference for hedging, while ours is defined *via* a preference for information precision. Thus, in Gilboa and Schmeidler (1989), uncertainty aversion is only indirectly revealed by a preference for hedging, while our approach is in some sense more direct. This is because we observe the preference for different "objective information". This point is of theoretical importance, as it allows us to define aversion towards ambiguity or imprecision as a reaction of the decision maker to a change in the information he possesses.

Our notion of comparative imprecision aversion could itself be compared to the one found in Epstein (1999) and Ghirardato and Marinacci (2002). The latter define comparative ambiguity aversion using constant acts. They therefore need to control for risk attitudes in a separate manner and in the end, can compare (with respect to their ambiguity attitudes) only decision makers that have the same utility functions.[2] Epstein (1999) uses in place of our bets in the definition of comparative uncertainty aversion, acts that are measurable with respect to an exogenously defined set of unambiguous events. As a consequence, in order to be compared, preferences of two decision makers have to coincide on the set of unambiguous events. If the latter is rich enough, utility functions then coincide. Our notion of comparative imprecision aversion, based on the comparison of bets under precise and imprecise information does not require utility functions to be the same when comparing two decision makers. Said differently, risk attitudes are simply irrelevant to the imprecision aversion comparison.

The more specific functional form that we axiomatize in Section 5 appears in some previous work (Gajdos, Tallon, and Vergnaud (2004) and Tapking (2004)). In independent work, Hayashi (2005) provides, in the same set up as ours, a different axiomatization of essentially the same decision criterion. His approach is different in at least one important direction. Hayashi's axiomatization of the equivalent of our general decision criterion rests on a notion of imprecision aversion that is based on gains via hedging, much as in Gilboa and Schmeidler (1989). Thus, imprecision aversion is not defined in terms of properties of the preferences when comparing various informational settings. Hence, one could argue that this definition does not take advantage of the full strength of the general setting adopted. On the other hand, Hayashi's main theorem is concerned with the more specific functional form discussed above, i.e., the convex combination of the minimum expected utility with respect to all the probability distributions in the information set, with the expected utility with respect to a particular probability distribution, which, in his approach, turns out to be the Steiner point of this set. His main extra axiom is a geometric axiom stating that the decision maker's preferences are "invariant to similarity reshuffles". The latter are a generalization of the notion of permutation. His axiom is difficult

[2] They actually mention that if one wants to compare two decision makers with different utility functions, one has first to completely elicit them.

to interpret as reflecting some kind of aversion towards imprecision even though it mechanically gives this result in the functional form axiomatized. We give in Section 5 an example of a decision maker whose preferences are compatible with the general representation but not with the specific functional form (thus violating Invariance to Similarity Reshuffles). As we argue then, it is not clear on what behavioral grounds such preferences should be ruled out. We believe our approach provides a deeper understanding of how imprecision of the data might or might not affect a decision maker's behavior.

Finally, we compare our approach with Klibanoff, Marinacci, and Mukerji (2005). They provide a fully subjective model of ambiguity aversion, in which attitude towards ambiguity is captured by a smooth function over the expected utilities associated with a set of priors. The latter, as in Gilboa and Schmeidler (1989) is subjective. Hence, although their model allows for a flexible and explicit modelling of ambiguity attitudes, there is no link between the subjective set of priors and the available information. Interestingly, part of Klibanoff, Marinacci, and Mukerji (2005)'s motivation is similar to ours, that is disentangling ambiguity attitude from the information the decision maker has. Formally, however, this separation holds in their model only if one makes the extra assumption that subjective beliefs coincide with the objective information available. In particular, comparative statics are more transparent in our model, as information can be exogenously changed. At a more conceptual level, Klibanoff, Marinacci, and Mukerji (2005)'s approach assumes that all uncertainty is eventually reduced to subjective probabilities, although on two different levels: essentially, the decision maker has in mind a second order probability distribution, but does not perform reduction of lotteries. The criterion they obtained is smooth and appeals only to probabilistic tools, which should make it easy to use in economic applications. Besides the different specific modelling choices, our conceptual departure from their approach is that we do not assume that, even subjectively, imprecise information can be reduced to probabilities (even of a second or higher order). In that sense we are more in line with Ellsberg (2001)'s view, that when a decision maker lacks a determinate probability distribution over states, "there will correspond [to any available option], in general, a set of expected utility values, among which he cannot discriminate in terms of definite probabilities".

2 Preliminaries

Let S be a countably infinite set of states of nature, that we will identify with $\mathbb{N}$. We assume that information in any given decision problem comes as a set of probability distributions over that state space, called the information set. Let $\mathbb{P}$ be the set of non-empty, closed (in the weak convergence topology) sets of priors with finite support, and $\mathbb{P}_C$ the set of convex elements of $\mathbb{P}$. Denote by $\mathcal{P}$ the generic element of $\mathbb{P}$, and $S(\mathcal{P}) = \cup_{p\in\mathcal{P}} Supp(p)$ the support of $\mathcal{P}$. We will assume throughout that $S(\mathcal{P})$ is finite.[3] For any subset E of S, let $\Delta(E)$ be the simplex on E,

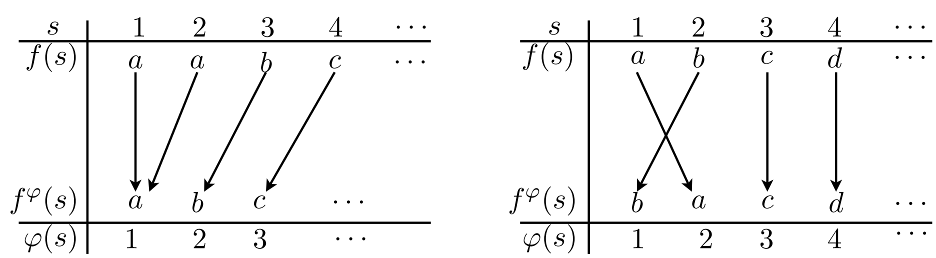
[3]See below why we need a countably infinite set of states of nature although $S(\mathcal{P})$ is assumed finite for any $\mathcal{P}$.

that is the set of probability distributions p with $Supp(p) \subset E$. For any $s \in S$, let δ_s be the probability measure putting weight one on s. For any $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$, $\alpha \in [0,1]$, define $\alpha\mathcal{P}+(1-\alpha)\mathcal{P}'$, the convex combination of $\mathcal{P}$ and $\mathcal{P}'$, to be the set:

$$\left\{q \,\middle|\, q = \alpha p + (1-\alpha)p',\, p \in \mathcal{P},\, p' \in \mathcal{P}'\right\}$$

Let $\mathcal{C}$ be the set of consequences. An act f is a mapping from S to $\mathcal{C}$. We denote by $\mathcal{A}$ the set of acts and $\mathcal{A}^c$ the set of constant acts. For $E \subset S$, let $f_E g$ be the act giving $f(s)$ if $s \in E$ and $g(s)$ otherwise. The decision maker's preferences is a binary relation $\succeq$ over $\mathcal{A} \times \mathbb{P}$, that is, on couples $(f, \mathcal{P})$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts, respectively, of $\succeq$. Compared to the standard setting, this amounts to enlarge the domain of preferences, to include information. It does not seem unrealistic to ascertain that such preferences could be elicited in experiments.

For any φ onto mapping from S to S, for any $f \in \mathcal{A}$, we say that f is φ-measurable if $f(s) = f(s')$ for all $s, s' \in S$ such that $\varphi(s) = \varphi(s')$. For a φ-measurable act f, define the act f^φ on $\varphi(S)$ by $f^\varphi(s) = f(s')$ where $s' \in \varphi^{-1}(s)$ for all $s \in S$. f^φ is the act f where either some states for which the outcome is the same are merged, or some states are permuted, or any combination of the two. Two examples of such transformations are depicted below, where a, b, c, d are elements of $\mathcal{C}$.

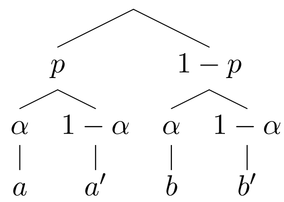


A similar operation for the available information can be defined as follows. For any $p \in \Delta(S)$ and $\mathcal{P} \in \mathbb{P}$ and φ onto mapping from S to S, p^φ is defined by $p^\varphi(s) = p(\varphi^{-1}(s))$ for all $s \in \varphi(S)$ and $\mathcal{P}^\varphi$ is defined by $\mathcal{P}^\varphi = \{p^\varphi | p \in \mathcal{P}\}$.

We cast our analysis in a Savage framework, in which the set of consequences $\mathcal{C}$ is unrestricted. On the other hand, we also have sets of probability distributions on S available. As a consequence, even though there is no given mixture operation on $\mathcal{C}$, one can *define* such an operation as we explain next. The idea is the following. Consider two acts f and g whose payoffs are described in the table below, where a, a', b, b' are elements of $\mathcal{C}$:

s	1	2 ...
f	a	b ...
g	a'	b' ...

and assume that $f(s) = g(s)$ for all $s > 2$. Let $\mathcal{P} = \Delta(\{1,2\})$, the simplex on the first two states, be the information set. Recall now that one usual interpretation of the $\alpha-$mixture of two acts in the Anscombe-Aumann setting is a randomization over the *consequences* of these acts with probabilities α and $(1-\alpha)$. According to this interpretation transposed to our setting, a randomization of $(f, \mathcal{P})$ and $(g, \mathcal{P})$ would look like the following:



with $p \in [0,1]$. Hence, intuitively, one would like to define $(\alpha f + (1-\alpha)g, \mathcal{P})$ to be the act that yields, in state 1, a with probability αp and a' with probability $(1-\alpha)p$ for all $p \in [0,1]$ and so on. In other words, the $\alpha-$mixture of $(f, \mathcal{P})$ and $(g, \mathcal{P})$ would be the couple $(h, \mathcal{Q})$ obtained by splitting any relevant state in two: state 1 is now split in two to become state 1 and 2, state 2 is now state 3 and 4 and so on.

s	1	2	3	4 ...
h	a	a'	b	b' ...
q(s)	αp	$(1-\alpha)p$	$\alpha(1-p)$	$(1-\alpha)(1-p)$ ...

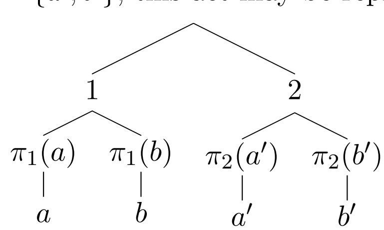
where $(p, 1-p)$ is any distribution in the simplex. This is precisely what our definition of a mixture achieves, for more general sets $\mathcal{P}$.

Definition 1 *For all $\mathcal{P} \in \mathbb{P}$, for all $\alpha \in [0,1]$, for all $f, g \in \mathcal{A}$, let $(\alpha f + (1-\alpha)g, \mathcal{P})$ be defined by $(h, \mathcal{Q})$ where*

- *$h(s) = f(\frac{s+1}{2})$ if s is odd, and $h(s) = g(\frac{s}{2})$ if s is even and,*
- *$\mathcal{Q} = \{q | \exists p \in \mathcal{P}$ s. th. $q(s) = \alpha p(\frac{s+1}{2})$ if s is odd and $q(s) = (1-\alpha)p(\frac{s}{2})$ if s is even$\}$*

Note that this definition requires that one is always able to essentially duplicate states of the world. This is the reason why, despite the fact that all acts considered will only involve finitely many states of the world (in the sense that the support of any $\mathcal{P}$ is finite) we need an infinite state space. The conclusion to be drawn from this construction is that, although our acts are Savage acts, the set probabilistic structure we consider makes it possible to define a mixture operation which is akin to the mixture in an Anscombe-Aumann setting.

As suggested by our definition of mixture, Anscombe-Aumann acts have a natural counterpart in our setting. Indeed, let $\tilde{f}$ be an Anscombe-Aumann act defined on the states $\{1,2\}$, with $\tilde{f}(1) = \pi_1$ and $\tilde{f}(2) = \pi_2$, where π_1 and π_2 are lotteries defined on $\mathcal{C}$. Assuming that $Supp(\pi_1) = \{a, b\}$ and $Supp(\pi_2) = \{a', b'\}$, this act may be represented as follows:



In our setting, when no further information is available on $\mathcal{S}$, (that is, the information the decision maker has is represented by $\Delta(\{1,2\})$), this act can intuitively be rewritten as follows:

s	1	2	3	4 ...
f	a	b	a'	b' ...
q	$\gamma(1)\pi_1(a)$	$\gamma(1)\pi_2(b)$	$\gamma(2)\pi_2(a')$	$\gamma(2)\pi_2(b')$ ...

where γ is any distribution in $\Delta(\{1,2\})$.

More generally, the decision maker could have some information on the state space, represented by a set Γ. If we restrict γ to belong to Γ, the couple (act,information) described above would be interpreted as follows: there is a set of possible probability distributions over the state space, and in each state, one obtains a lottery. Let us now describe more formally this kind of acts.

Definition 2 *Let I be a set of index and $\{\pi_i\}_{i\in I}$ a collection of probability distributions on S such that $Supp(\pi_i) \cap Supp(\pi_j) = \emptyset$ for all $i, j \in I$. Let $\Gamma \subset \Delta(I)$ and define $\Gamma \otimes (\pi_i)_{i\in I} = \left\{p | \exists \gamma \in \Gamma \text{ s. th. } p(s) = \sum_{i\in I} \gamma(i)\pi_i(s)\right\}$. We will call* generalized Anscombe-Aumann acts *couples (act, information) of the form $(f, \Gamma \otimes (\pi_i)_{i\in I})$.*

Example 1 To illustrate this construction, consider the following act and set of distributions:

s	1	2	3	4	5 ...
f	a	b	c	d	e ...
π_1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0 ...
π_2	0	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$ ...
$\gamma \otimes (\pi_i)_{i=1,2}$	$\gamma(1)\frac{1}{2}$	$\gamma(1)\frac{1}{2}$	$\gamma(2)\frac{1}{3}$	$\gamma(2)\frac{1}{2}$	$\gamma(2)\frac{1}{6}$ ...

where γ is any distribution in $\Gamma \subset \Delta(\{1,2\})$. $(f, \{\pi_1\})$ can be interpreted as a constant Anscombe-Aumann act yielding a with probability 1/2 and b with probability 1/2, and similarly for $(f, \{\pi_2\})$. Observe that for any $q \in \Gamma \otimes (\pi_i)_{i=1,2}$, the ratio $q(1)/q(2)$ is constant, equal to 1. Similarly, $q(3)/q(4) = 2/3$ and $q(3)/q(5) = 2$.

3 Multiple prior with information

3.1 Axioms

We now introduce our set of axioms. The first one is standard, although maybe even more demanding than usual in our setting in which preferences bear on pairs (act, information).

Axiom 1 *(Weak order) $\succeq$ is complete and transitive.*

We next turn to some invariance notion and define what it means for a couple $(f, \mathcal{P})$ to be equivalent to $(g, \mathcal{Q})$, with the idea that whenever $(f, \mathcal{P})$ is equivalent to $(g, \mathcal{Q})$, the decision maker will be indifferent between them. One simple idea would be to perform "reduction" and assess that these objects are equivalent if they yield the same distributions on consequences. However, this idea does not seem satisfactory. Consider the following example. An individual is facing two urns, and is told that the first urn contains Red and Black balls in unknown proportion, whereas the second urn contains Red, Black, Yellow, White and Blue balls, also in unknown proportion. He has to choose between betting \$100 on drawing a red ball in the first urn or in the second one. Although the induced distributions on consequences are the same for these two bets, it seems not unreasonable to assume that he will prefer to bet in the first urn. We therefore propose a weaker notion of equivalence, which is defined as follows.

Definition 3 *Let* $(f, \mathcal{P})$, $(g, \mathcal{Q}) \in \mathcal{A} \times \mathbb{P}$. $(f, \mathcal{P})$ *and* $(g, \mathcal{Q})$ *are said to be* equivalent *if there exists an onto mapping* φ *from* S *to* S *such that*

(i) f *is* φ*-measurable and* $f^{\varphi}(s) = g(s)$ *for all* $s \in S(\mathcal{Q})$,

(ii) $\mathcal{P}^{\varphi} = \mathcal{Q}$,

(iii) whenever $\left|\varphi^{-1}(s)\right| \geq 2$,

- *either* $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ *for all* $p, p' \in \mathcal{P}$,
- *or for all* $s' \in \varphi^{-1}(s)$, $p((s'|\varphi^{-1}(s)) = p'(s'|\varphi^{-1}(s))$ *for all* $p, p' \in \mathcal{P}$.

Let us illustrate what it means for $(f, \mathcal{P})$ and $(g, \mathcal{Q})$ to be equivalent. First, for all constant act $f \in \mathcal{A}^c$, for all $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$, $(f, \mathcal{P})$ is equivalent to $(f, \mathcal{P}')$. Second, if φ is a permutation of the states that leaves $\mathcal{P}$ unchanged (i.e., $\mathcal{P} = \mathcal{P}^{\varphi}$) then $(f, \mathcal{P})$ is equivalent to $(f^{\varphi}, \mathcal{P})$. If, for instance, $\mathcal{P}$ is a simplex, then any permutation of the states in its support will continue to yield the simplex, and therefore $(f, \mathcal{P})$ is equivalent to $(f^{\varphi}, \mathcal{P})$ for any act f.

There exist two more subtle instances of equivalence that we now illustrate. The first one states that "merging" two states on which an act gives the same outcome, *provided* the union of these two states is given a precise (i.e., unique) probability, is an operation that yields a new pair that is equivalent to the initial one: take $\mathcal{P} = \{(p, \frac{1}{2} - p, \frac{1}{2}, 0, ...)|p \in [0, 1/2]\}$ and define $\varphi : S \to S$ by $\varphi(1) = \varphi(2) = 1$ and $\varphi(i) = i - 1$, $i \geq 3$, which amounts to merge state 1 and 2. Then, for any f such that $f(1) = f(2)$, $(f, \mathcal{P})$ is equivalent to $(f, \mathcal{P})^{\varphi}$. The restriction that the union of the merged states has to be given a (precise) probability is what rules out the kind of reduction implicitly assumed when defining preferences over sets of lotteries.

The second one states that "merging" two states on which an act gives the same outcome, *provided* that the ratio of the probability of these two states is constant across all the possible probability distributions, is an operation that yields a new pair (act, information) that is equivalent to the initial one. The condition that the ratio be constant is equivalent to the fact that the distributions must have the same conditionals on $\varphi^{-1}(s)$. To illustrate this condition, take $\mathcal{P} = \{(p, \frac{1}{2}p, q, 0, ...)|\frac{3}{2}p + q = 1\}$ and take the same function $\varphi : S \to S$ that merges state 1 and

2 (i.e., $\varphi(1)=\varphi(2)=1$ and $\varphi(i)=i-1$, $i\geq 3$.) Then, for any f such that $f(1)=f(2)$, $(f,\mathcal{P})$ is equivalent to $(f,\mathcal{P})^\varphi$. It is easily seen that $p(1|12)=2/3$ and $p(2|12)=1/3$ for all $p\in\mathcal{P}$. Observe that $\mathcal{P}=\Delta(\{1,2\})\otimes((\frac{2}{3},\frac{1}{3}),\delta_3)$, and that $\mathcal{P}^\varphi=\Delta(\{1,2\})\otimes(\delta_1,\delta_2)$. Hence, viewing $(f,\mathcal{P})$ as a generalized Anscombe-Aumann act, φ merges states for which the outcome is the same within a roulette lottery.[4]

The second axiom states that acts that are equivalent according to Definition 3 are judged indifferent by the decision maker.

Axiom 2 *(Equivalence indifference) For all $(f,\mathcal{P})$, $(g,\mathcal{Q})\in\mathcal{A}\times\mathbb{P}$, if $(f,\mathcal{P})$ and $(g,\mathcal{Q})$ are equivalent then $(f,\mathcal{P})\sim(g,\mathcal{Q})$.*

The next axiom is an independence axiom in which the mixing operation bears on the information sets.

Axiom 3 *(Independence) For all $\mathcal{P}_1,\mathcal{Q}_1,\mathcal{P}_2,\mathcal{Q}_2\in\mathbb{P}$, and for all $f,g\in\mathcal{A}$,*

$$\left.\begin{array}{l}(f,\mathcal{P}_1)\succeq(\succ)(g,\mathcal{Q}_1)\\(f,\mathcal{P}_2)\succeq(g,\mathcal{Q}_2)\end{array}\right\}\Rightarrow(f,\alpha\mathcal{P}_1+(1-\alpha)\mathcal{P}_2)\succeq(\succ)(g,\alpha\mathcal{Q}_1+(1-\alpha)\mathcal{Q}_2)$$

When information sets are reduced to singletons this is the usual independence axiom. Its interpretation is the usual one: the set $\alpha\mathcal{P}_1+(1-\alpha)\mathcal{P}_2$ can be seen as the outcome of a process in which nature chooses the "true" probability distribution over S with probability α from $\mathcal{P}_1$ and $(1-\alpha)$ from $\mathcal{P}_2$.

Next, we state our continuity axiom, which is written in terms of mixture of acts as defined in Definition 1.

Axiom 4 *(Continuity) For all f, g, $h\in\mathcal{A}$, and all $\mathcal{P}\in\mathbb{P}$, if $(f,\mathcal{P})\succ(g,\mathcal{P})\succ(h,\mathcal{P})$, then there exist α and β in $(0,1)$ such that :*

$$(\alpha f+(1-\alpha)h,\mathcal{P})\succ(g,\mathcal{P})\succ(\beta f+(1-\beta)h,\mathcal{P}).$$

The next axiom is a monotonicity axiom, defined on the "roulette lottery part" of generalized Anscombe-Aumann acts.

Axiom 5 *(Lottery Monotonicity) Let $(f,\Gamma\otimes(\pi_i)_{i\in I})$ and $(g,\Gamma\otimes(\pi'_i)_{i\in I})$ be two generalized Anscombe-Aumann acts. If $(f,\{\pi_i\})\succeq(g,\{\pi'_i\})$ for all $i\in I$ then $(f,\Gamma\otimes(\pi_i)_{i\in I})\succeq(g,\Gamma\otimes(\pi'_i)_{i\in I})$.*

The next axiom simply requires that no matter what the available information is, there exists a pair of acts that are not indifferent.

[4] See also Example 1 where it is shown how some probability ratios are constant by construction of generalized Anscombe-Aumann acts.

Axiom 6 *(Non-degeneracy) For all $\mathcal{P} \in \mathbb{P}$, there exist f,$g \in \mathcal{A}$ such that $(f,\mathcal{P}) \succ (g,\mathcal{P})$.*

The next axiom is a Pareto axiom that states that if f is judged better than g according to any distribution $p \in \mathcal{P}$, then f is judged better according to the whole set $\mathcal{P}$.

Axiom 7 *(Pareto) For all $\mathcal{P} \in \mathbb{P}$, if for all $p \in \mathcal{P}$, we have $(f,\{p\}) \succeq (g,\{p\})$, then $(f,\mathcal{P}) \succeq (g,\mathcal{P})$.*

Our main and final axiom is an axiom of aversion towards imprecision. It compares an act in two different informational settings and states that the decision maker always prefers the more precise information. We therefore have to define a notion of imprecision on sets of probability distributions. The most natural definition would be that $\mathcal{P}$ is more precise than $\mathcal{Q}$ whenever $\mathcal{P} \subset \mathcal{Q}$. This is actually the definition proposed by Wang (2003). However, this definition turns out to be too strong for our purpose. Indeed, the idea that we want to push is that an imprecision averse decision maker should always prefer a more precise information, whatever the act under consideration. But consider an act f for which the worst outcome is obtained, say, in state 1. Then, Wang's notion of precision would force the decision maker to prefer $(f,\{(1,0,\ldots)\})$ to $(f,\Delta(\{1,2\}))$, which is very unlikely and unappealing. On the other hand, it is clear that a set being more precise than another has something to do with the inclusion of the former in the latter. We therefore need a definition that restricts the inclusion condition to some sets of probability distributions that are in some sense comparable, exactly as the comparison of two distributions in terms of risk focusses on distributions that have the same mean. We now propose one such definition.

Definition 4 *Let $\mathcal{P},\mathcal{Q} \in \mathbb{P}$. Say that $\mathcal{P}$ is* conditionally more precise *than $\mathcal{Q}$ if*

- *$\mathcal{P} \subset \mathcal{Q}$ and,*
- *there exists a partition $(E_1,\ldots,E_n)$ of S such that*

  (i) $\forall p \in \mathcal{P}$, $\forall q \in \mathcal{Q}$, $p(E_i) = q(E_i)$ for all $i = 1,\ldots,n$,

  (ii) $co\{p(.|E_i); p \in \mathcal{P}\} = co\{q(.|E_i); q \in \mathcal{Q}\}$ for all $i = 1,\ldots,n$.

Note that this notion is very weak in the sense that it is very incomplete. For instance, an n-dimensional simplex cannot be compared through this definition with any of its subsets. Indeed, two sets $\mathcal{P}$ and $\mathcal{Q}$, ordered by set inclusion, can be compared only if there exists a partition of the state space on which they agree *and* have precise probabilities (item (i) of the definition), and furthermore, conditionally on each cell of this partition, they give the same information (item (ii) of the definition). This means that the extra information contained in $\mathcal{P}$ is about some correlation between what happens in one cell E_i with what happens in another cell E_j.

Said differently, the extra information is orthogonal to the "initial" probabilistic information, reflected in the fact that the cells of the partition have precise probabilities attached to them.

Take for instance

$$\mathcal{Q} = \left\{ \left(p, \frac{1}{2} - p, q, \frac{1}{4} - q, \frac{1}{4}, 0, \ldots, \right) | p \in \left[0, \frac{1}{2}\right], q \in \left[0, \frac{1}{4}\right] \right\}$$

and consider

$$\mathcal{P}_\alpha = \left\{ \left(p, \frac{1}{2} - p, q, \frac{1}{4} - q, \frac{1}{4}, 0, \ldots, \right) | p \in \left[0, \frac{1}{2}\right], q \in \left[0, \frac{1}{4}\right], |q - \frac{p}{2}| \leq \alpha \right\}$$

where $\alpha \in [0, \frac{1}{4}]$.
One obviously has $\mathcal{P}_\alpha \subset \mathcal{P}_{\alpha'}$ for all $\alpha' \geq \alpha$, and $\mathcal{P}_{1/4} = \mathcal{Q}$. Furthermore, $\{\{1,2\},\{3,4\},\{5,6,\ldots\}\}$ is a partition of the state space such that $\forall p \in \mathcal{P}$, $\forall q \in \mathcal{Q}$, $p(E_i) = q(E_i)$ for all $i = 1, \ldots, n$; indeed, $p(\{1,2\}) = q(\{1,2\}) = \frac{1}{2}$, $p(\{3,4\}) = q(\{3,4\}) = \frac{1}{4}$, and $p(\{5,6,\ldots\}) = q(\{5,6,\ldots\}) = \frac{1}{4}$ $\forall p \in \mathcal{P}_\alpha$, $\forall q \in \mathcal{Q}$. It is also easily checked that the set of probabilities conditional on $\{1,2\}$ is the same when computed starting from $\mathcal{P}_\alpha$ and from $\mathcal{Q}$. The same is true for conditionals with respect to $\{3,4\}$, and $\{5,6,\ldots\}$. Thus, the two requisite of the definitions are met and we can assert that $\mathcal{P}_\alpha$ is conditionally more precise than $\mathcal{Q}$. The nature of the extra information that is present in $\mathcal{P}_\alpha$ is maybe clearest for $\alpha = 0$. In that case, one has $q = \frac{p}{2}$ and the extra information that is present in $\mathcal{P}_0$ is a strong correlation between the different cells of the partition. More generally, we can look at upper and lower probabilities for events according to $\mathcal{P}_\alpha$ and $\mathcal{Q}$. We know they agree on the partition $\{\{1,2\},\{3,4\},\{5,6,\ldots\}\}$. One can also check that the upper and lower probabilities on the events $\{1,3\}$ and $\{2,4\}$ are the same for the two sets (0 and 3/4 respectively). However, the lower and upper probability of events $\{2,3\}$ and $\{1,4\}$ do differ for the two sets. One has, with obvious notation, $\underline{p}_\alpha(\{2,3\}) = 1/4 - \alpha$ and $\bar{p}_\alpha(\{2,3\}) = 1/2 + \alpha$ while $\underline{q}(\{2,3\}) = 0$ and $\bar{q}(\{2,3\}) = 3/4$, and similarly for event $\{1,4\}$. The fact that $\underline{p}_\alpha > \underline{q}$ and $\bar{p}_\alpha < \bar{q}$, is another way to see that $\mathcal{P}_\alpha$ is more precise than $\mathcal{Q}$.

Having explained in detail our notion of imprecision, we can now state our final axiom.

Axiom 8 *(Aversion towards imprecision) Let $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$ be such that $\mathcal{P}$ is conditionally more precise than $\mathcal{Q}$, then for all $f \in \mathcal{A}$, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$.*

Compared to Gilboa and Schmeidler (1989)'s Uncertainty Aversion axiom and Hayashi (2005)'s Gains via Hedging axiom, ours deal with the problem in a more direct manner. According to their axiom, uncertainty aversion is revealed whenever the mixture of two indifferent acts is preferred to any of these acts. Our axiom of aversion towards imprecision directly points what kind of information the decision maker values to reduce imprecision of a set of probability distributions. This is in line with our view that aversion towards imprecision should be based on a notion of imprecision that has some content independently of the decision maker's preferences.

3.2 Representation theorem and discussion

Our main representation result asserts that our set of axioms characterizes a maxmin expected utility decision maker, whose "set of priors" (in Gilboa and Schmeidler (1989)'s terms) has a number of distinctive features. More precisely, we show that under our axioms, there exist a utility function and a transformation function $\mathcal{F}$ such that, when evaluation an act f with information $\mathcal{P}$, the decision maker is maxmin expected utility with respect to $\mathcal{F}(\mathcal{P})$, that is, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if, $\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp$. The function $\mathcal{F}$ has to satisfy a number of conditions. We introduce these conditions first and give the formal statement of the theorem afterwards.

The first condition states that the available information constitutes an "upper bound" as to which transformed set is admissible and comes rather straightforwardly from the Pareto condition embedded in Axiom 7.

Condition 1 $\mathcal{F}(\mathcal{P}) \subseteq co(\mathcal{P})$.

The second condition is a consequence of Axiom 2. It states that $\mathcal{F}$ is commutative with respect to a function φ leaving precise information unchanged.

Condition 2 *For all $\mathcal{P} \in \mathbb{P}$, for all onto mappings φ from S to S such that if $\left|\varphi^{-1}(s)\right| \geq 2$, then either $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$ or for all $s' \in \varphi^{-1}(s)$, $p((s'|\varphi^{-1}(s)) = p'(s'|\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$, one has $\mathcal{F}(\mathcal{P}^{\varphi}) = (\mathcal{F}(\mathcal{P}))^{\varphi}$.*

To illustrate this condition take φ to be a permutation of states in the support of $\mathcal{P}$. Then, the transformed set of the permutation of the information set is the permutation of the transformed set. Thus, if one starts with say the simplex on states 1, 2, and 3, the only admissible transformed sets will be sets that are invariant to a permutation of states 1, 2, and 3. In particular, they have to include the point $(1/3, 1/3, 1/3)$. Actually, an implication of this condition is that, in that example, the only singleton that is admissible is the point of equiprobability. We will come back on this point extensively in Section 5. Condition 2 also places some constraints on the link between the transformed set for families that are linked through one of the merging operation explained after Definition 3. It states that the transformed set when the merging of states has been performed (that is $\mathcal{F}(\mathcal{P}^{\varphi})$) is the same as the one obtained by performing the merging on the transformed set of the original set (that is, $(\mathcal{F}(\mathcal{P}))^{\varphi}$). In a nutshell, $\mathcal{F}$ preserves the notion of equivalence we defined.

Condition 3 states that $\mathcal{F}$ is linear and is a direct consequence of the independence axiom. This property is useful to extend properties from "well behaved" sets, like simplices, to sets that can be decomposed in these nicely behaved sets, like cores of beliefs functions.

Condition 3 *For all $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$, for all $\alpha \in [0, 1]$, $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) = \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$.*

Finally, Condition 4 expresses the fact that the function $\mathcal{F}$ preserves the order we introduced on information sets. This is important for applications since it gives some structure on the transformed sets when performing comparative static exercise on the precision of the information. It is also a distinctive feature of our approach that builds on our construction of a specific order on sets of probability distributions.

Condition 4 *For all $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$, if $\mathcal{P}$ is conditionally more precise than $\mathcal{Q}$ then $\mathcal{F}(\mathcal{P}) \subset \mathcal{F}(\mathcal{Q})$.*

This discussion can be summarized in the following theorem.

Theorem 1 *Axioms 1 to 8 hold if, and only if, there exist a unique (up to a positive linear transformation) function $u : \mathcal{C} \to \mathbb{R}$, and a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ satisfying Conditions 1 to 4 and such that for all $(f, \mathcal{P}), (g, \mathcal{Q}) \in \mathcal{A} \times \mathbb{P}$, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,*

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp.$$

Remark 1 When dropping his Invariance to Similarity Reshuffles, Hayashi (2005) derives a representation theorem that "includes our Theorem 1" (p.21).[5] Our result however differs in one important respect: our theorem puts explicit constraints on how the transformed set evolves with the initial information, while his has no bite when one performs comparative static exercise on the precision of the available information.

Remark 2 We adopted the neutral terminology "transformed set" for $\mathcal{F}(\mathcal{P})$ as its interpretation in terms of beliefs is not warranted. The function $\mathcal{F}$ is part of the representation and its properties should be interpreted only in so far as they can be related to properties of the underlying preferences.

4 Attitudes towards imprecision and risk

Based on our representation theorem, one can study how imprecision and risk attitudes are captured by the transformation and the utility functions.

4.1 Imprecision aversion

For $\bar{x}$ and $\underline{x}$ two consequences and for the event $E \subset S$, $\bar{x}_E\underline{x}$ denotes the act f in $\mathcal{A}$ that gives $\bar{x}$ for all s in E and $\underline{x}$ otherwise.

[5] Note however that the space of consequences in Hayashi (2005) is assumed compact metric and that acts are mappings from states of the world to distributions over this compact metric space, which is again compact metric. Thus the exact, technical, comparison in terms of "generality" is not warranted.

Definition 5 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$. Suppose there exist two consequences $\bar{x}$ and $\underline{x}$ in $\mathcal{C}$ such that both a and b strictly "prefer" the constant act $\bar{x}$ to the constant act $\underline{x}$. We say that $\succeq_b$ is* more averse to imprecision than $\succeq_a$ *if for all $E \subset S$, $\mathcal{P} \in \mathbb{P}$, and $\{p\} \in \mathbb{P}$,*

$$(\bar{x}_E\underline{x}, \{p\}) \succeq_a [\succ_a](\bar{x}_E\underline{x}, \mathcal{P}) \Rightarrow (\bar{x}_E\underline{x}, \{p\}) \succeq_b [\succ_b](\bar{x}_E\underline{x}, \mathcal{P})$$

That is, b is *more averse to imprecision than* a if whenever a prefers to bet on E with a precise probabilistic information rather than an imprecise one, b does as well. Note that this definition differs from definitions of comparative aversion to ambiguity that can be found in Ghirardato and Marinacci (2002), Epstein (1999), and subsequently in Klibanoff, Marinacci, and Mukerji (2005), or Hayashi (2005) for instance, in that we restrict attention to binary acts. This is essential to characterize this notion independently of risk attitudes, which are captured by the shape of the utility function.

Theorem 2 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$, satisfying Axioms 1 to 8. Then, the following assertions are equivalent:*
(i) $\succeq_b$ is more averse to imprecision than $\succeq_a$,
(ii) for all $\mathcal{P} \in \mathbb{P}$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$.

An interesting feature of this notion of aversion to imprecision is that it ranks preferences that do not necessarily have the same attitudes towards risk. This is of particular interest in applications if one wants to study the effects of risk aversion and imprecision aversion separately. For instance, one might want to compare portfolio choices of two decision makers, one being less risk averse but more imprecision averse than the other. This type of comparison cannot be done if imprecision attitudes can be compared only among preferences that have the same risk attitude, represented by the utility function. To the best of our knowledge, there is no available result in the literature that achieves this separation of the characterization of comparative ambiguity or imprecision attitudes from risk attitudes.

4.2 Risk aversion

In this subsection, we state for completeness a result on risk aversion, that shows that it is captured in our framework by concavity of the utility function. Take $\mathcal{C}$ to be equal to $[0, M] \subset \mathbb{R}$. For any act $f \in \mathcal{A}$ and $\{p\} \in \mathbb{P}$, denote $E_p f$ the expected value of act f.

Definition 6 *Let $\succeq$ be a preference relation defined on $\mathcal{A} \times \mathbb{P}$. Say that $\succeq$ is* risk averse *if for all $f \in \mathcal{A}$ and $\{p\} \in \mathbb{P}$, $(E_p f, \{p\}) \succeq (f, \{p\})$.*

In our setting, risk aversion is characterized through the restriction of preferences to situations in which the information is probabilistic (the information set is reduced to a singleton).

Definition 7 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$. Say that $\succeq_b$ is* more risk averse than $\succeq_a$ *if for all $f \in \mathcal{A}, x \in [0, M]$ and $\{p\} \in \mathbb{P}$,*

$$(x, \{p\}) \succeq_a [\succ_a](f, \{p\}) \Rightarrow (x, \{p\}) \succeq_b [\succ_b](f, \{p\})$$

We obtain the classical characterization:

Theorem 3 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$, satisfying Axioms 1 to 8.*

(i) $\succeq_a$ is risk averse if, and only if, u^a is concave on $[0, M]$,

(ii) $\succeq_b$ is more risk averse than $\succeq_a$ if, and only if, u^b is more concave than u^a on $[0, M]$.

Taken with Theorem 2, we thus obtain a clear cut separation of attitudes towards risk and imprecision, in which one can, for instance compare imprecision attitudes of two decision makers, one being risk seeking the other being risk averse. In our model, a decision maker is an expected utility maximizer whenever confronted to a situation of risk. Hence, there is no scope for for probabilistic risk aversion as captured say by the Rank Dependent Utility model.

5 Symmetrically decomposable sets of distributions

In this section, we restrict somewhat the sets of distributions we consider as information sets. First, define the notion of a symmetric set as follows. Say that a set of distributions $\mathcal{P}$ is *symmetric* if for any permutation φ on the support of $\mathcal{P}$, $\mathcal{P} = \mathcal{P}^\varphi$. Let $c_\mathcal{P}$, the *center* of $\mathcal{P}$, be the probability distribution in $\mathcal{P}$ that has the property that $c_\mathcal{P} = c_\mathcal{P}^\varphi = c_{\mathcal{P}^\varphi}$ for any permutation φ on states in the support of $\mathcal{P}$; it is the probability distribution putting weight $1/|S(\mathcal{P})|$ on any $s \in S(\mathcal{P})$ and 0 elsewhere.

Say that a set $\mathcal{P} \in \mathbb{P}$ is *symmetrically decomposable* if there exist

- a set of indices I,
- a collection of symmetric sets $\Gamma_j \subset \Delta(I)$, $j \in \{1, \cdots, m\}$,
- a vector of weights $(\alpha_1, \cdots, \alpha_m)$ with $\alpha_j \in [0, 1]$ and $\sum_{j=1}^m \alpha_j = 1$,
- a collection of probability distribution $\{\pi_i\}_{i \in I}$ on S such that $Supp(\pi_i) \cap Supp(\pi_j) = \emptyset$ for all $i \neq j \in I$

such that $\mathcal{P} = \Gamma \otimes (\pi_i)_{i \in I}$, where $\Gamma = \sum_{j=1}^m \alpha_j \Gamma_j$.

Define the center of a symmetrically decomposable set $\mathcal{P}$ to be the probability distribution $c_\mathcal{P}$ defined by $c_\mathcal{P}(s) = \sum_{j=1}^m \alpha_j \sum_{i \in I} c_{\Gamma_j}(i) \pi_i(s)$ where c_{Γ_j} is the center of Γ_j.

Let $\mathbb{SD}$ be the set of symmetrically decomposable sets of probability distributions. Observe that it includes the family of cores of beliefs functions.

5.1 Imprecision neutral decision maker

When the set of distributions $\mathcal{P}$ is in $\mathbb{SD}$, Conditions 2 and 3 introduced in Section 3 have further implications that we now explore. Consider first a set $\mathcal{P} \in \mathbb{SD}$ that is symmetric. Then, Condition 2 implies that the only singleton set that is admissible as a transformed set of a symmetric $\mathcal{P}$ is its center $c_{\mathcal{P}}$. In Ellsberg three-color urn example, this condition implies that Bayesian decision makers would have beliefs $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, as intuition suggests. More generally, Condition 2 asserts that the center of an information set made of finite convex combinations over symmetric sets will always belong to the transformed set. For Bayesian decision makers, this center is their subjective probabilistic beliefs. Note that for cores of beliefs functions this is nothing but the Shapley value.

Recall now Example 1, in which we constructed a probability distribution $\gamma \otimes (\pi_i)_{i=1,2} = (\gamma(1)\frac{1}{2}, \gamma(1)\frac{1}{2}, \gamma(2)\frac{1}{3}, \gamma(2)\frac{1}{2}, \gamma(2)\frac{1}{6}, 0, \dots)$ where $\gamma \in \Gamma \subset \Delta(\{1,2\})$. Assume now that $\Gamma = \{(\gamma, 1-\gamma) | \gamma \in [1/4, 3/4]\}$. Γ is a symmetric set and $\mathcal{P} = \Gamma \otimes (\pi_i)_{i=1,2}$ is symmetrically decomposable. Now, apply Condition 2 to this set. Observe that the function φ merging states 1 and 2 on the one hand, and states 3, 4, and 5 on the other satisfies the assumption of the Condition. We thus know that an imprecision neutral decision maker would put probability 1/2 on the bunched states 1 and 2 and similarly for $\{3,4,5\}$. Hence, $\mathcal{F}(\mathcal{P}^{\varphi}) = \{(1/2, 1/2)\}$ and therefore, we must have that $\mathcal{F}(\mathcal{P})^{\varphi}$ is also equal to $\{(1/2, 1/2)\}$. This in turn implies that an imprecision neutral decision maker (and hence subjective expected utility) revealed subjective probability would be $\{(1/2, 1/2)\} \otimes (\pi_i)_{i=1,2}$. More generally, the revealed subjective probability distribution for any set in $\mathbb{SD}$ will be equal to its "center" defined by $c_{\mathcal{P}}(s) = \sum_{j=1}^{m} \alpha_j \sum_{i \in I} c_{\Gamma_j}(i) \pi_i(s)$.

To conclude, we have established that for the class of symmetrically decomposable sets considered here, our representation theorem yields a sharp prediction concerning what beliefs an imprecision neutral, Bayesian, decision maker, would hold. It can be easily expressed as the appropriately defined "center" of the set under study. Armed with this (endogenously generated) benchmark distribution, we can proceed to define a notion of imprecision premium.

5.2 Imprecision premium

Here, we further characterize our notion of comparative imprecision aversion when the information sets are symmetrically decomposable. First, define a notion of *imprecision premium* which captures how much a decision maker is "willing to lose" when betting on an event in order to be in a probabilistically precise situation. Consider a preference relation $\succeq$ and let $\bar{x}$ and $\underline{x}$ be two consequences such that, with some abuse of notation, $\bar{x} \succ \underline{x}$. For any event $E \subset S$, let q be the probability distribution such that $(\bar{x}_E\underline{x}, \mathcal{P}) \sim (\bar{x}_E\underline{x}, \{q\})$. Under our set of axioms, such a probability distribution exists and is independent of $\bar{x}$ and $\underline{x}$, since $(\bar{x}_E\underline{x}, \mathcal{P}) \sim (\bar{x}_E\underline{x}, \{q\})$ if, and only if, $q(E) = \min_{p \in \mathcal{F}(\mathcal{P})} p(E)$. We can thus state the following definition.

Definition 8 *For any $\mathcal{P} \in \mathbb{S}$ and for any event $E \subset S$, let*

- *the* absolute imprecision premium, $\pi^A(E,\mathcal{P})$ *be defined by* $c_{\mathcal{P}}(E) - q(E)$ *where* q *is such that* $(\bar{x}_E \underline{x}, \mathcal{P}) \sim (\bar{x}_E \underline{x}, \{q\})$,
- *the* relative imprecision premium, $\pi^R(E,\mathcal{P})$ *be defined by* $\frac{\pi^A(E,\mathcal{P})}{c_{\mathcal{P}}(E) - Min_{p\in\mathcal{P}} p(E)}$ *whenever* $c_{\mathcal{P}}(E) \neq Min_{p\in\mathcal{P}} p(E)$.

Thus, the absolute imprecision premium, $c_{\mathcal{P}}(E) - q(E)$, can be interpreted as the mass of probability on the good event E that the decision maker is willing to forego (compared to the center of $\mathcal{P}$) in order to act on a precise, probabilistic, information rather than on the imprecise $\mathcal{P}$. An analogy with the risk premium can be drawn as follows: $c_{\mathcal{P}}$ plays the role of the expectation of the risky prospect while $q(E)$ plays the role of the certainty equivalent. The relative imprecision premium is defined to be the quantity $c_{\mathcal{P}}(E) - q(E)$ normalized by $c_{\mathcal{P}}(E) - Min_{p\in\mathcal{P}} p(E)$, which can be interpreted as the distance from the center to the border of the set in direction E.

An imprecision averse decision maker always exhibits positive imprecision premia. The relative premium is equal to zero for a Bayesian decision maker, and to one for an extremely averse decision maker. Note that the definition of the imprecision premia for any sets in $\mathbb{P}$ would require to fix a benchmark probability which would be the one used by Bayesian decision makers. Theorem 1 does not allow to identify uniquely such a benchmark outside of sets in $\mathbb{SD}$. Restricting our attention to $\mathbb{SD}$, we can now complete the previous result:

Theorem 4 *Let* $\succeq_a$ *and* $\succeq_b$ *be two preference relations defined on* $\mathcal{A} \times \mathbb{SD}$*, satisfying axioms 1 to 8. Then, the following assertions are equivalent:*
(i) $\succeq_b$ *is more averse to imprecision than* $\succeq_a$,
(ii) for all $\mathcal{P} \in \mathbb{SD}$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$,
(iii) for all $\mathcal{P} \in \mathbb{SD}$, *for all event* $E \subset S$, $\pi_b^A(E,\mathcal{P}) \geq \pi_a^A(E,\mathcal{P})$.

5.3 Unanimous order

In view of Theorem 1, one can define a ranking of information sets based on unanimity of decision makers: say that $\mathcal{P}$ is *unanimously more precise* than $\mathcal{Q}$ if for all preference relations that satisfy Axioms 1 to 8 and for all act f, $(f,\mathcal{P}) \succeq (f,\mathcal{Q})$.

This unanimous order can be characterized for decision makers satisfying an extra property, dubbed increasing absolute imprecision premium.

Definition 9 *A decision maker is said to have* increasing absolute imprecision premium *if for any* $\mathcal{P},\mathcal{Q} \in \mathbb{SD}$ *such that* $c_{\mathcal{P}} = c_{\mathcal{Q}}$ *and* $\mathcal{P} \subset \mathcal{Q}$*, for any event* $E \subset S$ *such that* $c_{\mathcal{Q}}(E) > 0$, $\pi^A(E,\mathcal{Q}) \geq \pi^A(E,\mathcal{P})$.

To illustrate this definition, consider an Ellsberg urn with 90 balls, 30 being Red and the remaining 60 being Black or Yellow. Lets $\mathcal{P}_n$ be the set of distributions corresponding to the information "there are n Yellow and n Black balls" ($n \leq 30$). A decision maker with an increasing absolute imprecision premium for betting on Yellow will have a premium $\pi^A(\{Y\}, \mathcal{P}_n)$ that is decreasing in n (when $n = 30$, the premium is equal to zero). The following proposition shows that for such decision makers, $(f, \mathcal{P}_n) \succeq (f, \mathcal{P}_{n'})$ for all f and all $n \geq n'$.

Proposition 1 *Assume that unanimity is based on preferences satisfying axioms 1 to 8 and increasing absolute imprecision premium, and let $\mathcal{P}, \mathcal{Q} \in \mathbb{SD}$. Then, $\mathcal{P}$ is unanimously more precise than $\mathcal{Q}$ if, and only if,*

(*i*) $c_{\mathcal{P}} = c_{\mathcal{Q}}$ *and,*

(*ii*) $\mathcal{P} \subset \mathcal{Q}$.

In Gajdos, Tallon, and Vergnaud (2004) we modelled information coming as a set of probability distributions $\mathcal{P}$ together with an anchor c. We introduced an order on couples $[\mathcal{P}, c]$ stating that $[\mathcal{P}_1, c_1]$ was a center preserving increase in imprecision over $[\mathcal{P}_2, c_2]$ whenever $c_1 = c_2$ and $\mathcal{P}_2 \subset \mathcal{P}_1$. The result we just derived shows that, under increasing absolute imprecision premium, the order on imprecision is the same as the one we assumed in our 2004 paper, with the twist that the anchor c is now endogenous.

5.4 Functional forms

The representation theorem we gave does not pin down a very specific functional form, as we did not establish a specific form for the mapping $\mathcal{F}$, although we were able to identify constraints on admissible such mappings. This general approach can be further specified to yield functional forms that are more "user friendly" for economic applications.

5.4.1 Contraction

An obvious way to construct the transformed set when starting from a simplex is to consider the homothetic reduction (or contraction) of that simplex around the point of equiprobability. Actually, this intuition could be extended to any symmetric set of probability distributions, and more generally, to arbitrary sets in $\mathbb{SD}$. However, the geometric intuition that the transformed set should have the same shape as the information set is not one that should be valid, as the transformed set does not have an interpretation outside of simply being part of the representation. We thus need to find interpretable conditions on the preferences to back this geometric "intuition".

We now turn in more detail to this possibility and give axiomatic foundation for a decision criterion in which the transformation function is the contraction of the information set around its center. The general approach we take here parallels the usual approach in expected utility theory, in which specific classes of utility functions are defined by characterizing some properties of the risk premium. We give two ways of approaching the behavioral property that yields the contraction. The first one is to impose a property, called constant relative imprecision premium, which states that the premium does not depend on the information set and does not depend on the betting event as well.

Definition 10 *A decision maker is said to have constant relative imprecision premium* θ *if for any* $\mathcal{P} \in \mathbb{SD}$ *and for any event* $E \subset S$ *such that* $c_{\mathcal{P}}(E) \neq Min_{p \in \mathcal{P}} p(E)$, $\pi^R(E, \mathcal{P}) = \theta$.

The second property that can be used to obtain the particular functional form described is a restatement of an extra axiom considered in Gajdos, Tallon, and Vergnaud (2004). Loosely speaking it states that whenever the expected utility of act f with respect to the center of information set $\mathcal{P}$ is higher than that of act g with respect to the center of information set $\mathcal{Q}$ and the *worst expected utility* of act f with information set $\mathcal{P}$ is higher than the worst expected utility of act g with information set $\mathcal{Q}$, then $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$.

Definition 11 *A decision maker is said to satisfy* dominance *if for all* $f, g \in \mathcal{A}$, $\mathcal{P}, \mathcal{Q} \in \mathbb{SD}$, *whenever* $(f, \{c_{\mathcal{P}}\}) \succeq (g, \{c_{\mathcal{Q}}\})$ *and for all* $p \in \mathcal{P}$ *there exists* $q \in \mathcal{Q}$ *such that* $(f, \{p\}) \succeq (g, \{q\})$, *then* $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$.

The next proposition states that these two particular manner of handling information sets are actually equivalent and give rise to the specific functional form that we called contraction.

Proposition 2 *Consider a decision maker satisfying Axioms 1 to 8. The following assertions are equivalent:*

(i) *the decision maker has constant relative imprecision premium* θ,

(ii) *for all* $\mathcal{P} \in \mathbb{SD}$, $\mathcal{F}(\mathcal{P}) = \theta \mathcal{P} + (1 - \theta) \{c_{\mathcal{P}}\}$,

(iii) *the decision maker satisfies dominance.*

Therefore, if a decision maker has constant relative imprecision premium θ, or satisfies dominance the representation theorem takes the form: $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\theta \min_{p \in \mathcal{P}} \int u \circ f dp + (1 - \theta) \int u \circ f dc_{\mathcal{P}} \geq \theta \min_{p \in \mathcal{Q}} \int u \circ g dp + (1 - \theta) \int u \circ g dc_{\mathcal{Q}}.$$

Thus, in this setting, $\succeq_b$ is more averse towards imprecision than $\succeq_a$ if, and only if, $\theta_a \leq \theta_b$. This parametrization of imprecision aversion is hence extremely simple and convenient to perform comparative static exercises in applications. This functional form was axiomatized in a different setup in Gajdos, Tallon, and Vergnaud (2004) and subsequently, in the same setup although in a different manner in Hayashi (2005).

Hayashi (2005) is more general from a technical point of view (although see footnote 5) in the sense that he considers any possible information sets, and not only those in $\mathbb{SD}$. We could also extend our result by defining our imprecision premium for any set of prior taking the Steiner point as the benchmark probability (note that it reduces to the appropriately defined center for sets in $\mathbb{SD}$.) However, this would be a bit artificial since the center emerges naturally from our axiomatic construction for symmetrically decomposable sets, while the Steiner point would have to be imposed as an exogenous reference point. In Hayashi's construction, Invariance to Similarity Reshuffle actually yields both the fact that for any set, its Steiner point is the distribution that is revealed by subjective expected utility decision makers *and* the fact that the functional form is a "contraction" of the information set (i.e., constant relative imprecision premium in our terminology). In our construction, these two issues are kept to some extent separate.

5.4.2 Other functional form

Although convenient, the functional form given in Proposition 2 does not have an axiomatic justification of the same nature as the general form of Theorem 1. Relative constant imprecision premium or dominance are not properties that have strong normative content. They should be viewed merely as testable properties that preferences might or might not satisfy.

To better understand what is implied by this functional form and the underlying axiom that is not implied more generally, consider the following example. Take $\mathcal{P} = \Delta(\{1,2,3\})$ and consider f and g such that $u(f(1)) = u(g(1)) = 0$, $u(f(2)) = 1$, $u(g(2)) = 3/2$, and $u(f(3)) = 2$, $u(g(3)) = 3/2$. Under the representation of Proposition 2, one has

$$\theta \min_{\mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}} = \theta \min_{\mathcal{P}} \int u \circ g dp + (1-\theta) \int u \circ g dc_{\mathcal{P}} = 1 - \theta$$

Consider now preferences that do not satisfy constant imprecision premium, giving rise for instance to the following transformed set:

$$\mathcal{F}(\Delta(\{1,2,3\})) = co\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(0,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$$

According to the functional form of Theorem 1 applied with this specific transformed set, one has $\min_{\mathcal{F}(\mathcal{P})} \int u \circ f dp = \frac{1}{2}u(f(1)) + \frac{1}{2}u(f(2)) = \frac{1}{2}$, and hence g is strictly better than f since $\min_{\mathcal{F}(\mathcal{P})} \int u(g)\, dp = \frac{3}{4}$. Note that in this case, the relative imprecision premium is not constant since $\pi^R(\{1\}, \Delta(\{1,2,3\})) = 1$ while $\pi^R(\{1,2\}, \Delta(\{1,2,3\})) = 1/4$. Both preferences seem reasonable. Hence, although the functional form of Proposition 2 has the nice feature of summarizing the attitude towards imprecision in a single parameter, the underlying axiom reflects an attitude towards imprecision that is not to be expected to hold for all decision makers.

The alternative transformation function introduced in this example can be given a more systematic treatment. Essentially, Theorem 1 states that, when the information set is a simplex over $\{1,\ldots,n\}$, any transformed set is admissible provided it is symmetric around the center of the simplex, that is the point of equiprobability. In particular, consider the family whose extreme points are all the possible permutations of the probability distribution $(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$. An act f such that $f(1) \leq f(2) \leq \cdots \leq f(n)$ together with the simplex is then evaluated by $\frac{1}{2}u(f(1)) + \frac{1}{2}u(f(2))$. This of course generalizes to any permutation of the probability distribution $(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$ for any $m \leq n$, yielding that that same act f is evaluated by $\sum_{i=1}^{m} \frac{1}{m}u(f(i))$. This type of functional would hence correspond to first truncating the act in its upper part (that is cutting its best consequences out) and then applying an expected utility computation with equal weights on the remaining states.

Observe that the transformed set described here is not connected in any obvious geometric way to the information set. This is an illustration of the fact that the "traditional" interpretation of the transformed set as a set of *priors* is misleading. The shape of the transformed set does not have a behavioral content of its own.

6 An application to portfolio choice

We develop in this section a simple application of our analysis to portfolio choice that is similar in spirit to Klibanoff, Marinacci, and Mukerji (2005)'s. There are three assets, a, b, and c. The following table gives the payoff matrix

s	1	2	3	4
a	k	k	k	k
b	$\bar{b}$	$\bar{b}$	1	1
c	$\bar{c}$	1	1	$\bar{c}$

We put the following restrictions on the parameters: $\bar{c} > \bar{b} > k > 1$. The information available is given by the set

$$\mathcal{P}_\alpha = \left\{ \left(p, \frac{1}{2} - p, q, \frac{1}{2} - q, 0, \ldots, \right) | p \in \left[0, \frac{1}{2}\right], q \in \left[0, \frac{1}{2}\right], |q - p| \leq \alpha \right\}$$

where $\alpha \in [0, \frac{1}{2}]$. Hence, the probability of $\{1, 2\}$ is precise, equal to $1/2$ and similarly for $\{3, 4\}$. α is a measure of how "imprecise the set is": a higher α corresponds to a higher degree of imprecision. Taken with this information, the assets have a natural interpretation: asset a is the safe asset, b is the "risky" asset as its payoffs are measurable with respect to the partition $\{\{1, 2\}, \{3, 4\}\}$, and asset c is the "imprecise" asset.

We consider a decision maker with CARA utility function $u(w) = -e^{-\gamma w}$, where γ is the coefficient of absolute risk aversion. The transformed set is given by:

$$\mathcal{F}(\mathcal{P}_\alpha) = \left\{ \left(p, \frac{1}{2} - p, q, \frac{1}{2} - q, 0, \ldots, \right) | p \in \left[\frac{1}{4} - \theta, \frac{1}{4} + \theta\right], q \in \left[\frac{1}{4} - \theta, \frac{1}{4} + \theta\right], |q - p| \leq \alpha \right\}$$

θ is the parameter of imprecision aversion, in the sense that it gives the rate of contraction for the simplex $\Delta(\{1, 2\})$. For simplicity, we assume that the constraint on the distance between p and q is the same in the transformed set as in the information set (it is easy to generalize to a constraint of the type $|q-p| \leq \beta(\alpha)$ with $\beta(.)$ increasing in α.) To make things interesting, we assume that $\theta \geq \alpha/2$, so that the constraint $|q-p| \leq \alpha$ is effective in the computation of the optimal portfolio (although see Remark 3 below.)

The decision maker has one unit of wealth that he has to allocate among the three assets. We allow for short sales. We consider successively three cases depending on which assets are actually available, the first case being the benchmark situation of choice between the safe and the risky asset.

Case 1: choice between safe and risky asset.

This case is the usual one and one gets that $b^\star = \frac{1}{\gamma(1-\bar{b})} log\left(\frac{k-1}{\bar{b}-k}\right)$, which is naturally independent from the parameters θ and α. Under the parameter restrictions, it is easy to see that increasing risk aversion decreases holding of the risky asset.

Case 2: choice between safe and imprecise asset.

The problem to be solved here is to find the optimal amount of the imprecise asset, i.e., the solution to: $\max_c \min_{\pi \in \mathcal{F}(\mathcal{P}_\alpha)} - \left[(\pi(1)+\pi(4))e^{-\gamma((1-c)k+c\bar{c})} + (\pi(2)+\pi(3))e^{-\gamma((1-c)k+c)}\right]$, or rewritten in terms of p and q:

$$\max_c \min_{\mathcal{F}(\mathcal{P}_\alpha)} - \left[(p+1/2-q))e^{-\gamma((1-c)k+c\bar{c})} + (1/2-p+q)e^{-\gamma((1-c)k+c)}\right]$$

As long as $c > 0$, $-e^{-\gamma((1-c)k+c\bar{c})} > -e^{-\gamma((1-c)k+c)}$ and hence the decision maker will "use" the probability in $\mathcal{F}(\mathcal{P}_\alpha)$ that put the highest weight on the event $\{2,3\}$ and lowest weight on $\{1,4\}$. Hence, one wants to minimize $p-q$. Let therefore $q = 1/4+\theta$ and $p = 1/4+\theta-\alpha$.[6] Solving for the optimal solution yields

$$c^\star = \frac{1}{\gamma(\bar{c}-1)} log\left(\frac{(\bar{c}-k)(1/2-\alpha)}{(k-1)(1/2+\alpha)}\right)$$

One can check that $c^\star$ is positive as conjectured if $(k-1)/(\bar{c}-k) < (1/2-\alpha)/(1/2+\alpha)$. Here, the comparative statics with respect to γ works as in the single risky asset case. What is more interesting, although intuitive, is that the imprecise asset holding is decreasing in α: an increase in imprecision of the information provided reduces the amount of asset the decision maker wants to hold. Note also that imprecise asset holding does not depend, in this example, on the imprecision aversion parameter θ (as long as $\theta \geq \alpha/2$).

Case 3: choice among all three assets.

This is the more general case and is a bit more tedious to study. Let's write u_s the utility of the portfolio in state s. As long as $b > 0$ and $c > 0$, one has that $u_1 > u_2$ and $u_4 > u_3$ and

[6] Actually, it is easy to see that this is not the only possible choice of a minimizing probability. $q = 1/4-\theta+\alpha$ and $p = 1/4-\theta$ would also minimize $p-q$. The optimal solution however does not depend on which one of these probability distributions is used, as the objective function depends only on $p-q$.

furthermore, $u_4 - u_3 > u_1 - u_2$. Hence, the minimizing probability that belongs to $\mathcal{F}(\mathcal{P}_\alpha)$ is given by $p = 1/4 + \theta - \alpha$ and $q = 1/4 + \alpha$.

Let $K = \frac{(\bar{c}-k)(\bar{b}-1)}{(\bar{c}-\bar{b})(k-1)}$. Under our assumption, $K > 1$. Then, the optimal solution can be written:

$$
\begin{aligned}
b^\star &= \frac{1}{\gamma(\bar{b}-1)} log \left[(K-1) \frac{1/4 - \theta + \alpha}{1/4 + \theta} \right] \\
c^\star &= \frac{1}{\gamma(\bar{c}-1)} log \left[\frac{\bar{c} - \bar{b}}{\bar{b} - 1} \left((K-1) \frac{1/4 - \theta}{1/4 + \theta} + \frac{1/4 + \theta - \alpha}{1/4 - \theta + \alpha} \right) \right]
\end{aligned}
$$

Under some further (uninteresting) restrictions on the parameters, one can check that $b^\star > 0$ and $c^\star > 0$ as conjectured when picking the minimizing probability distribution.

One can thus perform comparative statics exercises. As α increases, that is as the information available is less precise, the decision maker will hold *more* of the risky asset and *less* of the imprecise asset. Thus, there is some form of substitution among assets as imprecision increases. This suggests that the observed under diversification of decision makers' portfolio might be a consequence of how imprecision affects different assets. More specifically, consider parameter values such that $b^\star > c^\star$ (in our toy example this is the case for a large range of parameter values.) Note that if one were to ignore the effect of uncertainty on asset holding by wrongly setting $\alpha = 0$, the predicted holding of the risky asset would be lower than $b^\star$ while the predicted holding of the imprecise asset would be higher than $c^\star$, i.e., the predicted holdings would appear to be more diversified. Thus if one fails to identify which assets are affected by imprecision, one could overestimate the predicted weight of these assets in the optimal portfolio.

Finally, it is also easy to show that the holdings of the risky as well as the imprecise assets are decreasing in the risk aversion parameter γ, as well as with the imprecision aversion parameter θ. This might help explaining phenomenon like the equity premium puzzle, as imprecision aversion essentially reinforces the effect of risk aversion. Interestingly, these two very tentative hint as how to account for the under-diversification puzzle *and* the equity premium puzzle in our model are linked to two different parameters (imprecision and imprecision aversion) and could therefore be incorporated in the *same* model.

Remark 3 The comparative static exercises performed were done under the assumption that $\theta \geq \alpha/2$. If this were not the case, then one can show that the minimizing probability used to evaluate the portfolio returns does not depend on α (when looking at the choice among all three assets.) Hence, over the full range of parameters there is a discontinuity in how imprecision affects holding of the risky and imprecise assets.

Remark 4 Note that all the action in this example does not take place because of the non-differentiability introduced by the min operator, as for instance in Epstein and Wang (1994) or Mukerji and Tallon (2001). Rather, the comparative statics were done at points where, locally, the decision maker behaves like an expected utility maximizers. More precisely, in usual maxmin expected utility models, decision makers look like expected utility maximizers away from the 45 line and there is no sense in which one can change the set of priors as there is no explicit link with the available information. In our model, there is some leverage in that respect even away from the kinks, as we have a way to link changes in the set of priors to changes in available information and to changes in imprecision attitudes. Thus, although non smooth, our model remains tractable in applications.

Appendix

Proof of Theorem 1

In order to prove Theorem 1, some additional definitions are needed. Let $\mathcal{H}$ be the set of horse lotteries, i.e., the set of mappings F from S to Y, where Y is the set of distributions over $\mathcal{C}$ with finite supports. Denote by $\mathcal{H}^c$ the subset of constant horse lotteries. For all $c \in \mathcal{C}$ and all $s \in S$, we note $\pi_{F(s)}(c)$ the probability of c according to the lottery $F(s)$. The following definition establishes a link between $\mathcal{H} \times \mathbb{P}$ and $\mathcal{A} \times \mathbb{P}$.

Definition 12 *For any* $(F, \mathcal{P}) \in \mathcal{H}\times\mathbb{P}$*, we say that* $(f, \mathcal{Q}) \in \mathcal{A}\times\mathbb{P}$ *is* Savage equivalent *to* $(F, \mathcal{P})$ *if there exist:*

- *a partition* $(E_1, \ldots, E_n)$ *of* $S(\mathcal{Q})$*,*
- *a collection of probability distributions on* S*,* $\{\pi_i\}_{i\in\{1,..,n\}}$*, such that* $Supp(\pi_i) \subseteq E_i$ *for all* $i \in \{1, \ldots, n\}$*,*
- *a one to one mapping* $\psi : S(\mathcal{P}) \to \{1, .., n\}$*,*

such that

- *for all* $s \in S(\mathcal{P})$*, for all* c*,* $\pi_{\psi(s)}\left(f^{-1}(c)\right) = \pi_{F(s)}(c)$*,*
- $\mathcal{Q} = \Gamma \otimes (\pi_i)_{i\in I}$ *where* $\Gamma = \left\{q \in \Delta(\{1, .., n\}) | \exists p \in \mathcal{P} \text{ s.t } \forall i \in \{1, .., n\}, q(i) = p\left(\psi^{-1}(i)\right)\right\}$*.*

Note that for all $(F, \mathcal{P}) \in \mathcal{H}\times\mathbb{P}$, there exists $(f, \mathcal{Q}) \in \mathcal{A}\times\mathbb{P}$ such that $(f, \mathcal{Q})$ is Savage equivalent to $(F, \mathcal{P})$. Furthermore, Axiom 2 implies that for any $(f, \mathcal{Q})$ and $(f', \mathcal{Q}')$ which are both Savage equivalent to $(F, \mathcal{P})$, $(f, \mathcal{Q}) \sim (f', \mathcal{Q}')$.

Let us define the preference relation $\succeq^{AA}$ on $\mathcal{H}\times\mathbb{P}$ as follows: $(F, \mathcal{P}) \succeq^{AA} (F', \mathcal{P}')$ if and only if there exist $(f, \mathcal{Q})$ and $(f', \mathcal{Q}')$ that are Savage equivalent to $(F, \mathcal{P})$ and $(F', \mathcal{P}')$, respectively, such that $(f, \mathcal{Q}) \succeq (f', \mathcal{Q}')$.

The proof of the sufficiency part of the theorem goes through several lemma. First, we show that $\succeq^{AA}$ can be represented by a multiple-priors functional.

Lemma 1 *Assume that Axioms 1 to 6, and 8 hold. Then, there exist a unique (up to a positive linear transformation) affine function $U : Y \to \mathbb{R}$, and a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$, such that for all $(F, \mathcal{P}), (G, \mathcal{Q})$ in $\mathcal{H} \times \mathbb{P}$,*

$$(F, \mathcal{P}) \succeq^{AA} (G, \mathcal{Q}) \Leftrightarrow \min_{p \in \mathcal{F}(\mathcal{P})} \int U \circ F dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int U \circ G dp.$$

Proof.

Step 1. We first prove that $\succeq^{AA}$ satisfies Gilboa and Schmeidler (1989)'s axioms.

Gilboa and Schmeidler (1989)'s axioms can be restated formally in our framework as follows.

Axiom 9 *$\succeq^{AA}$ is complete and transitive.*

Axiom 10 *There exist $(F, \mathcal{P})$ and $(G, \mathcal{Q})$ in $\mathcal{H} \times \mathbb{P}$ such that $(F, \mathcal{P}) \succ^{AA} (G, \mathcal{Q})$.*

Axiom 11 *For all $F, G \in \mathcal{H}$, and all $\mathcal{P} \in \mathbb{P}$, if $(F(s), \{\delta_s\}) \succeq^{AA} (G(s), \{\delta_s\})$ for all $s \in S(\mathcal{P})$, then $(F, \mathcal{P}) \succeq^{AA} (G, \mathcal{P})$.*

Axiom 12 *For all $F, G, H \in \mathcal{H}$, and all $\mathcal{P} \in \mathbb{P}$, if $(F, \mathcal{P}) \succ^{AA} (G, \mathcal{P}) \succ^{AA} (H, \mathcal{P})$, then there exist α and β in $(0, 1)$ such that:*

$$(\alpha F + (1 - \alpha)H, \mathcal{P}) \succ^{AA} (G, \mathcal{P}) \succ^{AA} (\beta F + (1 - \beta)H, \mathcal{P}).$$

Axiom 13 *For all $F, G \in \mathcal{F}$, $H \in \mathcal{H}^c$, $\mathcal{P} \in \mathbb{P}$, and $\alpha \in (0, 1)$,*

$$(F, \mathcal{P}) \succeq^{AA} (G, \mathcal{P}) \Leftrightarrow (\alpha F + (1 - \alpha)H, \mathcal{P}) \succeq^{AA} (\alpha G + (1 - \alpha)H, \mathcal{P}).$$

Axiom 14 *For all $F, G \in \mathcal{H}$, $\mathcal{P} \in \mathbb{P}$, $\alpha \in]0, 1[$,*

$$(F, \mathcal{P}) \sim^{AA} (G, \mathcal{P}) \Rightarrow (\alpha F + (1 - \alpha)G, \mathcal{P}) \succeq^{AA} (F, \mathcal{P}).$$

Clearly, Axioms 1, 5, and 6 imply Axioms 9, 10, and 11.

Let us prove that Axiom 12 holds. For any $(F, \mathcal{P})$, $(G, \mathcal{P})$ and $(H, \mathcal{P})$ in $\mathcal{H} \times \mathbb{P}$, one can find f, g, and h in $\mathcal{A}$ and $\mathcal{Q}$ in $\mathbb{P}$ such that $(f, \mathcal{Q})$, $(g, \mathcal{Q})$, and $(h, \mathcal{Q})$ are Savage equivalent to $(F, \mathcal{P})$, $(G, \mathcal{P})$, and $(H, \mathcal{P})$, respectively. Suppose that $(F, \mathcal{P}) \succ^{AA} (G, \mathcal{P}) \succ^{AA} (H, \mathcal{P})$. Then by definition, $(f, \mathcal{Q}) \succ (g, \mathcal{Q}) \succ (h, \mathcal{Q})$, and by Axiom 4 there exist α and β in $(0, 1)$ such that:

$$(\alpha f + (1 - \alpha)h, \mathcal{Q}) \succ (g, \mathcal{Q}) \succ (\beta f + (1 - \beta)h, \mathcal{Q}).$$

Note that $(\alpha f + (1 - \alpha)h, \mathcal{Q})$ and $(\beta f + (1 - \beta)h, \mathcal{Q})$ are Savage equivalent to $(\alpha F + (1 - \alpha)H, \mathcal{P})$ and $(\beta F + (1 - \beta)H, \mathcal{P})$, respectively. Therefore,

$$(\alpha F + (1 - \alpha)H, \mathcal{P}) \succ^{AA} (G, \mathcal{P}) \succ^{AA} (\beta F + (1 - \beta)H, \mathcal{P}),$$

which proves that $\succeq^{AA}$ satisfies Axiom 12.

We now turn to Axiom 13. Let $F, G \in \mathcal{H}$, and $\mathcal{P} \in \mathbb{P}$ be such that $(F, \mathcal{P}) \succeq^{AA} (G, \mathcal{P})$. Consider any $H \in \mathcal{H}^c$, and $\alpha \in]0,1[$. We can find $(f, \Gamma \otimes (\pi_i)_{i\in I})$, $(g, \Gamma \otimes (\pi_i')_{i\in I})$ and $(h, \Gamma \otimes (\pi_i'')_{i\in I})$ which are Savage equivalent to $(F, \mathcal{P})$, $(G, \mathcal{P})$ and $(H, \mathcal{P})$, respectively, such that for all $i, j, k \in I$, $Supp(\pi_i) \cap Supp(\pi_k'') = Supp(\pi_j') \cap Supp(\pi_k'') = \emptyset$. It is easily checked that

$$\left(f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h, \Gamma \otimes (\alpha\pi_i + (1-\alpha)\pi_i'')_{i\in I}\right)$$

and

$$\left(g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h, \Gamma \otimes (\alpha\pi_i' + (1-\alpha)\pi_i'')_{i\in I}\right)$$

are Savage equivalent to $(\alpha F + (1-\alpha)H, \mathcal{P})$ and $(\alpha G + (1-\alpha)H, \mathcal{P})$, respectively.

Let $\varphi : S \to S$ be an onto mapping satisfying the three following conditions:

(i) $\varphi(s) = s$ for all $s \in S\setminus \cup_{i\in I} Supp(\pi_i'')$,

(ii) $\varphi(s) \in \cup_{i\in I} Supp(\pi_i'')$ for all $s \in \cup_{i\in I} Supp(\pi_i'')$,

(iii) $\varphi(s) = \varphi(s')$ for all $s, s' \in \cup_{i\in I} Supp(\pi_i'')$, such that $h(s) = h(s')$.

Note that $f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h$ and $g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h$ are φ-measurable. Furthermore, for all s such that $\left|\varphi^{-1}(s)\right| \geq 2$, $p(\varphi^{-1}(s)) = (1-\alpha)\pi_H(h(\varphi^{-1}(s)))$ for all p in $\Gamma \otimes (\alpha\pi_i + (1-\alpha)\pi_i'')_{i\in I}$ or in $\Gamma \otimes (\alpha\pi_i' + (1-\alpha)\pi_i'')_{i\in I}$. Let $\pi'' = (\pi_i'')^\varphi$ (indeed we have that $(\pi_i'')^\varphi = \left(\pi_j''\right)^\varphi$ $\forall i, j \in I$). We have that $(\Gamma \otimes (\alpha\pi_i + (1-\alpha)\pi_i'')_{i\in I})^\varphi = \alpha\left(\Gamma \otimes (\pi_i)_{i\in I}\right) + (1-\alpha)\left\{\pi''\right\}$ as well as $(\Gamma \otimes (\alpha\pi_i' + (1-\alpha)\pi_i'')_{i\in I})^\varphi = \alpha\left(\Gamma \otimes (\pi_i')_{i\in I}\right) + (1-\alpha)\left\{\pi''\right\}$. Therefore, by Axiom 2,

$$\left(f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h, \Gamma \otimes (\alpha\pi_i + (1-\alpha)\pi_i'')_{i\in I}\right) \sim \left(f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \alpha\left(\Gamma \otimes (\pi_i)_{i\in I}\right) + (1-\alpha)\left\{\pi''\right\}\right)$$

$$\left(g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h, \Gamma \otimes (\alpha\pi_i + (1-\alpha)\pi_i'')_{i\in I}\right) \sim \left(g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \alpha\left(\Gamma \otimes (\pi_i')_{i\in I}\right) + (1-\alpha)\left\{\pi''\right\}\right)$$

$$\begin{aligned}
\left(f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \Gamma \otimes (\pi_i)_{i\in I}\right) &\sim (f, \Gamma \otimes (\pi_i)_{i\in I}) \\
\left(g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \Gamma \otimes (\pi_i')_{i\in I}\right) &\sim (g, \Gamma \otimes (\pi_i')_{i\in I}) \\
\left(f_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \left\{\pi''\right\}\right) &\sim \left(g_{S\setminus\cup_{i\in I} Supp(\pi_i'')}h^\varphi, \left\{\pi''\right\}\right) \sim \left(h, \Gamma \otimes (\pi_i'')_{i\in I}\right)
\end{aligned}$$

By Axiom 3,

$$\left(f_{S\setminus\cup_{i\in I}Supp(\pi''_i)}h^{\varphi}, \alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left\{\pi''\right\}\right)$$
$$\succeq \left(g_{S\setminus\cup_{i\in I}Supp(\pi''_i)}h^{\varphi}, \alpha\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)+(1-\alpha)\left\{\pi''\right\}\right)$$
$$\Leftrightarrow \left(f_{S\setminus\cup_{i\in I}Supp(\pi''_i)}h^{\varphi}, \Gamma\otimes(\pi_i)_{i\in I}\right)\succeq\left(g_{S\setminus\cup_{i\in I}Supp(\pi''_i)}h^{\varphi}, \Gamma\otimes(\pi'_i)_{i\in I}\right)$$

which proves that Axiom 13 holds.

Finally, we prove that Axiom 14 holds. Let consider $F, G \in \mathcal{H}$, $\mathcal{P}\in\mathbb{P}$, and $\alpha\in]0,1[$ such that $(F,\mathcal{P})\sim^{AA}(G,\mathcal{P})$. We can find $(f,\Gamma\otimes(\pi_i)_{i\in I})$ and $(g,\Gamma\otimes(\pi'_i)_{i\in I})$ which are Savage equivalent respectively to $(F,\mathcal{P})$ and $(G,\mathcal{P})$ such that for all $i,j\in I$, $Supp(\pi_i)\cap Supp(\pi'_j)=\emptyset$. Then $\left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\right)$ is Savage equivalent to $(\alpha F+(1-\alpha)G,\mathcal{P})$.

Observe that $\Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}$ is conditionally more precise than $\alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)$. Indeed, it is clear that

$$\Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\subseteq\alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right).$$

Furthermore, consider the partition (E_f,E_g) of S where $E_f=\cup_{i\in I}Supp(\pi_i)$. We have that for all $p\in\alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)$, $p(E_f)=\alpha$, $p(E_g)=1-\alpha$ and

$$\begin{aligned}\{p(.|E_f)|p\in\Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\} &= \{p(.|E_f)|p\in\alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)\}\\ &= \left(\Gamma\otimes(\pi_i)_{i\in I}\right),\end{aligned}$$

$$\begin{aligned}\{p(.|E_g)|p\in\Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\} &= \{p(.|E_g)|p\in\alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)\}\\ &= \left(\Gamma\otimes(\pi'_i)_{i\in I}\right).\end{aligned}$$

By Axiom 8,

$$\left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\right)\succeq\left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)\right).$$

Since, by Axiom 2:

$$\begin{aligned}(f,\Gamma\otimes(\pi_i)_{i\in I}) &\sim \left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \left(\Gamma\otimes(\pi_i)_{i\in I}\right)\right),\\ \left(g,\Gamma\otimes(\pi'_i)_{i\in I}\right) &\sim \left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \left(\Gamma\otimes(\pi'_i)_{i\in I}\right)\right),\end{aligned}$$

Axiom 3 implies:

$$(f,\Gamma\otimes(\pi_i)_{i\in I})\sim\left(g,\Gamma\otimes(\pi'_i)_{i\in I}\right)\sim\left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \alpha\left(\Gamma\otimes(\pi_i)_{i\in I}\right)+(1-\alpha)\left(\Gamma\otimes(\pi'_i)_{i\in I}\right)\right).$$

Therefore,

$$\left(f_{S\setminus\cup_{i\in I}Supp(\pi'_i)}g, \Gamma\otimes(\alpha\pi_i+(1-\alpha)\pi'_i)_{i\in I}\right)\succeq(f,\Gamma\otimes(\pi_i)_{i\in I}),$$

which proves that

$$(\alpha F + (1-\alpha)G, \mathcal{P}) \succeq^{AA} (F, \mathcal{P}).$$

By Gilboa and Schmeidler (1989)'s Theorem we know that there exist a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$, and for all $\mathcal{P} \in \mathbb{P}$ a unique (up to a positive linear transformation) affine function $U_{\mathcal{P}} : Y \to \mathbb{R}$, such that $\forall (F,\mathcal{P}), (G,\mathcal{P}) \in \mathcal{H} \times \mathbb{P}$, $(F,\mathcal{P}) \succeq^{AA} (G,\mathcal{P})$ if and only if $\min_{p\in\mathcal{F}(\mathcal{P})} \int U_{\mathcal{P}} \circ F dp \geq \min_{p\in\mathcal{F}(\mathcal{P})} \int U_{\mathcal{P}} \circ G dp$.

Step 2. We now extend the representation obtained in Step 2 to compare acts associated with different information sets.

First, note that for all $H \in \mathcal{H}^c$ for all $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$, Axiom 2 implies that $(H,\mathcal{P}) \sim^{AA} (H,\mathcal{Q})$. Therefore, $U_{\mathcal{P}}$ and $U_{\mathcal{Q}}$ represent the same expected utility over constant acts in $\mathcal{H}^c$. Hence, they can be chosen such that $U_{\mathcal{P}} = U_{\mathcal{Q}} = U$. We denote by $\succeq_c^{AA}$ the restricted preferences over $\mathcal{H}^c$.

To show that the representation can be extended to compare acts associated to different sets $\mathcal{P}$, let $(F,\mathcal{P}) \succeq^{AA} (G,\mathcal{Q})$. Since $S(\mathcal{P})$ and $S(\mathcal{Q})$ are finite and $F(s)$ and $G(s)$ have finite support, there exist $\overline{x}$ and $\underline{x}$ in $\mathcal{C}$ such that for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $k_{\overline{x}} \succeq_c^{AA} k_{F(s)}$ and $k_{G(s)} \succeq_c^{AA} k_{\underline{x}}$ where $k_{\overline{x}}$ (resp. $k_{\underline{x}}$) is the constant act giving $\delta_{\overline{x}}$ (resp. $\delta_{\underline{x}}$) in all states, and $k_{F(s)}$ (resp. $k_{G(s)}$) is the constant act giving the lottery $F(s)$ (resp. $G(s)$) in all states

By Axiom 11 we know that $(k_{\overline{x}},\mathcal{P}) \succeq^{AA} (F,\mathcal{P}) \succeq^{AA} \left(k_{\underline{x}},\mathcal{P}\right)$, and $(k_{\overline{x}},\mathcal{Q}) \succeq^{AA} (G,\mathcal{Q}) \succeq^{AA} \left(k_{\underline{x}},\mathcal{Q}\right)$. Since $\succeq^{AA}$ is a continuous weak order, there exists λ such that $(F,\mathcal{P}) \sim^{AA} (\lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}},\mathcal{P})$. Similarly, there exists μ such that $(G,\mathcal{Q}) \sim^{AA} (\mu k_{\overline{x}} + (1-\mu)k_{\underline{x}},\mathcal{Q})$. Thus,

$$\begin{aligned}(F,\mathcal{P}) \succeq^{AA} (G,\mathcal{Q}) &\Leftrightarrow (\lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}},\mathcal{P}) \succeq^{AA} (\mu k_{\overline{x}} + (1-\mu)k_{\underline{x}},\mathcal{Q}) \\ &\Leftrightarrow \lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}} \succeq_c^{AA} \mu k_{\overline{x}} + (1-\mu)k_{\underline{x}}.\end{aligned}$$

But by Step 1, $(F,\mathcal{P}) \sim^{AA} (\lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}},\mathcal{P})$ implies that $\min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ F dp = U(\lambda\delta_{\overline{x}} + (1-\lambda)\delta_{\underline{x}})$. We also have that $\min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ G dp = U(\mu\delta_{\overline{x}} + (1-\mu)\delta_{\underline{x}})$ and $U(\lambda\delta_{\overline{x}} + (1-\lambda)\delta_{\underline{x}}) \geq U(\mu\delta_{\overline{x}} + (1-\mu)\delta_{\underline{x}})$, which implies that

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ F dp \geq \min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ G dp.$$

■

We now prove, through the next four lemma, that Conditions 1 to 4 in Theorem 1 are satisfied.

Lemma 2 $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ *satisfies Condition 1.*

Proof. Let $p^* \in \mathcal{F}(\mathcal{P})$ and suppose that $p^*(S(\mathcal{P})) = q \neq 1$. Consider $\overline{x}$ and $\underline{x}$ in $\mathcal{C}$ such that $U(k_{\overline{x}}) > U(k_{\underline{x}})$ and let F be defined by $F(s) = \delta_{\overline{x}}$ for all $s \in S(\mathcal{P})$, $F(s) = \delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$, and G by $G(s) = \delta_{\overline{x}}$ for all $s \in S$. Then,

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int U \circ F dp \leq \int U \circ F dp^* = qU(k_{\overline{x}}) + (1-q)U(k_{\underline{x}}) < U(k_{\overline{x}}) = \min_{p \in \mathcal{F}(\mathcal{P})} \int U \circ G dp.$$

Hence, $(G, \mathcal{P}) \succ^{AA} (F, \mathcal{P})$. But any $(f, \mathcal{Q})$ which is Savage equivalent to $(F, \mathcal{P})$ is also Savage equivalent to $(G, \mathcal{P})$. Therefore, $(F, \mathcal{P}) \sim^{AA} (G, \mathcal{P})$, a contradiction. We thus have proved that $S(\mathcal{F}(\mathcal{P})) \subseteq S(\mathcal{P})$.

Assume now that there exists $p^* \in \mathcal{F}(\mathcal{P})$ such that $p^* \notin co(\mathcal{P})$. Since $co(\mathcal{P})$ is a convex set, using a separation argument, we know that there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in co(\mathcal{P})} \int \phi dp$. Since $Supp(p^*) \subseteq S(\mathcal{P})$ and since $S(\mathcal{P})$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P})$, $(a\phi(s) + b) \in U(Y)$. Then, for all $s \in S(\mathcal{P})$ there exists $y(s) \in Y$ such that $U(y(s)) = a\phi(s) + b$. Define F by $F(s) = y(s)$ for all $s \in S(\mathcal{P})$, $F(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P})$, where $x \in \mathcal{C}$. Note that $\min_{p \in co(\mathcal{P})} \int (a\phi + b) dp \in U(Y)$ and thus there exists y^* such that $U(y^*) = \min_{p \in co(\mathcal{P})} \int (a\phi + b) dp$. Define G by $G(s) = y^*$ for all $s \in S$.

By construction,

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int U \circ F dp \leq \int U \circ F dp^* = \int (a\phi + b) dp^* < \min_{p \in co(\mathcal{P})} \int (a\phi + b) dp = U(y^*) = \min_{p \in \mathcal{F}(\mathcal{P})} \int U \circ G dp,$$

and thus $(F, \mathcal{P}) \prec^{AA} (G, \mathcal{P})$

However, observe that for all $p \in co(\mathcal{P})$,

$$\int U \circ F dp \geq \min_{p \in co(\mathcal{P})} \int U \circ F dp = \min_{p \in co(\mathcal{P})} \int (a\phi + b) dp = U(y^*) = \int U \circ G dp.$$

Therefore, for all $p \in \mathcal{P}$, $(F, \{p\}) \succeq^{AA} (G, \{p\})$. Note that we can find some adequately chosen f, g and $\Gamma \otimes (\pi_i)_{i \in I}$ such that $(f, \Gamma \otimes (\pi_i)_{i \in I})$ and $(g, \Gamma \otimes (\pi_i)_{i \in I})$ are Savage equivalent to $(F, \mathcal{P})$ and $(G, \mathcal{P})$, respectively. For all $q \in \Gamma$, there exists $p \in \mathcal{P}$, such that $(f, \{q\} \otimes (\pi_i)_{i \in I})$ and $(g, \{q\} \otimes (\pi_i)_{i \in I})$ are Savage equivalent respectively to $(F, \{p\})$ and $(G, \{p\})$. Therefore, for all $p \in \Gamma \otimes (\pi_i)_{i \in I}$, $(f, \{q\} \otimes (\pi_i)_{i \in I}) \succeq (g, \{q\} \otimes (\pi_i)_{i \in I})$, which implies by Axiom 7 that $(f, \Gamma \otimes (\pi_i)_{i \in I}) \succeq (g, \Gamma \otimes (\pi_i)_{i \in I})$, and thus $(F, \mathcal{P}) \succeq^{AA} (G, \mathcal{P})$, which yields a contradiction. ∎

Lemma 3 *$\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ satisfies Condition 2.*

Proof. Let $\mathcal{P} \in \mathbb{P}$ and φ be an onto mapping from S to S such that if $\left|\varphi^{-1}(s)\right| \geq 2$ either $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$, or for all $s' \in \varphi^{-1}(s)$, $p((s'|\varphi^{-1}(s)) = p'(s'|\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$.

We first prove that $\mathcal{F}(\mathcal{P}^\varphi) \subseteq (\mathcal{F}(\mathcal{P}))^\varphi$. Assume that there exists $p^* \in \mathcal{F}(\mathcal{P}^\varphi)$ such that $p^* \notin (\mathcal{F}(\mathcal{P}))^\varphi$. Since $\mathcal{F}(\mathcal{P})$ is a convex set, $(\mathcal{F}(\mathcal{P}))^\varphi$ is also convex. Hence, using a separation argument, we know that there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in (\mathcal{F}(\mathcal{P}))^\varphi} \int \phi dp$.

Since $S(\mathcal{P}^{\varphi})$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}^{\varphi})$, $(a\phi(s) + b) \in U(Y)$. Then, for all $s \in S(\mathcal{P}^{\varphi})$ there exists $y(s) \in Y$ such that $U(y(s)) = a\phi(s)+b$. Define F by $F(s) = y(s)$ for all $s \in S(\mathcal{P}^{\varphi})$, $F(s) = \delta_x$ for all $s \in S\backslash S(\mathcal{P}^{\varphi})$, where $x \in \mathcal{C}$. Define G by $G^{\varphi} = F$. Since for all $p \in \mathcal{F}(\mathcal{P})$, $\int U \circ G dp = \int U \circ G^{\varphi} dp^{\varphi}$, we have:

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ G dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int U \circ G^{\varphi} dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int U \circ F dp$$

Condition 1 implies that $\min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int U \circ F dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int (a\phi + b) dp$. But:

$$\min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int (a\phi + b) dp > \int (a\phi + b) dp^* \geq \min_{p\in\mathcal{F}(\mathcal{P}^{\varphi})} \int U \circ F dp,$$

and therefore $(G, \mathcal{P}) \succ^{AA} (F, \mathcal{P}^{\varphi})$.

Let $(g, \Gamma \otimes (\pi_i)_{i\in I})$ be a Savage equivalent to $(G, \mathcal{P})$ with a one to one mapping ψ from $S(\mathcal{P})$ to $\{1, .., n\}$ such that for all $s \in S(\mathcal{P})$, for all c, $\pi_{\psi(s)}\left(g^{-1}(c)\right) = \pi_{G(s)}(c)$ and $\Gamma = \left\{q \in \Delta(\{1, .., n\}) | \exists p \in \mathcal{P} \text{ s.t } \forall i \in \{1, .., n\}, q(i) = p\left(\psi^{-1}(i)\right)\right\}$.

Let us define a mapping χ from S to S such that, for all $s, s' \in S$,

- $\chi(s) = \chi(s')$ if $s \in Supp(\pi_i)$, $s' \in Supp(\pi_j)$, $\varphi\left(\psi^{-1}(i)\right) = \varphi\left(\psi^{-1}(j)\right)$ and $g(s) = g(s')$,
- $\chi(s) \neq \chi(s')$ otherwise.

First note that g is χ-measurable and the properties of φ entail that if $\left|\chi^{-1}(s)\right| \geq 2$, either $p(\chi^{-1}(s)) = p'(\chi^{-1}(s))$ for all $p, p' \in \Gamma \otimes (\pi_i)_{i\in I}$, or $p((s'|\chi^{-1}(s)) = p'(s'|\chi^{-1}(s))$ for all $s' \in \chi^{-1}(s)$ and $p, p' \in \Gamma \otimes (\pi_i)_{i\in I}$. By Axiom 2, $(g, \Gamma \otimes (\pi_i)_{i\in I}) \sim (g^{\chi}, (\Gamma \otimes (\pi_i)_{i\in I})^{\chi})$. One can check that $(g^{\chi}, (\Gamma \otimes (\pi_i)_{i\in I})^{\chi})$ is Savage equivalent to $(F, \mathcal{P}^{\varphi})$. Therefore, $(G, \mathcal{P}) \sim^{AA} (F, \mathcal{P}^{\varphi})$ which yields a contradiction. ■

Lemma 4 *$\mathcal{F} : \mathbb{P} \rightarrow \mathbb{P}_C$ satisfies Condition 3.*

Proof. First, note that it can be easily proved that Axiom 3 holds also for the preference relation $\succeq^{AA}$. Consider $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$ and $\alpha \in [0, 1]$.

Step 1. $\mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q}) \supseteq \alpha\mathcal{F}(\mathcal{P}) + (1 - \alpha)\mathcal{F}(\mathcal{Q})$

Suppose that there exist $p^* \in \mathcal{F}(\mathcal{P})$ and $q^* \in \mathcal{F}(\mathcal{Q})$ such that $r^* = \alpha p^* + (1 - \alpha)q^* \notin \mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})$. Since $\mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})$ is a convex set, using a separation argument, we know that there exists a function $\phi : S \rightarrow \mathbb{R}$ such that $\int \phi dr^* < \min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int \phi dp$. Since $S(\mathcal{P})$ and $S(\mathcal{Q})$ are finite sets, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$ there exists $y(s) \in Y$ such that $U(y(s)) = a\phi(s) + b$. Define F by $F(s) = y(s)$ for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $F(s) = \delta_x$ for all

$s \in S \setminus (S(\mathcal{P}) \cup S(\mathcal{Q}))$, where $x \in \mathcal{C}$. Since for all $p \in \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})$, $p\,(S(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})) = p\,(S(\mathcal{P}) \cup S(\mathcal{Q})) = 1$,

$$\begin{aligned}\min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int U \circ F dp &= \min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int (a\phi + b)dp \\ &> \int (a\phi+b)dr^* = \alpha \int U \circ F dp^* + (1-\alpha)\int U \circ F dq^* \end{aligned} \tag{1}$$

Since $\int U \circ F dp^* \in U(Y)$ and $\int U \circ F dq^* \in U(Y)$ there exist $y_1, y_2 \in Y$ such that $U(y_1) = \int U \circ F dp^*$ and $U(y_2) = \int U \circ F dq^*$. Let $\varphi : S \to S$ be a bijective mapping such that $\varphi(S(\mathcal{P}) \cap S(\mathcal{Q})) = \emptyset$. Define G by $G(s) = y_1$ for all $s \in \varphi(S(\mathcal{P}))$, $G(s) = y_2$ for all $s \in S(\mathcal{Q})$ and $G(s) = \delta_x$ for all $s \in S \setminus (S(\mathcal{P}) \cup \varphi(S(\mathcal{Q})))$, with $x \in \mathcal{C}$. We have:

$$\min_{p\in\mathcal{F}(\mathcal{P}^\varphi)} \int U \circ G dp = U(y_1) = \int U \circ F dp^* \geq \min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ F dp$$

and

$$\min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ G dp = U(y_2) = \int U \circ F dq^* \geq \min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ F dp$$

Therefore, $(G, \mathcal{P}^\varphi) \succeq^{AA} (F, \mathcal{P})$ and $(G, \mathcal{Q}) \succeq^{AA} (F, \mathcal{Q})$. Therefore, by Axiom 3,

$$(G, \alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}) \succeq^{AA} (F, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}). \tag{2}$$

On the other hand,

$$\alpha \int U \circ F dp^* + (1-\alpha)\int U \circ F dq^* = \alpha U(y_1) + (1-\alpha)U(y_2) = \min_{p\in\mathcal{F}(\alpha\mathcal{P}^\varphi+(1-\alpha)\mathcal{Q})} \int U \circ G dp,$$

where the last equality follows from Condition 1.

Therefore, equation (1) implies $(F, \alpha\mathcal{P}+(1-\alpha)\mathcal{Q}) \succ^{AA} (G, \alpha\mathcal{P}^\varphi+(1-\alpha)\mathcal{Q})$, which contradicts equation (2).

Step 2. Assume that $S(\mathcal{P}) \cap S(\mathcal{Q}) = \emptyset$. Then, $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \subseteq \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$.

Assume that there exists $r^* \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ such that $r^* \notin \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$. By Condition 1, there exist $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ such that $r^* = \alpha p^* + (1-\alpha)q^*$. Assume, for instance, that $p^* \notin \mathcal{F}(\mathcal{P})$. Since $\mathcal{F}(\mathcal{P})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in\mathcal{F}(\mathcal{P})} \int \phi dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers a, b with $a > 0$ such that $(a\phi(s) + b) \in U(Y)$ for all $s \in S(\mathcal{P})$. Then, for all $s \in S(\mathcal{P})$, there exists $y(s) \in Y$ such that $U(y(s)) = a\phi(s) + b$. There also exists $y^* \in Y$ such that $U(y^*) = \min_{p\in\mathcal{P}} \int a\phi + bdp$. Define F by $F(s) = y(s)$ for all $s \in S(\mathcal{P})$, $F(s) = y^*$ for all $s \in S \setminus S(\mathcal{P})$, and define G by $G(s) = y^*$ for all $s \in S$. Since Condition 1 applies, we have:

$$\begin{aligned}\min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ F dp &= \min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ F dp \\ \min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ G dp &= \min_{p\in\mathcal{F}(\mathcal{Q})} \int U \circ G dp = U(y^*)\end{aligned}$$

Thus, $(F,\mathcal{P}) \sim^{AA} (G,\mathcal{P}) \sim^{AA} (F,\mathcal{Q}) \sim^{AA} (G,\mathcal{Q}) \sim^{AA} (F_{S(\mathcal{P})}G,\mathcal{P}) \sim^{AA} (F_{S(\mathcal{P})}G,\mathcal{Q})$. Axiom 3 implies $(F_{S(\mathcal{P})}G, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim^{AA} (F, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ and $(F_{S(\mathcal{P})}G, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim^{AA} (G, \alpha\mathcal{P} + (1-\alpha)\mathcal{P})$, establishing that:

$$(F, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim^{AA} (G, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}). \tag{3}$$

Since G is a constant act, we have $\min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int U \circ G dp = U(y^*)$. Yet,

$$\begin{aligned}
\min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int U \circ F dp &\leq \int U \circ F dr^* = \alpha \int U \circ F dp^* + (1-\alpha) \int U \circ F dq^* \\
&= \alpha \int (a\phi + b) dp^* + (1-\alpha) U(y^*) \\
&< \alpha \min_{p\in\mathcal{F}(\mathcal{P})} \int (a\phi + b) dp + (1-\alpha) U(y^*) = U(y^*)
\end{aligned}$$

which contradicts equation (3).

Step 3. $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \subseteq \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$.

Assume that there exists $r^* \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ such that $r^* \notin \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$. By Condition 1, there exist $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ such that $r^* = \alpha p^* + (1-\alpha) q^*$. Since $\alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that

$$\begin{aligned}
\int \phi dr^* &= \alpha \int \phi dp^* + (1-\alpha) \int \phi dq^* \\
&< \min_{p\in\alpha\mathcal{F}(\mathcal{P})+(1-\alpha)\mathcal{F}(\mathcal{Q})} \int \phi dp \\
&= \alpha \min_{p\in\mathcal{F}(\mathcal{P})} \int \phi dp + (1-\alpha) \min_{p\in\mathcal{F}(\mathcal{Q})} \int \phi dp.
\end{aligned} \tag{4}$$

Since $S(\mathcal{P}) \cup S(\mathcal{Q})$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $(a\phi(s) + b) \in U(Y)$. Then, for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$ there exists $y(s) \in Y$ such that $U(y(s)) = a\phi(s) + b$. Let F be defined by $F(s) = y(s)$ for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, and $F(s) = \delta_x$, with $x \in \mathcal{C}$, for all $s \notin S(\mathcal{P}) \cup S(\mathcal{Q})$.

Let φ and ψ be two bijective mappings on S, such that: $\varphi(S(\mathcal{P})) \cap (S(\mathcal{P}) \cup S(\mathcal{Q}) \cup \psi(S(\mathcal{Q}))) = \emptyset$ and $\psi(S(\mathcal{Q})) \cap (S(\mathcal{P}) \cup S(\mathcal{Q}) \cup \varphi(S(\mathcal{P}))) = \emptyset$. Define G by $G(s) = F(\varphi^{-1}(s))$ if $s \in \varphi(S(\mathcal{P}))$, $G(s) = F(\psi^{-1}(s))$ if $s \in \psi(S(\mathcal{Q}))$, and $G(s) = \delta_x$, with $x \in \mathcal{C}$ otherwise. We have $(G, \mathcal{P}^\varphi) \sim^{AA} (F, \mathcal{P})$ since any $(f, \mathcal{P}')$ which is Savage equivalent to $(F, \mathcal{P})$ is also Savage equivalent to $(G, \mathcal{P}^\varphi)$. Similarly, $(G, \mathcal{Q}^\psi) \sim^{AA} (F, \mathcal{Q})$. Therefore, the Axiom 3 implies:

$$(F, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim^{AA} (G, \mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}^\psi). \tag{5}$$

On the other hand, since $S(\mathcal{P}^\varphi) \cap S(\mathcal{Q}^\psi) = \emptyset$, Steps 1 and 2 imply:

$$\mathcal{F}(\alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}^\psi) = \alpha\mathcal{F}(\mathcal{P}^\varphi) + (1-\alpha)\mathcal{F}(\mathcal{Q}^\psi) = \alpha(\mathcal{F}(\mathcal{P}))^\varphi + (1-\alpha)(\mathcal{F}(\mathcal{Q}))^\psi$$

where the last equality follows from Condition 2. Therefore:

$$\begin{aligned}
\min_{p\in\mathcal{F}(\alpha\mathcal{P}^{\varphi}+(1-\alpha)\mathcal{Q}^{\psi})}\int U\circ Gdp &= \min_{p\in\alpha\mathcal{F}(\mathcal{P}^{\varphi})+(1-\alpha)\mathcal{F}(\mathcal{Q})^{\psi}}\int U\circ Gdp \\
&= \alpha\min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}}\int U\circ Gdp+(1-\alpha)\min_{p\in(\mathcal{F}(\mathcal{Q}))^{\psi}}\int U\circ Gdp \\
&= \alpha\min_{p\in\mathcal{F}(\mathcal{P})}\int U\circ Fdp+(1-\alpha)\min_{p\in\mathcal{F}(\mathcal{Q})}\int U\circ Fdp \\
&> \alpha\int(a\phi+b)dp^{*}+(1-\alpha)\int(a\phi+b)dq^{*} \\
&= \alpha\int U\circ Fdp^{*}+(1-\alpha)\int U\circ Fdq^{*} \\
&\geq \min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})}\int U\circ Fdp,
\end{aligned}$$

where the strict inequality follows from equation (4). Therefore, $(G,\alpha\mathcal{P}^{\varphi}+(1-\alpha)\mathcal{Q}^{\psi})\succ^{AA}(F,\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})$, which contradicts equation (5). ■

Lemma 5 *$\mathcal{F}:\mathbb{P}\rightarrow\mathbb{P}_C$ satisfies Condition 4.*

Proof. First, note that it can be easily proved that Axiom 8 holds also for the preference relation $\succeq^{AA}$. Consider $\mathcal{P},\mathcal{Q}\in\mathbb{P}$ and $\alpha\in[0,1]$, such that $\mathcal{P}$ is conditionally more precise than $\mathcal{Q}$ with respect to the partition $(E_1,\ldots,E_n)$ of S.

Assume that $\mathcal{F}(\mathcal{P})\nsubseteq\mathcal{F}(\mathcal{Q})$. Then using a separation argument we can exhibit an act F such that $(F,\mathcal{Q})\succ^{AA}(F,\mathcal{P})$, which yields a contradiction with Axiom 8. ■

We now complete the proof of the sufficiency part of theorem 1.

Define $u:\mathcal{C}\rightarrow\mathbb{R}$ as follows: for all $x\in\mathcal{C}$, $u(x)=U(\delta_x)$. For all $f\in\mathcal{A}$, denote by $AA(f)$ the act in $\mathcal{H}$ such that $\forall s\in S$, $AA(f)(s)=\delta_{f(s)}$. For all $f\in\mathcal{A}$, $\mathcal{P}\in\mathbb{P}$, $(f,\mathcal{P})$ is Savage equivalent to $(AA(f),\mathcal{P})$. Therefore, for all $f,g\in\mathcal{A}$, and $\mathcal{P},\mathcal{Q}\in\mathbb{P}$,

$$\begin{aligned}
(f,\mathcal{P})\succeq(g,\mathcal{P}) &\Leftrightarrow (AA(f),\mathcal{P})\succeq^{AA}(AA(g),\mathcal{Q}) \\
&\Leftrightarrow \min_{p\in\mathcal{F}(\mathcal{P})}\int U\circ AA(f)dp\geq\min_{p\in\mathcal{F}(\mathcal{Q})}\int U\circ AA(g)dp \\
&\Leftrightarrow \min_{p\in\mathcal{F}(\mathcal{P})}\int u\circ fdp\geq\min_{p\in\mathcal{F}(\mathcal{Q})}\int u\circ gdp.
\end{aligned}$$

Therefore, we proved the existence of the multiple-prior representation.

It is clear that u is unique up to a positive linear transformation. Observe for instance that for any $y\in Y$ we can find a couple $(f,\{p\})\in\mathcal{A}\times\mathbb{P}$ which is a Savage equivalent. Therefore, u is constrained by the von Neumann-Morgenstern's uniqueness conditions.

The uniqueness of $\mathcal{F}(\mathcal{P})$ is also clear. Indeed, assume that $\mathcal{F}'$ is another mapping that can be used in the representation functional, and that $\mathcal{F}(\mathcal{P}) \neq \mathcal{F}'(\mathcal{P})$ for some $\mathcal{P}$. Then using a separation argument, we can find $F, G \in \mathcal{H}$ such that

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ F dp > \min_{p\in\mathcal{F}(\mathcal{P})} \int U \circ G dp,$$

while

$$\min_{p\in\mathcal{F}'(\mathcal{P})} \int U \circ F dp < \min_{p\in\mathcal{F}'(\mathcal{P})} \int U \circ G dp.$$

Abusing notation, we can find $f, g \in \mathcal{A}$, $\mathcal{P} \otimes (\pi_i)_{i\in I}$ and $\mathcal{P} \otimes (\pi_i')_{i\in I}$ such that $(f, \mathcal{P} \otimes (\pi_i)_{i\in I})$ and $(g, \mathcal{P} \otimes (\pi_i')_{i\in I})$ are Savage equivalent to $(F, \mathcal{P})$ and $(G, \mathcal{P})$, respectively. Condition 2 implies that $\mathcal{F}(\mathcal{P} \otimes (\pi_i)_{i\in I}) = \mathcal{F}(\mathcal{P}) \otimes (\pi_i)_{i\in I}$, $\mathcal{F}(\mathcal{P} \otimes (\pi_i')_{i\in I}) = \mathcal{F}(\mathcal{P}) \otimes (\pi_i')_{i\in I}$, $\mathcal{F}'(\mathcal{P} \otimes (\pi_i)_{i\in I}) = \mathcal{F}'(\mathcal{P}) \otimes (\pi_i)_{i\in I}$, $\mathcal{F}'(\mathcal{P} \otimes (\pi_i')_{i\in I}) = \mathcal{F}'(\mathcal{P}) \otimes (\pi_i')_{i\in I}$. We thus have:

$$\min_{p\in\mathcal{F}(\mathcal{P}\otimes(\pi_i)_{i\in I})} \int u \circ f dp > \min_{p\in\mathcal{F}(\mathcal{P}\otimes(\pi_i')_{i\in I})} \int u \circ g dp$$
$$\Leftrightarrow (f, \mathcal{P} \otimes (\pi_i)_{i\in I}) \succ (g, \mathcal{P} \otimes (\pi_i')_{i\in I}),$$

while

$$\min_{p\in\mathcal{F}'(\mathcal{P}\otimes(\pi_i)_{i\in I})} \int U \circ F dp < \min_{p\in\mathcal{F}'(\mathcal{P}\otimes(\pi_i')_{i\in I})} \int U \circ G dp$$
$$\Leftrightarrow (f, \mathcal{P} \otimes (\pi_i)_{i\in I}) \prec (g, \mathcal{P} \otimes (\pi_i')_{i\in I}),$$

a contradiction.

Finally, the necessity part of the Theorem is easily verified.

Proof of Theorem 2

$[(i) \Rightarrow (ii)]$

Let $\mathcal{P} \in \mathbb{P}$ and assume that $\mathcal{F}^a(\mathcal{P}) \not\subset \mathcal{F}^b(\mathcal{P})$, i.e., there exists $p^* \in \mathcal{F}^a(\mathcal{P})$ such that $p^* \notin \mathcal{F}^b(\mathcal{P})$. Using a separation argument, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \underset{p\in\mathcal{F}^b(\mathcal{P})}{min} \int \phi dp$. Note that we can choose by normalization u_a and u_b such that $u_a(\bar{x}) = u_b(\bar{x}) > u_a(\underline{x}) = u_b(\underline{x})$. Since $S(\mathcal{P})$ is a finite set, there exist numbers $k > 0$ and ℓ, such that for all $s \in S(\mathcal{P})$, $k\phi(s) + \ell \in [u_a(\underline{x}), u_a(\bar{x})]$. W.l.o.g, suppose that $S(\mathcal{P}) = \{1, .., n\}$. Let $\alpha_i = \frac{k\phi(i)+\ell-u_a(\underline{x})}{u_a(\bar{x})-u_a(\underline{x})}$.

Define:

$$\mathcal{Q} = \left\{ q \;\middle|\; \begin{array}{l} \exists p \in \mathcal{P} \text{ s.t } \forall s \in \{1, .., 2n\}, \\ q(s) = \alpha_{\frac{s+1}{2}} p\left(\frac{s+1}{2}\right) \text{ if } s \text{ is odd,} \\ q(s) = (1 - \alpha_{\frac{s}{2}}) p\left(\frac{s}{2}\right) \text{ if } s \text{ is even,} \\ \forall s > 2n,\; q(s) = 0 \end{array} \right\}.$$

Condition 2 implies that:

$$\mathcal{F}^a(\mathcal{Q}) = \left\{ q \left| \begin{array}{l} \exists p \in \mathcal{F}^a(\mathcal{P}) \text{ s.t } \forall s \in \{1,..,2n\}, \\ q(s) = \alpha_{\frac{s+1}{2}} p\left(\frac{s+1}{2}\right) \text{ if } s \text{ is odd,} \\ q(s) = (1-\alpha_{\frac{s}{2}}) p\left(\frac{s}{2}\right) \text{ if } s \text{ is even,} \\ \forall s > 2n,\ q(s) = 0 \end{array} \right. \right\},$$

$$\mathcal{F}^b(\mathcal{Q}) = \left\{ q \left| \begin{array}{l} \exists p \in \mathcal{F}^b(\mathcal{P}) \text{ s.t } \forall s \in \{1,..,2n\}, \\ q(s) = \alpha_{\frac{s+1}{2}} p\left(\frac{s+1}{2}\right) \text{ if } s \text{ is odd,} \\ q(s) = (1-\alpha_{\frac{s}{2}}) p\left(\frac{s}{2}\right) \text{ if } s \text{ is even,} \\ \forall s > 2n,\ q(s) = 0 \end{array} \right. \right\}.$$

Define q^* by $\forall s \in \{1,..,2n\}$, $q^*(s) = \alpha_{\frac{s+1}{2}} p\left(\frac{s+1}{2}\right)$ if s is odd, $q^*(s) = (1-\alpha_{\frac{s}{2}}) p\left(\frac{s}{2}\right)$ if s is even, and $q^*(s) = 0\ \forall s > 2n$. Therefore, $q^* \in \mathcal{F}^a(\mathcal{Q})$ while $q^* \notin \mathcal{F}^b(\mathcal{Q})$.

Let $E = \{s \in S |\ s \text{ is even}\}$. Observe that

$$\begin{aligned} \min_{p\in\mathcal{F}^b(\mathcal{Q})} \int u_b \circ (\bar{x}_E \underline{x})\, dp &= \min_{p\in\mathcal{F}^b(\mathcal{P})} \int \phi dp \\ &> \int \phi dp^* = \int u_a \circ (\bar{x}_E \underline{x})\, dq^* \geq \min_{p\in\mathcal{F}^a(\mathcal{Q})} \int u_a \circ (\bar{x}_E \underline{x})\, dp. \end{aligned}$$

Thus $(\bar{x}_E \underline{x}, \{q^*\}) \succeq_a (\bar{x}_E \underline{x}, \mathcal{Q})$ while $(\bar{x}_E \underline{x}, \{q^*\}) \prec_b (\bar{x}_E \underline{x}, \mathcal{Q})$ which yields a contradiction with the fact that $\succeq_b$ is more averse to imprecision than $\succeq_a$.

$[(ii) \Rightarrow (i)]$ Straightforward.

Proof of Theorem 4

$[(i) \Leftrightarrow (ii)]$

This equivalence was proved in Theorem 2.

$[(ii) \Rightarrow (iii)]$

Consider $\mathcal{P} \in \mathbb{SD}$, and $E \subset S$. Since $\pi_a^A(E,\mathcal{P}) = c_{\mathcal{P}}(E) - Min_{p\in\mathcal{F}^a(\mathcal{P})} p(E)$ and $\pi_b^A(E,\mathcal{P}) = c_{\mathcal{P}}(E) - Min_{p\in\mathcal{F}^b(\mathcal{P})} p(E)$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$ implies that $\pi_b^A(E,\mathcal{P}) \geq \pi_a^A(E,\mathcal{P})$.

$[(iii) \Rightarrow (i)]$

Consider prizes $\bar{x}$ and $\underline{x}$ in $\mathcal{C}$ such that both a and b strictly prefer $\bar{x}$ to $\underline{x}$, and let $\mathcal{P} \in \mathbb{SD}$, and $E \subset S$. For any p, for any agent $i = a, b$, $(\bar{x}_E \underline{x}, \{p\}) \succeq_i [\succ_i](\bar{x}_E \underline{x}, \mathcal{P})$ if, and only if, $\pi_i^A(E,\mathcal{P}) \geq [\succ] c_{\mathcal{P}}(E) - p(E)$. Therefore since $\pi_b^A(E,\mathcal{P}) \geq \pi_a^A(E,\mathcal{P})$, this implies that we have

$$(\bar{x}_E \underline{x}, \{p\}) \succeq_a [\succ_a](\bar{x}_E \underline{x}, \mathcal{P}) \Rightarrow (\bar{x}_E \underline{x}, \{p\}) \succeq_b [\succ_b](\bar{x}_E \underline{x}, \mathcal{P}),$$

which completes the proof that $\succeq_b$ is more averse to imprecision than $\succeq_a$.

Proof of Proposition 1

1. Necessity.

Assume that $\mathcal{P}, \mathcal{P}' \in \mathbb{SD}$ satisfy conditions (i) and (ii). We show that for any agent satisfying increasing absolute imprecision premium, $\mathcal{F}(\mathcal{P}') \subset \mathcal{F}(\mathcal{P})$. Consider an agent such that $\mathcal{F}(\mathcal{P}') \nsubseteq \mathcal{F}(\mathcal{P})$. Let $p^* \in \mathcal{F}(\mathcal{P}') \backslash \mathcal{F}(\mathcal{P})$. There exists a function $\phi : S \to \mathbb{R}$ such that $\underset{p\in\mathcal{F}(\mathcal{P}')}{min} \int \phi dp \leq \int \phi dp^* < \underset{p\in\mathcal{F}(\mathcal{P})}{min} \int \phi dp$.

Using the notation and definitions introduced in the proof of Theorem 2, we have that $(\bar{x}_E\underline{x}, \mathcal{Q}) \succ (\bar{x}_E\underline{x}, \{q^*\}) \succsim (\bar{x}_E\underline{x}, \mathcal{Q}')$. Therefore,

$$\pi^A(E, \mathcal{Q}) = c_{\mathcal{Q}}(E) - \min_{p\in\mathcal{F}(\mathcal{Q})} p(E) < c_{\mathcal{Q}}(E) - q^*(E),$$

while

$$\pi^A(E, \mathcal{Q}') = c_{\mathcal{Q}'}(E) - \min_{p\in\mathcal{F}(\mathcal{Q}')} p(E) \geq c_{\mathcal{Q}'}(E) - q^*(E).$$

Note that $c_{\mathcal{Q}}(E) = c_{\mathcal{Q}'}(E)$ and therefore

$$c_{\mathcal{Q}}(E) - q^*(E) = c_{\mathcal{Q}'}(E) - q^*(E),$$

which proves that $\pi^A(E, \mathcal{Q}) < \pi^A(E, \mathcal{Q}')$. Therefore, such an agent does not satisfy increasing absolute imprecision premium.

2. Sufficiency.

Given that a Bayesian decision maker has an increasing absolute imprecision premium, condition (i) must hold. Note also that an extremely imprecision averse decision maker, that is, for whom $\mathcal{F}(\mathcal{P}) = \mathcal{P}$ for all $\mathcal{P}$, also has increasing absolute imprecision premium. Therefore, condition (ii) must also hold.

Proof of Proposition 2

$[(i) \Rightarrow (ii)]$

Let $\mathcal{P} \in \mathbb{SD}$, and p be a boundary point p of $co(\mathcal{P})$. Define:

$$\overline{\theta} = Sup\left\{\theta' | \theta' \in [0,1] \text{ s.th. } \left(\theta' p + (1-\theta)c_{\mathcal{P}}\right) \in \mathcal{F}(\mathcal{P})\right\}.$$

Then $\overline{p} = \overline{\theta}p + (1-\overline{\theta})c_{\mathcal{P}}$ is a boundary point of $\mathcal{F}(\mathcal{P})$ since $\mathcal{F}(\mathcal{P})$ is closed. Since it is convex as well, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi d\overline{p} = \underset{p\in\mathcal{F}(\mathcal{P})}{min} \int \phi dp$.

Using the notation and definitions introduced in the proof of Theorem 2 in order to define $\bar{x}_E\underline{x}$, $q, \overline{q}$ and $\mathcal{Q}$, we have that $(\bar{x}_E\underline{x}, \{\overline{q}\}) \sim (\bar{x}_E\underline{x}, \mathcal{Q})$. Note that $\overline{q} = \overline{\theta}q + (1-\overline{\theta})c_{\mathcal{Q}}$. Thus

$$\pi^R(E, \mathcal{Q}) = \frac{c_{\mathcal{Q}}(E) - \overline{q}(E)}{c_{\mathcal{Q}}(E) - Min_{q\in\mathcal{Q}}q(E)} \leq \frac{c_{\mathcal{Q}}(E) - \overline{q}(E)}{c_{\mathcal{Q}}(E) - q(E)} = \overline{\theta}.$$

If $\theta > \overline{\theta}$ we get a contradiction with the fact that $\pi^R(E, \mathcal{P}^n) = \theta$. Therefore, for any boundary point p of $co(\mathcal{P})$, $\overline{\theta}\,(p) = Sup\,\{\theta' | \theta' \in [0,1] \text{ s.th. } (\theta' p + (1-\theta) c_{\mathcal{P}}) \in \mathcal{F}(\mathcal{P})\}$ is such that $\overline{\theta}\,(p) \geq \theta$. Let p^* be a boundary point of $co(\mathcal{P})$ such that $\overline{\theta}\,(p^*) \geq \overline{\theta}\,(p)$ for all boundary point p of $co(\mathcal{P})$. Then, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* = \underset{p \in \mathcal{P}}{min} \int \phi dp$. Define $\overline{p^*} = \overline{\theta}(p^*) p^* + (1 - \overline{\theta}(p^*)) c_{\mathcal{P}}$ and consider now $p' \in \mathcal{F}(\mathcal{P})$. There exists a boundary point p of $co(\mathcal{P})$ and $\theta' < \overline{\theta}(p)$ such that $p' = \theta' p + (1 - \theta') c_{\mathcal{P}}$.

Let us use again the notation and definition introduced in the proof of Theorem 2. Since $\int u \circ (\bar{x}_E \underline{x})\, dq^* \leq \int u \circ (\bar{x}_E \underline{x})\, dq$ and $\int u \circ (\bar{x}_E \underline{x})\, dq^* \leq \int u \circ (\bar{x}_E \underline{x})\, dc_{\mathcal{Q}}$, we have that $\int u \circ (\bar{x}_E \underline{x})\, d\overline{q^*} \leq \int u \circ (\bar{x}_E \underline{x})\, dq'$. Thus $\int u \circ (\bar{x}_E \underline{x})\, d\overline{q^*} = \underset{r \in \mathcal{F}(\mathcal{Q})}{min} \int u \circ (\bar{x}_E \underline{x})\, dr$ while $\int u \circ (\bar{x}_E \underline{x})\, dq^* = \underset{r \in \mathcal{Q}}{min} \int u \circ (\bar{x}_E \underline{x})\, dr$. Therefore

$$\pi^R(E, \mathcal{Q}) = \frac{c_{\mathcal{Q}}(E) - \overline{q^*}(E)}{c_{\mathcal{Q}}(E) - Min_{q \in \mathcal{Q}} q(E)} = \overline{\theta}\,(p^*)\,,$$

and thus $\overline{\theta}\,(p^*) = \theta$. Hence, for all boundary point p of $co(\mathcal{P})$, $\overline{\theta}\,(p) = \theta$ which proves that $\mathcal{F}(\mathcal{P}) = \theta \mathcal{P} + (1-\theta)\,\{c_{\mathcal{P}}\}$.

$[(ii) \Rightarrow (i)]$

Consider $\mathcal{P} \in \mathbb{SD}$ and $E \subset S$ such that $c_{\mathcal{P}}(E) \neq Min_{p \in \mathcal{P}} p(E)$. We have

$$\min_{p \in \mathcal{F}(\mathcal{P})} p(E) = \theta \min_{p \in \mathcal{P}} p(E) + (1-\theta) c_{\mathcal{P}}(E),$$

and therefore

$$\pi^R(E, \mathcal{P}) = \frac{c_{\mathcal{P}}(E) - Min_{p \in \mathcal{F}(\mathcal{P})} p(E)}{c_{\mathcal{P}}(E) - Min_{p \in \mathcal{P}} p(E)} = \theta.$$

$[(ii) \Leftrightarrow (iii)]$

The proof given in Gajdos, Tallon, and Vergnaud (2004) can be adapted in this framework.

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