Efficient Allocations under Ambiguous Model Uncertainty*

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Abstract

We investigate consequences of ambiguity on efficient allocations in an exchange economy. Ambiguity is embodied in the model uncertainty perceived by the consumers: they are unsure what would be the appropriate probability measure to apply to evaluate consumption and keep in consideration a set $\mathcal{P}$ of alternative probabilistic laws. Consumers are heterogeneously ambiguity averse with smooth

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ambiguity preferences and $\mathcal{P}$ is point identified, and the aggregate risk is ambiguous. Our analysis addresses, in particular, the full range of set-ups where under expected utility the efficient consumption sharing rule is a linear function of the aggregate endowment. We identify the systematic differences ambiguity aversion introduces to efficient sharing rules in these environments. We also characterize the representative consumer and use it to find implications of heterogeneity in ambiguity aversion for the pricing kernel. The pricing kernel is shown to be qualitatively different under heterogeneity and has the empirically compelling implication that the Sharpe ratio is counter-cyclical.

1 Introduction

We investigate consequences of ambiguity on efficient allocations in an exchange economy. The ambiguity we consider is embodied in the model uncertainty perceived by the consumers: they are unsure what would be the appropriate probability measure to apply to evaluate consumption contingent on a state space Ω and keep in consideration a set $\mathcal{P}$ of alternative probabilistic laws. We study the case where the typical consumer in the economy is ambiguity averse with smooth ambiguity preferences (Klibanoff et al. (2005)) and $\mathcal{P}$ is point identified, i.e., the true law can be recovered empirically from events in Ω.

Our particular aim is to bring to light the systematic difference that ambiguity aversion brings to the nature of efficient allocations. Our analysis addresses, in particular, the full range of set-ups where under expected utility the efficient consumption sharing rule is a linear function of the aggregate endowment. In these environments, if either aggregate risk is unambiguous or consumers are homogeneously ambiguity averse, ambiguity aversion does not change the nature of efficient sharing rules. Different from the literature which studies efficient allocations under ambiguity (see the references in the related literature section at the end of the introduction), we allow for the case where the aggregate risk is ambiguous and consumers are heterogeneously ambiguity averse. Given these allowances, we show, ambiguity aversion affects the efficient sharing rules in systematic ways.

Much of the recent work on the impact of ambiguity aversion in financial markets has been in macro-finance.[1] Indeed, arguably, this is the area where

[1]See, e.g., Hansen and Sargent (2010), Ju and Miao (2012), Drechsler (2013),

ambiguity aversion has been demonstrated to have more significant impact. These studies analyze economies with ambiguous aggregate risk in which the single consumer invests in the aggregate equity, with ambiguous dividends, and the price supporting the equilibrium decision determines equilibrium asset returns. These macro-finance studies, so far, miss a representative consumer foundation. We provide such a foundation and use it to show how heterogeneity in ambiguity attitudes affects the pricing kernel. We find this heterogeneity makes the pricing kernel qualitatively different, endowing it with the empirically compelling feature that the associated Hansen-Jagannathan bound, the market price of risk, is counter-cyclical.

The stochastic setting we are analyzing may be interpreted as one of structured model uncertainty. Each $P \in \mathcal{P}$ is a probabilistic forecast based on a structured theory, a model. Crucially, models are assumed to leave physical imprints: the true parameters and distinctive mechanisms driving a particular model can be identified by publicly observable events in Ω, and consumption may be made contingent on all observable events. Next, we look at some more concrete examples of the setting.[2]

Consider a farmer deciding on plans for her orchards over a 20-30 year planning horizon: e.g., what type of fruit trees to plant, what complementary investments to make. The decision depends on the climate forecast for the planning horizon, in particular the annual *distribution* of variables like rainfall, temperature, sunshine. Given the planning horizon, the desiderata for the investment decision is the forecast of the distribution rather than the actual realization of these variables in a particular season. However, due to climate change in the offing, the climate forecast, that is the forecast as to which *distribution* will realize, is far from confident: a set of possibilities can be identified along with a rough guess about the chance of any one of them being realized. The forecast will become confident and pinned down as understanding of scientific processes at play and relevant public policy decisions become evident through observations of technical parameters (e.g., dynamics of concentrations of various gases at various reaches of the atmosphere, dynamics of ocean currents and temperatures,etc) and policy decisions.[3] Once

Bidder and Dew-Becker (2016), Collin-Dufresne et al. (2016), Collard et al. (2018).

[2]In Remarks 1 and 2 we explain how our results robustly extend to a setting where models are partially (or, set) identified.

[3]Schlenker and Taylor (2021) documents the fact that futures market closely follows advances in the climate literature. Market expectations, as measured by futures prices when weather outcomes are unknown, have been changing at the same annual rate as

the set of observations reaches a stage that the forecast becomes firm, the observations can be regarded as an event which identifies the probability law used to evaluate the payoffs from the investment decision.[4]

A second example is of a stochastic environment commonly applied in macro-finance studies, which specifies the data generating process in the macroeconomy as a regime-switching process, with an consumer in the economy typically unsure what regime would apply in a forthcoming period. Cecchetti et al. (2000), Kandel and Stambaugh (1991), Mehra and Prescott (1985), Ju and Miao (2012), for example, specify aggregate growth as driven by a latent/hidden state process that follows a two-state Markov chain, identifying one regime with a boom and the other with recession. In the former, the growth distribution has a larger mean and a smaller variance. The modeling echoes Hansen (2007)'s suggestion that one should put econometricians and economic agents on comparable footings in terms of statistical knowledge. Econometricians are able to pin down accurate estimates of growth distribution parameters *conditional* on a regime but do not assume to know what the regime will be in the short/medium term. This uncertainty can be substantial for extended periods over the business cycle. Think of a firm deciding on a product line for the next several years: the decision depends crucially on the state of the market which, in turn, is determined by whether the macroeconomy is in a boom or a recession over the planning horizon. Note, the determination of whether the economy was (or, was not) in a recession in past periods is (eventually) made by a group of experts (e.g., NBER) based on observations of several variables from different sectors of economy, labor and financial markets, inventory levels etc. In this example, we may consider the event comprising of the various outcomes of these indicator variables that lead to the pronouncement identifying the (business cycle) state of the economy, or the announcement itself, as an event that identifies the probability law relevant to the firm's decision.

A third example is that of decision making in the face of a contagion engendered by a novel virus.[5] The decision maker might be a business choosing an action plan to best adapt to the prospect of the contagion. The question is how will the contagion progress? The probabilistic forecast of an epidemi-

temperature projections made from publicly available models.

[4]For an application of the smooth ambiguity model to analysis of policy to address climate change, see Barnett et al. (2022a).

[5]For an application of the smooth ambiguity model to analysis of policy to address COVID contagion, see Barnett et al. (2022b).

ological model is contingent on a host of assumptions ranging from values of parameters describing characteristics of the virus to assumptions about behavioral responses to policy and information, etc. Fits of the model with various historical episodes give reason to have confidence in the probabilistic forecast *conditional* on such parameters, public policy and mechanisms. However, the relative novelty of the virus does not allow a similarly confident judgement regarding the values of the conditioning variables till enough has been learnt about them, though decisions have to be made in advance of this learning.

A fourth example of observables identifying parameters underlying a relevant model uncertainty is the VIX (and related indices, e.g., VXN and VXD). Option prices observed in the market at a point of time can be used to infer the "implied volatility", the market's estimate of the volatility of the underlying stock price for the forthcoming period, conditional on the market's information (since the Black-Scholes-Merton option pricing formulae gives a one-to-one relationship between the implied volatility and option prices). The VIX, e.g., is an index of the impled volatility of 30-day options on the S&P 500 calculated from a range of calls and puts.

Under expected utility, the characterizing condition for efficient allocations is that individual consumption is comonotone with aggregate consumption. We show, if distributions on aggregate endowment induced by models are ordered by FOSD, the additional bit required to characterize efficiency in the smooth ambiguity economy is a property we call *Expected Utility-comonotonicity*. Since ambiguity averse consumers also care about smoothing welfare across models it seems natural that efficiency would require that consumers' expected utility move in the same direction across models, which is essentially what EU-comonotonicity entails. Our further characterization results restrict the parametric form of functions describing attitudes toward uncertainty but not the class of models.Efficient allocations in an expected utility economy with a common belief satisfy a linear sharing rule if and only if consumers' utility functions exhibit linear risk tolerance (LRT) with the same marginal risk tolerance. We characterize representative consumers for such economies showing consumers agree on the ranking of the models (i.e., probability laws in $\mathcal{P}$) and this common ranking arises endogenously, as a consequence of efficiency, without assumptions on $\mathcal{P}$. Ambiguity aversion is shown to affect the linear sharing rule only when it is heterogeneous across consumers. The rule is then adjusted by tilting one of the two parameters of the linear rule to (relatively) favor the more ambiguity averse consumers in worse ranked models, ensuring that the more ambiguity averse consumer has a smoother expected utility across models. In the case consumers' utility functions are in the constant risk tolerance class it is he intercept parameter (of the sharing rule) that adjusts, while for the non-zero marginal risk tolerance case it is the slope parameter. In the latter case, the most relatively ambiguity averse consumers get protected with extra share at the worst models, the "middling" relative ambiguity averse consumers get extra at the "middling" models and the least relatively ambiguity averse ones get compensated by extra shares at the best models. Also, when some consumers are ambiguity neutral, we find ambiguity is not entirely borne by such consumers, an interesting contrast to the expected utility case and demonstrating that the effect of ambiguity aversion is robust to the presence of expected utility consumers.

As the assortative matching between ambiguity aversion and worse ranked models in the efficient allocation may suggest, if consumers (with common non-zero marginal risk tolerance) are heterogeneously ambiguity averse then the representative consumer's relative ambiguity aversion is *decreasing*, not constant as assumed in common practice. We consider economies where aggregate growth is described by log-normal distributions, whose parameters are (ambiguously) uncertain. The modeling strategy follows the common modeling practice in macro-finance, where economies are assumed to be subject to different growth (distribution) regimes over the business cycle. It is shown that in such economies the decreasing ambiguity aversion of the representative consumer implies that the market price of risk varies more pronouncedly between states associated with worse models (think recessions) and states associated with more optimistic models: the market price of risk is higher in recessionary states and lower in good states. This is empirically compelling since the Sharpe ratio for the U.S. aggregate stock market is countercyclical and highly volatile. As Lettau and Ludvigson (2010) put it, "the data imply a 'Sharpe ratio variability puzzle' that remains to be explained". Leading consumption-based asset pricing models that can rationalize a time-varying price of risk to some extent work by effectively allowing individual consumers to behave with different degrees of risk aversion over the business cycle. In contrast, here the Sharpe ratio varies not because individual consumers change their ambiguity aversion over the business cycle but because of the systematic pattern by which resources are allocated that makes the economy as a whole behave as if it were more ambiguity averse in recessionary times.

Related literature. Efficient risk sharing in expected utility economies was first studied by Borch (1962), further refined for the LRT class of utility functions by Wilson (1968), Cass and Stiglitz (1970) and Hara et al. (2007) among others. Under ambiguity, Chateauneuf et al. (2000) extended the comonotonicity result obtained under expected utility to Choquet expected utility with common capacity. Billot et al. (2000), Rigotti et al. (2008) and Ghirardato and Siniscalchi (2018) further studied the case in which aggregate endowment is non risky and preferences are more general than Choquet expected utility preferences (including, for the two latter references, the smooth ambiguity model). Strzalecki and Werner (2011) and De Castro and Chayeauneuf (2011) characterized properties of efficient risk-sharing when the endowment is risky but not ambiguous. Beißner and Werner (2022) extends some of these results to cases where agents have possibly heterogeneous, non-convex ambiguity sensitive preferences. Wakai (2007) proves that, under LRT with common risk tolerance, a two-fund theorem holds for maxmin expected utility economy (and hence efficient allocations are comonotonic). To the best of our knowledge, no paper has studied risk-sharing with *ambiguous* endowments and *heterogeneous* ambiguity aversion. Gollier (2011) studied the effect of ambiguity aversion on the pricing kernel in an economy with a smooth ambiguity representative consumer but not how heterogeneity in ambiguity aversion affects the pricing kernel. Cvitanic et al. (2012) considered in a continuous time, geometric Brownian motion setup, an expected utility economy with heterogeneity in three dimensions: risk aversion, discount rates and beliefs. Heterogeneity in risk aversion alone, they show, is sufficient to give rise to counter-cyclical market price of risk (Corollary 4.2).

The paper is organized as follows. Section 2 introduces and explains the setting and some preliminary results about efficient risk and uncertainty sharing that apply generally to smooth ambiguity economies. Section 3.2 specializes the analysis to the widely studied class of LRT utility functions and provides a characterization of the efficient allocations as well as the derivation of a representative consumer for this class of utility functions. Section 4 derives implications for asset pricing via properties of the kernel pricing derived from the representative consumer. Proofs are gathered in an Appendix. An Online Appendix contains further material supporting the formal results.

2 Setting and preliminary results

2.1 Preferences and beliefs

In this (sub-)section we introduce and discuss the key assumptions on preferences and beliefs. We consider a pure exchange economy under uncertainty, captured by the state space Ω, assumed to be finite. There are finitely many consumers, $i = 1, \ldots, I$, and one good (money). Consumers have smooth ambiguity-averse preferences (Klibanoff et al. (2005)) with a *common* second order prior $\mu \in \Delta(\Delta(\Omega))$ whose support is denoted $\mathcal{P}$. The domain $\mathcal{P}$ is a subjective statistical frame adopted by the consumer as a guide for making decisions, and each measure $P \in \mathcal{P}$ corresponds to a possible law governing the states. The belief μ is a prior over the true law, by analogy with the framework of Bayesian statistics.

Consumer i evaluates contingent consumption $X_i : \Omega \to \mathbb{R}_+$ through the functional:

$$U_i(X_i) = \int_{\mathcal{P}} \phi_i\left(E^P u_i(X_i(\omega))\right) \mu(dP). \tag{1}$$

where E^P is the expectation operator with respect to $P \in \mathcal{P}$, $u_i : \mathbb{X}_i \to \mathbb{R}$ is the Bernoulli utility function, assumed continuously differentiable, strictly increasing and strictly concave on its domain of definition $\mathbb{X}_i \subset \mathbb{R}$ for all i. The function $\phi_i : \mathbb{R} \to \mathbb{R}$, assumed continuously differentiable, strictly increasing and concave for all i captures ambiguity attitudes. Consumer i is *strictly ambiguity averse* if $\phi_i'' < 0$ and *ambiguity neutral* if $\phi_i'' = 0$.

We further assume that the smooth ambiguity representation (1) is *identifiable* (Denti and Pomatto (2022)), in that the support of μ is itself identifiable from observations, i.e., there exists a *kernel* function $k : \Omega \to \mathcal{P}$ such that for all $P \in \mathcal{P}$, $P(\{\omega : k(\omega) = P\} = 1$. In terms of the statistics literature, essentially, the kernel k is a consistent estimator for $\mathcal{P}$, and we are invoking the common statistical assumption of $\mathcal{P}$ being point-identified. To ease exposition, we set $\Omega = S \times M$: a state of the world $\omega = (s, m)$ determines the realization of the profile of individual endowments via the map $X : S \to \mathbb{R}^I_{++}$ *and* the probabilistic model P_m, where $P_m = k(s, m)$. Hence, each P_m assigns probability one to the event $S \times \{m\}$.

The identifiable smooth representation was introduced and axiomatized by Cerreia-Vioglio et al. (2013) using the formalism of a Dynkin structure, taking $\mathcal{P}$ as a primitive. Denti and Pomatto (2022) provides an axiomatic foundation where $\mathcal{P}$ is endogenously revealed. The tuple $\left\{\Omega, 2^\Omega, \mathcal{P}\right\}$

may be seen to comprise a Dynkin space in which the sub-σ-algebra $\mathcal{G} = \sigma(S \times \{m\} : m \in M)$ plays a crucial role. As Cerreia-Vioglo, et al. put it: "In the theory of choice under uncertainty, ambiguity is described as 'lack of information' that prevents the agent from forming a unique probabilistic model. The Dynkin space structure allows one to formally describe this 'missing information' through $\mathcal{G}$." Indeed, if information $\mathcal{G}$ were received (i.e., an element of $\mathcal{G}$ becomes known) the consumer would form, by updating, a single probabilistic model of the world, P_m, corresponding to the revealed $m \in M$. The kernel function approach and the Dynkin structure formalism reflect the same idea: there is some information, i.e., potentially observable events, that allow the decision maker to resolve their model uncertainty about the true law governing the state of the world.[6] This idea is fundamental to the interpretation of our setting and analysis.

Given the consumers' preferences, ambiguity aversion manifests as a *revealed belief* about events $\mathcal{E}_m \equiv S \times \{m\}$ that match with a non-trivial interval of values around $\mu(P_m)$, rather than with the point value $\mu(P_m)$. This can be seen from the following standard probability matching exercise. Consider a bet on $\mathcal{E}_m$ paying c^* on the event and c_* off it, and a lottery ℓ^π which pays c^* with probability π and c_* with probability $1-\pi$, where $c^* > c_*$. For a strictly ambiguity averse consumer there will be an interval $[\underline{\pi}, \bar{\pi}]$, $\underline{\pi} < \mu(P_m) < \bar{\pi}$, such that for $\pi \in [\underline{\pi}, \bar{\pi}]$, the bet on $\mathcal{E}_m$ is less desirable than ℓ^π and the complementary bet (on $\neg\mathcal{E}_m$) is *also* less desirable than $\ell^{1-\pi}$. Furthermore, the interval is wider, the more ambiguity averse the preference.[7] Hence the consumer acts *as if* their belief that the model m is true is described by a probability interval: $\mathcal{E}_m$ is an *ambiguous* event (KMM, Section 4).[8] Thus, the two assumptions, that there is a common prior μ and that ambiguity aversion is heterogeneous, together imply that a consumer's revealed belief on models is described by a set of probabilities and that these sets are different across consumers with a non-empty intersection

[6]To set ideas, it might be helpful to refer back to the Introduction for examples of events which identify structured models.

[7]see Appendix A for calculations.

[8]Decision makers treating the prior probability (on parameters/models) as unreliable is the distinctive feature of the approach taken in the Robust Bayesian statistics literature. As Hodges and Lehman (1952) p. 396 put it, "On the one hand, one does frequently have a good idea as to the range of [the parameter], and as to which values in this range are more or less likely. On the other hand, such information cannot be expected to be either sufficiently precise or sufficiently reliable to justify complete trust in the Bayes approach."

(containing $\{\mu\}$). It is as if there is shared information about the likelihood of models but this information is acted upon with differing degrees of trust by heterogeneously ambiguity averse consumers.

Recall, $P_m \in \mathcal{P}$ and assigns probability 1 to $\mathcal{E}_m$. Abusing notation twice, we now write $P_m(s) = P_m(s, m)$ and then $P(s)$, $Q(s), \ldots$ instead of $P_m(s), P_{m'}(s), \ldots$ Denote by $X_i^P(s)$ the consumption by individual i in s under model P and write X_i^P for the vector $\left(X_i^P(1), \ldots, X_i^P(S)\right)$ and $X_i = (X_i^P)_{P \in \mathcal{P}}$. Consumer i's preferences are therefore described by

$$U_i\left(X_i\right) = \int_{\mathcal{P}} \phi_i\left(E^P u_i\left(X_i^P\right)\right) \mu(dP). \tag{2}$$

Importantly, the restriction to identifiable preferences ensures that making consumption contingent on s *and* P is meaningful. In particular, implementing efficiency in a decentralized economy would require that claims contingent on events in $\mathcal{G}$ are traded. For instance, taking the second example in the Introduction, a consumer may hedge against the growth distribution that comes with a recession by selling a claim on the complementary event, that the NBER *does not* announce a recession.

Remark 1 *Denti and Pomatto (2020) also provides an axiomatization for partially identifiable preferences, where $\mathcal{P}$ is not point-identified but only set-identified. The functional representing such preferences is*

$$U_i\left(X_i\right) = \int_{\mathcal{C}} \phi_i\left(\min_{P \in C} E^P u_i\left(X_i^C\right)\right) \mu(dC), \tag{3}$$

where $\mathcal{C}$ is the set of set-identified models. One can think of the collection of sets C partitioning $\mathcal{C}$. For each C, the decision maker is Maxmin Expected Utility à la Gilboa and Schmeidler (1989) with respect to models in C; he then aggregates these Maxmin expected utilities through the smooth ambiguity aggregator ϕ.

2.2 Efficient allocations

We say that X_i is *model-independent* if $X_i^P(s) = X_i^Q(s)$ for all $s \in S$ and all $P, Q \in \mathcal{P}$. If, furthermore, *distributions* of consumption levels are model-independent, i.e., $P[X_i^P(s) = z] = Q[X_i^Q(s) = z]$ for all s, all z and all P, Q, we say that X_i is *unambiguous*. We assume that the aggregate endowment, denoted $\bar{X}$, is model-independent that is, $\bar{X}^P(s) = \bar{X}^Q(s) = \bar{X}(s)$ for all P, Q and all s. A model affects the distribution over possible endowments but not the endowment itself. Note, this is not to say that the aggregate endowment is unambiguous. An allocation $(X_i)_{i\in I}$ is *feasible* if $X_i(s) \in \mathbb{X}_i$ for all i and s, and $\sum_{i\in I} X_i^P(s) = \bar{X}(s)$ for all s, P, or simply $\sum_{i\in I} X_i^P = \bar{X}$ for all P.

Definition 1 *Let $(X_i)_{i\in I}$ be a feasible allocation. We say that $(X_i)_{i\in I}$ is*

- efficient *if there is no feasible allocation $(Y_i)_{i\in I}$ s.th. $U_i(X_i) \le U_i(Y_i)$ for every i, with at least one strict inequality;*
- P-conditionally efficient, *for $P \in \mathcal{P}$, if the allocation $\left(X_i^P\right)_i$ is Pareto efficient under model P, that is, there is no feasible allocation $\left(Y_i^P\right)_i$ s.th. $E^P\left(u_i\left(X_i^P\right)\right) \le E^P\left(u_i\left(Y_i^P\right)\right)$ for every i, with at least one strict inequality;*
- conditionally efficient *if $(X_i^P)_{i\in I}$ is P-conditionally efficient for all P.*

The following utilitarian welfare (Negishi) maximization problem characterizes efficient allocations for suitable individual weights $\lambda_i \ge 0$:

$$\begin{array}{rcl} V(\bar{X}) \equiv & \max\limits_{\left(X_i^P\right)_{P\in\mathcal{P}, i\in I}} & \sum\limits_i \lambda_i U_i\left(\left(X_i^P\right)_{P\in\mathcal{P}}\right) \\ & \text{subject to} & \sum\limits_i X_i^P \le \bar{X} \quad \text{for all } P \in \mathcal{P}. \end{array} \tag{4}$$

Denote by V the *representative consumer*'s *utility*, which we define to be the value function of problem (4).[9] Conditional on a model P, we consider the maximization problem :

$$\begin{array}{rcl} V^P(\bar{X}^P) \equiv & \max\limits_{\left(X_i^P\right)_{i\in I}} & \sum\limits_i \lambda_i \phi_i\left(E^P u_i(X_i^P)\right) \\ & \text{subject to} & \sum\limits_i X_i^P \le \bar{X}^P. \end{array} \tag{5}$$

As the ambiguity attitudes ϕ_i are strictly increasing functions, equation (5) solves the social welfare problem for a *P-conditional economy*; an economy

[9] We allow for model-dependent aggregate endowment in problem (4), so that the value function $V((\bar{X}^P)_{P\in\mathcal{P}})$ is defined on all (possibly model-dependent) allocations.

without ambiguity in which consumers are expected utility maximizers with common beliefs P, i.e., a von Neumann-Morgenstren (vNM) economy. Denote the *P-conditional representative consumer's utility* by V^P, the value function of the problem (5).
Since μ is common and the $U_i(.)$s are additively separable across models,

$$\max_{(X_i)_{i\in I}} \sum_i \lambda_i U_i(X_i) = \int_{\mathcal{P}} \max_{\left(X_i^P\right)_{P\in\mathcal{P},i\in I}} \sum_{i\in I} \lambda_i \phi_i \left(E^P u_i \left(X_i^P\right)\right) \mu(dP) .$$

Given that the constraint in problem (4) is separable across models, we can determine the efficient allocation "model by model" as recorded in the following proposition.

Proposition 1 *The representative consumer's utility is a μ-average of* P-*conditional representative consumer's utilities:*

$$V\left(\left(\bar{X}^P\right)_P\right) = \sum_{P\in\mathcal{P}} \mu(P) V^P\left(\bar{X}^P\right).$$

The proposition implies that if $(X_i)_{i\in I}$ is an efficient allocation, then it is conditionally efficient.

Corollary 1

1. *If $(X_i)_{i\in I}$ is an interior efficient allocation, then for any fixed $P \in \mathcal{P}$, the allocation $(X_i^P)_{i\in I}$ is comonotone, that is $X_i^P(s) \leq X_i^P(s')$ if and only if $X_j^P(s) \leq X_j^P(s')$ for all i, j and s, s'.*

2. *If $(X_i^P)_{i\in I}$ is an interior efficient allocation in the P-conditional economy, then $(X_i^P)_{i\in I}$ is efficient in the Q-conditional economy as well, where $P, Q \in \mathcal{P}$.*

3. *If the aggregate endowment $\bar{X}$ is unambiguous and $(X_i^\star)_{i\in I} \in \Pi_{i\in I}\mathbb{X}_i^S$ is an interior efficient allocation in a P-contingent economy, then the replica $((X_i^\star)_{i\in I}, \ldots, (X_i^\star)_{i\in I}) \in \Pi_{i\in I}\mathbb{X}_i^{S|\mathcal{P}|}$ is an efficient, unambiguous, allocation.*

Part 3 is saying that if the aggregate endowment is unambiguous, there is no point in making the allocation depend on P; indeed, a P-conditional efficient allocation, when replicated across models, is also efficient overall.[10] Hence, taking all three parts together, if the endowment is unambiguous an efficient allocation is comonotone, just as under expected utility.

In the rest of the paper, we focus only on *interior* efficient allocations which allocate strictly positive amount to every consumer in every state. The first-order necessary and sufficient condition for a feasible interior allocation $(X_i)_{i\in I}$ to be efficient is that there is a $\psi : S \times \mathcal{P} \to \mathbb{R}_{++}$ such that for every i, there are weights $\lambda_i > 0$ such that,

$$\psi(s,P) = \lambda_i \phi_i' \left(E^P u_i\left(X_i^P\right)\right) u_i'\left(X_i^P(s)\right) \tag{6}$$

for every (s,P).

A P-conditional efficient allocation $(X_i^P)_{i\in I}$ is comonotone for each P, as Corollary 1 reminds us. This comonotonicity is, essentially, a consequence of risk aversion. Analogously, when consumers also care about smoothing welfare across models it seems natural that efficiency would require that consumers' welfare move in the same direction across models.

Definition 2 *An allocation $(X_i)_{i\in I}$ is* expected-utility-comonotone *if for every $i,j \in I$ and P,$Q \in \mathcal{P}$, $E^P u_i(X_i^P) \leq E^Q u_i(X_i^Q)$ if and only if $E^P u_j(X_j^P) \leq E^Q u_j(X_j^Q)$.*

The following proposition provides a sufficient condition for EU-comonotonicity to be satisfied by an efficient allocation: that the distributions on aggregate endowment induced by models are ordered by FOSD.

Proposition 2 *Let $(X_i)_{i\in I}$ be an efficient allocation. Let $P,Q \in \mathcal{P}$. Suppose that $P \circ \bar{X}^{-1}$ is first-order stochastically dominated by $Q \circ \bar{X}^{-1}$. Then $E^P u_i\left(X_i^P\right) \leq E^Q u_i\left(X_i^Q\right)$ for every i.*

The proposition does not require the set $\{P \circ \bar{X}^{-1} \mid P \in \mathcal{P}\}$ to be totally ordered by FOSD. It merely states that if two elements in this set are FOSD-ordered, then EU-comonotonicity is obtained between the two. If the set $\{P \circ \bar{X}^{-1} \mid P \in \mathcal{P}\}$ is totally ordered by FOSD, then EU-comonotonicity is obtained over the entire $\mathcal{P}$.

[10]Strzalecki and Werner (2011), corollary 6 provides a proof of this. One can also infer from that corollary that every efficient allocation is a replication of an efficient allocation of a P-conditional economy.

Proposition 10 in Online Appendix 1 provides a converse to Proposition 2 (assuming in addition that $\{P \circ \bar{X}^{-1} \mid P \in \mathcal{P}\}$ is totally ordered by FOSD). Under some technical conditions it shows that, if an allocation is conditionally efficient and satisfies EU-comonotonocity given a profile $(u_i)_{i \in I}$, then there exists a profile of concave and twice-differentiable $(\phi_i)_{i \in I}$ such that the allocation is efficient. Hence, essentially, under the FOSD assumption, an allocation is efficient in a smooth ambiguity economy with common beliefs if and only if it is EU-comonotone and conditionally efficient. Under the FOSD condition, the additional bit, on top of conditional efficiency, required to characterize efficiency in the smooth ambiguity economy is EU-comonotonicity. In the next section we characterize how the aggregate endowment may be shared to implement this additional requirement. We do this by restricting the parametric form of functions describing attitudes toward uncertainty but without restricting the class of models.

3 Efficient sharing rules in linear risk tolerance economies

We characterize the effect on efficient allocations of introducing (smooth) ambiguity aversion to those economies where, under expected utility, efficient risk sharing is achieved by a linear sharing rule. Proposition 16.13 in Magill and Quinzii (1996), based on the classic contributions by Borch (1962), Wilson (1968) and Cass and Stiglitz (1970), asserts that efficient allocations in an expected utility economy with a common belief satisfy a linear sharing rule if and only if consumers' utility functions exhibit linear risk tolerance (LRT) with the same marginal risk tolerance (see also Hara et al. (2007)). A consumer i's utility function u_i satisfies LRT with parameters (b_i, a_i) if

$$-\frac{u_i'(x)}{u_i''(x)} = a_i + b_i x, i = 1, ..., I \tag{7}$$

is defined on the domain $a_i + b_i x_i > 0$. Consumers have LRT utility function with *common cautiousness* or *same marginal risk tolerance* if $b_i = b$ for all i. The LRT class is also sometimes referred to the *HARA family* of Bernoulli utility functions. When $b > 0$ and $a_i = 0$ the risk tolerance is proportional to income or alternatively, the utility function exhibits constant relative risk aversion (CRRA with index $\frac{1}{b}$). For $b = 0$ risk tolerance and absolute risk aversion are constant (CARA with index $\frac{1}{a_i}$). Quadratic utility functions correspond to the case $b = -1$. The HARA family thus covers, pretty much, the entire gamut of utility functions considered in economics, finance and asset pricing. We begin by explaining how this family facilitates the analytical tractability of the problem of finding efficient allocations under smooth ambiguity.

3.1 A nested Negishi approach

Observe that the smooth ambiguity model can be written in two different, equivalent, ways: as an expected utility of expected utilities as in (2) or as an expected utility of certainty equivalents

$$U_i\left(X_i\right) = \int_{\mathcal{P}} v_i\left(u_i^{-1}\left(E^P u_i\left(X_i^P(.)\right)\right)\right)\mu(dP)$$

by setting $v_i \equiv \phi_i \circ u_i$. In what follows, we consider the subclass of economies where u_i is restricted to the HARA with common marginal risk tolerance class, which we know leads in vNM economies to a linear sharing rule.

Notice, in a vNM economy with belief P we may find the efficient allocations by solving, for each $s \in S$, the following program:

$$\begin{array}{rl} u(\bar{X}^P(s)) \equiv \max\limits_{\left(X_i^P(s)\right)_{i\in I}} & \sum\limits_{i\in I} \lambda_i u_i(X_i^P(s)) \\ \text{subject to} & \sum\limits_{i\in I} X_i^P(s) \leq \bar{X}^P(s). \end{array} \tag{8}$$

As is well-known,[11] when the u_i's exhibit HARA with common marginal linear risk tolerance the value function u does not depend on the λ_is. Importantly, u is independent of the common belief and s. The representative consumer in the vNM economy has expected utility preferences, represented by $E^P u(\bar{X}^P)$ with Bernoulli utility u. Let $c_i^P(X_i^P) = u_i^{-1}\left(E^P u_i(X_i^P)\right)$ and $c^P(\bar{X}) = u^{-1}\left(E^P u(\bar{X})\right)$ be the certainty equivalents of consumer i and the representative consumer, respectively, at an allocation $(X_i^P)_{P,i}$. The following lemma notes another property of these economies that is key to our analysis.

Lemma 1 *Let $(X_i^P)_{P,i}$ be a conditionally efficient allocation where u_i's exhibit HARA with common marginal risk tolerance. Let u be the representative*

[11]See, e.g., LeRoy and Werner (2014), section 16.8.

consumer's utility function in a P-conditional economy. Then, $\sum_{i\in I} c_i^P(X_i^P) = c^P(\bar{X})$ *for all P.*

Lemma 1 delivers additivity of the certainty equivalents at conditionally efficient allocations. By Proposition 1, an efficient allocation is also conditionally efficient. Hence, given Lemma 1, we may characterize an efficient allocation $(X_i)_{i\in I}$ in two steps. First, solve program (5) for P-conditional efficient allocations (the inner program), yielding aggregate P-certainty equivalents that can be then allocated across models by solving the following Negishi (outer) programs:

$$\begin{aligned} v(c) \equiv \max_{(c_i)_{i\in I}} \quad & \sum_{i\in I} \lambda_i v_i\left(c_i\right) \\ \text{subject to} \quad & \sum_{i\in I} c_i = c. \end{aligned} \tag{9}$$

If the v_i's satisfy LRT with common marginal risk tolerance, then the value function v is independent of the weights (λ_i).

Assuming that u_i's satisfy LRT with common marginal risk tolerance has allowed us to enormously simplify the original Negishi problem through this *nested* approach. After solving for aggregate P-certainty equivalents in the inner program, the outer program implements efficient sharing of P-contingent aggregate certainty equivalents in a way like solving for the efficient allocation in a vNM economy (program (8)): think of $\left(c_i^P\right)_i$ as an efficient allocation of the aggregate resource c^P in the "state P" across consumers with Bernoulli utility v_i. Notice, the efficient allocation problem in state P is independent of the problem in state P' (it also does not matter what the common belief about P is). Moreover, we know that in the vNM problem we will have an expected utility representative consumer whose utility function is the value function v. Let $v(c^P) = v(u^{-1}(E^P u(\bar{X})))$ and hence, setting $\phi = v \circ u^{-1}$, $V^P(\bar{X}) = v(c^P) = \phi(E^P u(\bar{X}))$, where $V^P(\bar{X})$ is as in Proposition 1. Also, as in the vNM economy problem, the allocation $(c_i^P)_{P\in\mathcal{P}, i\in I}$, solution of program (9) will be comonotone with respect to c^P. It then follows that the efficient allocation $(X_i)_{i\in I}$ is expected-utility comonotone. Hence, we obtain the following proposition:[12]

Proposition 3 *Let* $(X_i)_{i\in I}$ *be an efficient allocation and assume* $u_i s$ *are LRT with common marginal risk tolerance. Then,*

[12]Notice, neither the validity of the nested Negishi approach (9) nor Proposition 3 requires conditions on v_i (beyond concavity). All that is needed is that u_i's be of the HARA with common linear risk tolerance type.

1. *the representative consumer has a smooth ambiguity utility function, i.e., there exist u, ϕ and v such that V, defined in (4), satisfies:*

$$\forall \bar{X}, \quad V(\bar{X}) = \int_{\mathcal{P}} \phi\left(E^P u(\bar{X})\right) \mu(dP) \quad \textit{with} \quad \phi = v \circ u^{-1}. \tag{10}$$

   Furthermore, $\phi'' \leq 0$, and $\phi'' = 0$ if and only if $\phi_i'' = 0$ for all i.

2. *$(X_i)_{i \in I}$ is expected-utility comonotone.*[13]

Remarkably, EU-comonotonicity of efficient allocations, which means that all consumers rank models in the same way at such allocations, is achieved here without restricting the class of models. Recall, in Proposition 2 the class of models was exogenously restricted to ensure that the set of distributions on aggregate endowments were ordered by FOSD.

The nested Negishi approach shows how, in the class of economies we consider here, the problem of risk sharing and the problem of sharing ambiguous risk have a common analytical core –the key to the results we establish. The program also holds insights about an important property that is not common. In risk sharing, we know that if some consumers were risk neutral, they would bear all the risk. This is evident from the program (8) since u_i would be linear for these consumers. Now consider ambiguity sharing, as embedded in program (9):, v_i is now equal to $\phi_i \circ u_i$. Since ϕ_i is concave, v_i cannot be linear unless u_i is. Hence, ambiguity neutral (i.e., with linear ϕ_i) consumers do not take on all ambiguity because v_i is concave. Of course, if some consumers are both risk and ambiguity neutral, then v_i would indeed be linear and such consumers would take on all uncertainty. Thus, if there is a strictly ambiguity averse consumer in the economy, then the representative consumer is also strictly ambiguity averse.

Unlike ϕ_i, which is defined on the domain of (expected) utilities, v_i and u_i are defined on a common domain, the consumption space. In the sequel, we proceed to characterize efficient allocations by imposing the LRT restriction on both u_i and v_i.

[13]Note, EU-comonotonicity of an allocation is equivalent to the comonotonocity of the associated certainty equivalents.

3.2 Constant risk tolerance

We study in this sub-section an economy where u_i and v_i satisfy LRT with zero marginal risk tolerance.[14]

Assumption 1

(i) $\forall i \in I$, $u_i : (-\infty, \infty) \to \mathbb{R}$ *is LRT with parameters* $(0, \frac{1}{\alpha_i})$, $\alpha_i > 0$.

(ii) $\forall i \in I$, $v_i : (-\infty, \infty) \to \mathbb{R}$ *is LRT with parameters* $(0, \frac{1}{\gamma_i})$, $\gamma_i \geq \alpha_i$.

Let $\phi_i = v_i \circ u_i^{-1}$, so $\phi_i(t) \propto -(-t^{\gamma_i/\alpha_i})$. Hence, our economy consists of smooth ambiguity averse consumers with heterogeneous risk aversion and heterogeneous ambiguity aversion, parameterized by CARA Bernoulli utilities with risk aversion coefficient $\alpha_i > 0$ and by a power function with index $\frac{\gamma_i}{\alpha_i} \geq 1$, respectively. The following proposition shows how uncertainty is efficiently shared in this economy.

Proposition 4 *Let* $(X_i^P)_{P,i}$ *be an efficient allocation of an economy that satisfies Assumption 1. Let* $\alpha = \left(\sum_i \alpha_i^{-1}\right)^{-1}$ *and* $\gamma = \left(\sum_i \gamma_i^{-1}\right)^{-1}$. *Then,*

1. *For each* P*, there are constants* $(\tau_i^P)_{i \in I}$ *s.th.* $\sum_i \tau_i^P = 0$ *and* $X_i^P = (\alpha/\alpha_i)\bar{X} + \tau_i^P$ *for every* i.

2. *For every* i*, there is a function* $\tau_i : (-\infty, \infty) \to (-\infty, \infty)$ *and constants* κ_i *such that* $\tau_i(c) = \frac{\gamma}{\gamma_i}\left(1 - \frac{\gamma_i/\alpha_i}{\gamma/\alpha}\right) c + \kappa_i$ *with* $\sum_i \kappa_i = 0$ *and*

$$\tau_i^P = \tau_i(c^P) \tag{11}$$

   with $c^P = u^{-1}(E^P u(\bar{X}))$*, where* u*, the representative consumer's utility function, is CARA with absolute risk aversion coefficient* α.

3. *In the smooth ambiguity representative consumer's utility* (10), $\phi(t) \propto -(-t^{\gamma/\alpha})$ *and* $v = \phi \circ u$ *is CARA with parameter* γ.

[14] While this class of utility functions is usually not the one considered in the DSGE literature, it admits an easy representation for the efficient allocations and the representative consumer's utility function, while allowing for heterogeneity.

The proposition characterizes how the efficient allocation adjusts contingent on models and the uncertainty attitudes of the smooth ambiguity representative consumer. The smooth ambiguity representative consumer's ambiguity aversion and risk aversion are described respectively by the function $-(-t^{\gamma/\alpha})$ and $-\frac{1}{\alpha}\exp(-\alpha x)$ where $\alpha = \left(\sum_i \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_i \gamma_i^{-1}\right)^{-1}$. Note here, the value function v corresponding to the Negishi outer program (9) is independent of the weights (λ_i)'s since the v_i's satisfy LRT with common (zero) marginal risk tolerance.

The nature of the efficient allocation rule is as follows: as P varies, the allocation rule adjusts by varying the intercept term of the linear sharing rule, τ_i^P, a term denoting transfers that sum to zero across all the consumers. The function τ_i^P is itself linear in the aggregate certainty equivalent.

To understand what determines τ_i^P, it is useful to be able to compare the ambiguity aversion of the representative with that of consumer i. However, since the representative consumer and the individuals will typically not share the same risk preferences, we cannot apply the comparative notion of ambiguity aversion defined in Klibanoff et al. (2005). Instead, we appeal to Theorem 6 of Wang (2019) which allows a comparison irrespective of risk preferences and shows that a smooth ambiguity averse consumer 1 is more ambiguity averse than consumer 2 if ϕ_1 is a concave transform of ϕ_2 (see also Baillon et al. (2012) and Hara (2020) for a similar conclusion for the case under consideration). So, if consumer i is more (less) ambiguity averse than the representative consumer then the linear τ_i^P function has a negative (resp., positive) slope. Hence, consumers more ambiguity-averse than "average" are protected from the variability of the certainty equivalents of the aggregate consumption by making their model-contingent transfer move in opposite direction to changes of the model-contingent certainty equivalent.

How does the allocation vary between consumers *purely* on account of the difference in their ambiguity aversion? Consider a pair (i, j), such that i is more ambiguity averse than j but shares the same risk aversion. We will have, $\tau_i^P - \tau_i^Q < \tau_j^P - \tau_j^Q$, when P is a (strictly) higher ranked model than Q: the extra transfer a consumer gets, compared to another who is less ambiguity averse, decreases as we go to better models, thus ensuring that the more ambiguity averse consumer has a smoother *expected* utility across models. Figure (1) gives a graphical depiction showing how τ_i^P varies as a function of the representative consumer's certainty equivalent for two consumers in this economy as established in Proposition 4.

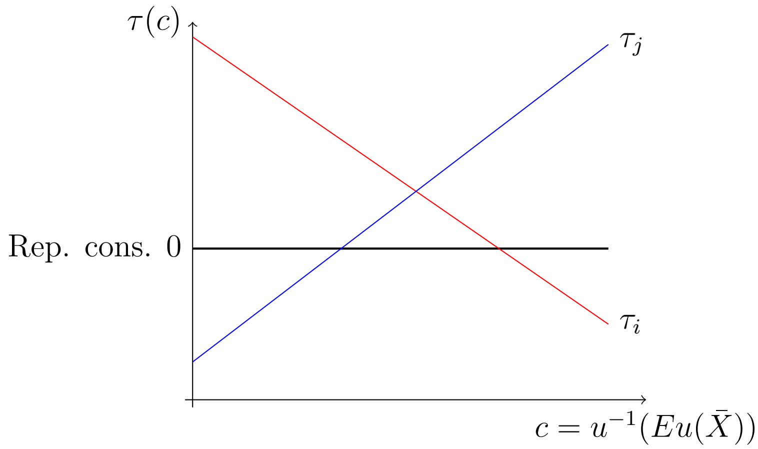

Figure 1: Constant risk tolerance case. The Figure shows the transfers as a function of the certainty equivalents for two consumers, i and j. Consumer i is more ambiguity averse than, and j is less ambiguity averse than, the representative consumer.

If ambiguity attitudes were homogeneous, i.e., $\gamma_i/\alpha_i = \gamma_j/\alpha_j$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected utility consumers: for all i, τ_i^P is independent of P.

3.3 Non-zero marginal risk tolerance

We now consider an economy where u_i and v_i satisfy LRT with non-zero marginal risk tolerance. Furthermore, the profile $(u_i)_{i\in I}$ is required to have a *common* marginal risk tolerance (though, not v_i). This still allows for heterogeneity in risk attitudes in $(u_i)_{i\in I}$ through the heterogeneity of the intercept term (a_i in (7)) in the LRT function.

Assumption 2

(i) $\forall i \in I$, $u_i : \mathbb{X}_i \to \mathbb{R}$ *is LRT with parameters* $(\frac{1}{\alpha}, -\frac{\zeta_i}{\alpha})$, $\alpha > 0$.

(ii) $\forall i \in I$, $v_i : \mathbb{X}_i \to \mathbb{R}$ *is LRT with parameters* $(\frac{1}{\gamma_i}, -\frac{\zeta_i}{\gamma_i})$ *with* $\gamma_i \geq \alpha$.

Note, between Assumptions 1 and 2, we cover the entire LRT with common positive cautiousness class of utility functions u_i. The functions u_i and

v_i that satisfy Assumption 2 can be represented by the *shifted power* family, i.e., for u_i:

$$u_i(x_i) = \begin{cases} \frac{\alpha}{1-\alpha}\left(\frac{x_i-\zeta_i}{\alpha}\right)^{1-\alpha} & \text{if} \quad \alpha \neq 1, \\ \ln(x_i - \zeta_i) & \text{otherwise,} \end{cases}$$

and so, $-\frac{u_i''(x)}{u_i'(x)} = \frac{\alpha}{x-\zeta_i}$.[15][16] Hence, the relative risk aversion coefficient, relative to effective consumption $z \equiv x - \zeta_i$, is α. Define the *relative ambiguity aversion* coefficient, *relative to effective consumption*, for consumer i by:[17]

$$-\frac{\phi_i''(u_i(z+\zeta_i))}{\phi_i'(u_i(z+\zeta_i))}u_i'(z+\zeta_i)z \equiv RAA_{\phi_i}(z).$$

Relative ambiguity aversion of consumer i is then equal to $\gamma_i - \alpha$, assumed positive for all $i \in I$. Proposition 5 shows how uncertainty is efficiently shared in this economy and characterizes the representative consumer's utility.

Proposition 5 *Let $(X_i^P)_{P\in\mathcal{P},i\in I}$ be an efficient allocation. Under Assumption 2:*

1. *For all $i \in I$ and models $P \in \mathcal{P}$ there exist constants $(\theta_i^P)_{P\in\mathcal{P},i\in I}$ s.th. $\sum_i \theta_i^P = 1$, $\theta_i^P > 0$ and*

$$X_i^P = \theta_i^P(\bar{X} - \zeta) + \zeta_i \quad \text{where} \quad \zeta = \sum_i \zeta_i.$$

The representative consumer's utility function in a P-conditional economy, u, is LRT with parameters $(\frac{1}{\alpha}, -\frac{\zeta}{\alpha})$.

[15] See Back (2017) section 1.3, for a discussion of shifted power utility.

[16] By concentrating on positive values of the common marginal risk tolerance, we exclude the quadratic case. However, it can be shown that our results in Proposition 5 hold for quadratic utility functions u_i's where $\zeta_i - X_i^P$ replaces $X_i^P - \zeta_i$. ζ_i is now a bliss point and $\zeta_i - X_i^P$ represents the shortfall from the bliss point.

[17] In Appendix B, we motivate this definition analogous to the way the coefficient of relative risk aversion is motivated. We take a quadratic approximation of ambiguity premium for an ambiguous prospect proportional to wealth and identify the part of the premium that is separate from belief aspects. We also obtain that:

$$-\frac{\phi_i''(u_i(z+\zeta_i))}{\phi_i'(u_i(z+\zeta_i))}u_i'(z+\zeta_i)z = -\frac{v_i''(z+\zeta_i)z}{v_i'(z+\zeta_i)} - \left(-\frac{u_i''(z+\zeta_i)z}{u_i'(z+\zeta_i)}\right). \tag{12}$$

2. *$\forall i \in I$, there is an infinitely differentiable function $\theta_i : (0,\infty) \to (0,1)$ s.th. $\forall z > 0$, $\sum_i \theta_i(z) = 1$, $\forall i, P$, $\theta_i^P = \theta_i\left(u^{-1}\left(E^P u(\bar{X})\right) - \zeta\right)$, where u, the representative consumer's utility function, is the same as that in P-conditional economies and*

   (a) *$\forall i, j$, and $\forall z > 0$, $\frac{d}{dz}\left(\frac{\theta_j(z)}{\theta_i(z)}\right) \gtreqless 0$ if and only if $\gamma_i \gtreqless \gamma_j$.*

   (b) *$\theta_i(z) \to 0$ as $z \to 0$ if $\gamma_i \neq \bar{\gamma} \equiv \max_{i\in I}\gamma_i$ and $\theta_i(z) \to 0$ as $z \to \infty$ if $\gamma_i \neq \underline{\gamma} \equiv \min_{i\in I}\gamma_i$.*

3. *Let $(\lambda_i)_{i\in I}$ be s.th. $(X_i^P)_{P,i}$ is a solution to (4) . Let $v(z+\zeta) \equiv \sum_i \lambda_i v_i(\theta_i(z)z + \zeta_i)$ $\forall z > 0$ and $\phi = v \circ u^{-1}$. Then, the representative consumer's relative ambiguity aversion $RAA_\phi(z)$ is equal to $b(z) - \alpha$ where $b(z) = -\frac{v''(z+\zeta)z}{v'(z+\zeta)}$ has the following properties:*

   (a) *$\forall i \in I$ and $z > 0$, $\frac{1}{b(z)} = \sum_i \theta_i(z)\frac{1}{\gamma_i}$. Furthermore, $b(z) \to \bar{\gamma}$ as $z \to 0$, and $b(z) \to \underline{\gamma}$ as $z \to \infty$.*

   (b) *If $\underline{\gamma} < \bar{\gamma}$, then $b'(z) < 0$ $\forall z > 0$.*

   (c) *$\forall i$ and $z > 0$, $1/\gamma_i \lesseqgtr 1/b(z)$ if and only if $\theta_i'(z) \lesseqgtr 0$.*

Part 1 notes that contingent on a model P the efficient allocation is given by a linear sharing rule whose slope coefficient, θ_i^P, may vary with P. Part 2 shows that the share θ_i^P is a function of the certainty equivalent of the aggregate consumption (in excess of ζ) and notes properties of this function Part 3 characterizes the representative consumer

Part 2(a) shows how the allocation varies between consumers because of the difference in their relative ambiguity aversion. Consider a pair (i, j), such that $\gamma_i > \gamma_j$, so i is more relatively ambiguity averse than j. Let P be a model with a higher aggregate certainty equivalent than Q. Then, the ratio of the share going to i to the share going to j, decreases as we go to better models. This ensures that the more relatively ambiguity averse consumer has a smoother expected utility across models. Part 2(b) says that the share contingent on the worst (best) model goes entirely to consumers with the highest (lowest) relative ambiguity aversion.

The representative consumer's risk and ambiguity attitude are described respectively, by u, LRT with parameters $(\frac{1}{\alpha}, -\frac{\zeta}{\alpha})$ where $\zeta = \sum_i \zeta_i$, and a concave ϕ. Note, here the value function v corresponding to the Negishi outer program (9) is not independent of the weights (λ_i)'s since v_i's may not have common marginal risk tolerance. However, we are able to derive crucial properties of the representative consumer's utility, with significant implications for the sharing rule and for asset prices, that hold irrespective of the particular specification of the welfare weights.

Parts 3(a) establishes that $b(x)$ is a weighted harmonic mean of γ_i, weighted by i's share of the aggregate certainty equivalent at an efficient allocation. Taken together with part 2(a), this implies that as we go to better models, i.e., models with higher aggregate certainty equivalents, the representative consumer's relative ambiguity aversion is influenced more by consumers with lower relative ambiguity aversion. Hence, as 3(b) notes, the relative ambiguity aversion of the representative consumer, embodied in $\phi(u(c^P))$, declines as models P get better. Remarkably, even though individual consumers have constant relative ambiguity aversion, the nature of efficient allocation is such that the representative consumer has decreasing relative ambiguity aversion, so long as there is heterogeneity in relative ambiguity aversion (by 3(a), if relative ambiguity aversion is homogeneous ($\underline{\gamma} = \bar{\gamma}$) the representative consumer's relative ambiguity aversion is constant).

Part 3(c) shows that how θ_i^P varies with P depends on how i's relative ambiguity aversion stands in relation to that of the representative consumer's. It implies that, as we move from worse to better models, a consumer whose relative ambiguity aversion is greater (smaller) than that of the representative consumer around $\xi = c^P$ will see their share decrease (resp. increase) for models with certainty equivalents marginally greater than ξ. This property allows us to get a complete qualitative characterization of the functions θ_i, graphed in Figure 2. Consider consumers with the largest relative ambiguity aversion in the economy. By part 3(a), their relative ambiguity aversion is greater than that of the representative consumer (at all c^P). By part 3(c), for these consumers i, θ_i will be negatively sloped, globally. Analogously, the consumers with the lowest relative ambiguity aversion in the economy will have a θ_I that is positively sloped globally. From 2(b), the most relatively ambiguity averse consumers get all of $\bar{X} - \zeta$ at the worst models. Therefore, at these models the representative consumer's relative ambiguity aversion is $\bar{\gamma} - \alpha$. Hence, as shown on Figure 2, by part 3(c), any consumer i with relative ambiguity aversion less than $\bar{\gamma} - \alpha$ will have their share increasing at least initially. Since the representative consumer has decreasing relative ambiguity aversion, we will reach a model, identified by $\hat{c}_i$ in Figure 2, where the representative consumer's relative ambiguity aversion falls below i's; hence, i's share is decreasing to the right of $\hat{c}_i$. For a consumer j relatively less ambiguity averse than i, the representative consumer's ambiguity aversion has to decrease further before j's share peaks. Hence, as shown in the figure, $\hat{c}_j$ is to the right of $\hat{c}_i$. Taken together, the most relatively ambiguity averse consumers get protected with extra share at the worst models, the "middling" relative ambiguity averse consumers get extra at the "middling" models and the least relatively ambiguity averse ones get compensated by extra shares at the best models. This kind of assortative matching suggests that, if better models are associated with better distributions of aggregate output, the relative ambiguity aversion of the economy as a whole will be counter-cyclical–a feature that has very significant implications for asset prices, as we show in the next section.

Finally, note that if relative ambiguity aversion were homogeneous, i.e., $\gamma_i - \alpha = \gamma_j - \alpha$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected utility consumers: for all i, θ_i is a constant function. Hence, efficient allocations under ambiguity aversion are different from those under expected utility *only* when there is heterogeneity in ambiguity attitudes.

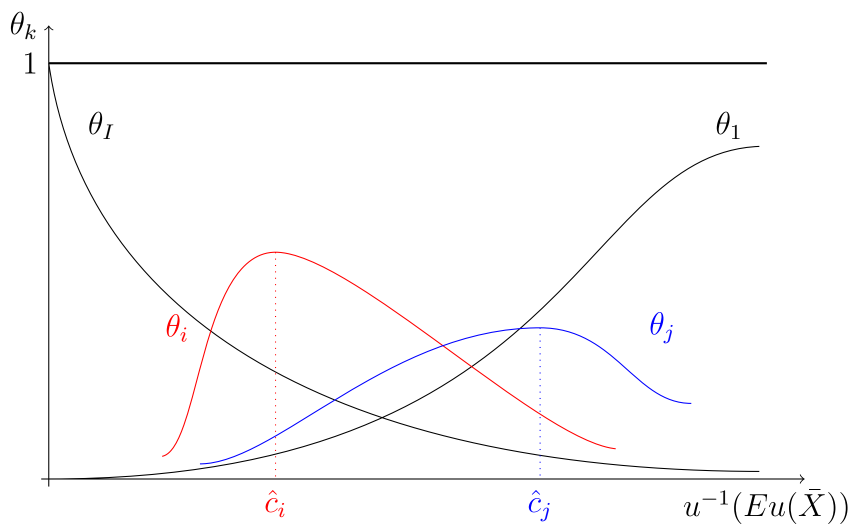

Figure 2: Comparing consumption shares θ_k under Assumption 2. Consumer I (resp. 1) is the most (resp. the least) relatively ambiguity averse. i is more relatively ambiguity averse than consumer j.

Remark 2 *In Remark 1, we defined a preference functional which accommodates partially identified models. Interestingly, the results in this section go through for this more general class of preferences. As Wakai (2007) shows, when consumers have Maxmin Expected Utility preferences with LRT Bernoulli utility functions, the Pareto optimal allocations are comonotone and there is a representative consumer with LRT utility function. Armed with this result, we can then "replace" each set-identified model by its worst prior according to the representative consumer and apply our analysis to this economy and obtain analogous results. In this sense, our analysis extends robustly to the case of partially identified models.*

4 The pricing kernel

Throughout this section, we place ourselves in economies where either Assumption 1 or 2 holds. Our objective here is to understand the effect ambiguity aversion has on asset prices at equilibria that implement efficient allocations in these economies. In terms of key results, we show, first, that ambiguity aversion increases the elasticity of the pricing kernel with respect to aggregate consumption and, secondly, that heterogeneity in ambiguity aversion makes elasticity a decreasing function of aggregate consumption levels. The former increases the Hansen-Jagannathan (H-J) bounds (Hansen and Jagannathan (1991)) and the market price of risk. The latter makes the H-J bounds, the market price of risk, vary counter-cyclically.

4.1 The pricing kernel and the Hansen-Jagannathan bound

The sharing rules characterized in Propositions 4 and 5 imply that decentralized implementation is possible if it is feasible for consumers to trade (1) assets which deliver a unit of the consumption good contingent on model P for all $P \in \mathcal{P}$; (2) assets which deliver a share of the market portfolio contingent on model P for all $P \in \mathcal{P}$; and their endowments lie in the span of these assets.[18] Nonetheless, in the following, we assume there is a com-

[18]It might be of interest to compare with analogous sufficient conditions in these economies when all consumers are ambiguity neutral. These are that consumers can trade, the risk free asset, the market portfolio, and their endowments lie in the span of these assets.

plete set of (s, P)-contingent claims or Arrow securities that can be traded in markets. The propositions also show how the smooth ambiguity representative consumer's risk and ambiguity attitudes depend on those of individual consumers'.

The advantage of being able to construct a representative consumer is that properties of the equilibrium vector of state prices in the decentralized economy can be deduced from properties of the representative consumer's utility functional, in particular its gradient vector at the aggregate endowment, without having to explicitly solve the equilibrium problem. The gradient vector determines the pricing kernel which summarizes all relevant asset pricing information of the decentralized equilibrium. We may thus deduce the effect of individual preferences on asset prices via our knowledge of how individual preferences affect the representative consumer's preferences.[19]

In this section we analyze a representative consumer economy where S is the set of possible realizations of the aggregate endowment; specifically, $S = \mathbb{R}$ and $\bar{X}(s) = \exp(s)$. The tuple (μ, ϕ, u, v) describes components of the representative consumer's smooth ambiguity preferences with $\phi \circ u = v$. The equilibrium price for an (s, P)-contingent commodity is proportional to marginal utility. That is, it is given by

$$\lambda\phi'\left(E^P u\left(\bar{X}(s)\right)\right) P(s) u'(\bar{X}(s))\mu(P),$$

for some $\lambda > 0$ that we can normalize to be equal to one. We wish to benchmark the implications for the pricing kernel against the findings in the macro-finance literature,[20] where the kernel is posited as a function of the aggregate resources in the economy (often proxied by the realized returns of the market portfolio). To that end, we focus on securities contingent on s rather than on (s, P). An s-contingent claim delivers a unit of the good if s occurs, no matter what P is, and hence its price is the sum over models of the price for (s, P)-contingent claims:

$$\int_{\mathcal{P}} \phi'\left(E^P u\left(\bar{X}(s)\right)\right) P(s) u'(\bar{X}(s))\mu(dP).$$

[19] As was noted, the representative consumer in an economy satisfying Assumption 2 will depend on the efficient allocation under consideration. However, the qualitative properties we demonstrate in the propositions in this section hold irrespective of which actual efficient allocation is considered.

[20] See, e.g., Campbell (2018), chapter 4 & 6, and Hens and Reichlin (2013).

Divide this price by the probability of s with respect to the reduced measure, $P^\star(s) = \int_{\mathcal{P}} Q(s)\mu(dQ)$, to obtain the *$s$-contingent pricing kernel*:[21]

$$s \mapsto \tilde{\pi}_{u,\phi}(s) \equiv \int_{\mathcal{P}} \frac{P(s)}{P^\star(s)} \phi'\left(E^P u\left(\bar{X}\right)\right) u'(\bar{X}(s))\mu(dP). \tag{13}$$

If ϕ' were constant, we have ambiguity neutrality and since $\int_{\mathcal{P}} \frac{P(s)}{P^\star(s)}\mu(dP) = 1$, we get that $\tilde{\pi}_{u,id}(s) = u'(\bar{X}(s))$ where the subscript *id* denotes the identity function. Notice, we may write the kernel[22] as

$$\tilde{\pi}_{u,\phi}(s) = u'(\bar{X}(s))h(s,\mu), \tag{14}$$

to identify the component h which encapsulates the effect of ambiguity aversion,

$$h(s,\mu) \equiv \int_{\mathcal{P}} \frac{P(s)}{P^\star(s)} \phi'\left(E^P u\left(\bar{X}\right)\right) \mu(dP). \tag{15}$$

From the s-contingent pricing kernel $\tilde{\pi}_{u,\phi}$, one can define another pricing kernel on endowments $\bar{X}$ using the fact $s = \ln(\bar{X})$. Let $\pi_{u,\phi}(x) = \tilde{\pi}_{u,\phi}(\ln(x))$; $\pi_{u,\phi}$ is the *pricing kernel in terms of aggregate consumption* levels or simply, the *pricing kernel.* Think of $y : \mathbb{R}_{++} \to \mathbb{R}$ as the payoff of an asset, a portfolio of assets, or a contingent claim written on aggregate endowment. From the endowment distribution P on S, one can define the endowment distribution $\check{P}$ on $\mathbb{R}_{++}$ via $\check{P}(x) = P(\bar{X}^{-1}(x)) = P(\ln(x))$. Then the price of y is equal to

$$E\left[\pi y\right] = \int_{\mathbb{R}_{++}} \pi_{u,\phi}(x) y\left(x\right) d\check{P}^\star(x),$$

and its expected gross (one plus) return, $E\left[R_y\right]$ is equal to $E\left[y\right]/E\left[\pi y\right]$. The *Sharpe ratio* of the return on y is the expectation of y's excess return divided by its standard deviation. The highest Sharpe ratio of all asset returns is referred to as the *market price of risk* (though, in the present context, "risk" is interpreted more broadly to include ambiguous uncertainty).

We now introduce two concepts, well-known in the finance literature, which we will apply in the sequel to articulate how the pricing kernel and

[21]If the representative consumer were both risk and ambiguity neutral, the price of this s-contingent claim would be equal to $P^\star(s)$.

[22]In all rigor, we should write $\tilde{\pi}_{u,\phi}(s,\mu)$, but we drop the reference to μ to save on notation when μ is a constant in the background.

hence, asset prices, are affected by ambiguity aversion. The first is the *elasticity of the pricing kernel* $\pi_{u,\phi}$ at x, given by $\varepsilon(x;\pi_{u,\phi}) \equiv -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)}$. This elasticity is a measure of the kernel's variability: it shows how responsive the kernel is to proportional changes in the state variable, the aggregate consumption/endowment. The second, the H-J bound of the pricing kernel $\pi_{u,\phi}$, is the ratio between the standard deviation of the pricing kernel and its expectation $\sigma\left[\pi_{u,\phi}\right]/E\left[\pi_{u,\phi}\right]$. In the case of complete markets, as in the economy we consider here, the H-J bound equals the market price of risk. Sharpe ratios, and hence the market price of risk are (in principle) deducible from returns data. Therefore, the bound provides test of an asset pricing theory and a way to compare theories: a higher bound shows a greater potential to accommodate market volatility and to explain larger equity premia and Sharpe ratios.

We proceed with our analysis under assumptions that impose different degrees of discipline on consumers' beliefs.

4.2 General case of MLR orderings

The first restriction on beliefs we consider is one where the family of models are ordered in terms of monotone likelihood ratio.

Assumption 3 *$\mathcal{P}$ is totally ordered according to the strict monotone likelihood ratio property: $\forall P, Q \in \mathcal{P}$, $P \neq Q$, $P(x)/Q(x)$ is strictly increasing or decreasing in x.*

Notice, $\mu(P|x) \equiv \frac{\mu(P)\check{P}(x)}{\int_{\mathcal{P}}\check{Q}(x)\mu(dQ)} = \frac{\mu(P)\check{P}(x)}{\check{P}^{\star}(x)}$ is the conditional probability of model P given a consumption level x. Hence, the assumption implies that this conditional probability has a monotone likelihood property: as x increases, the conditional probabilities are shifted from models with worse distributions to models with better distributions, that is, given the strict ordering on $\mathcal{P}$, $\mu(.|x)$ first-order stochastically dominates $\mu(.|x')$ whenever $x > x'$.[23]

The following proposition shows that the MLR property guarantees that the elasticity of the pricing kernel is greater when the representative consumer is strictly ambiguity averse than when he is of the expected utility type with the same Bernoulli utility function and the same beliefs, and furthermore, the H-J bound corresponding to the former type is larger.

[23]See the proof of Proposition 6 for a formal argument.

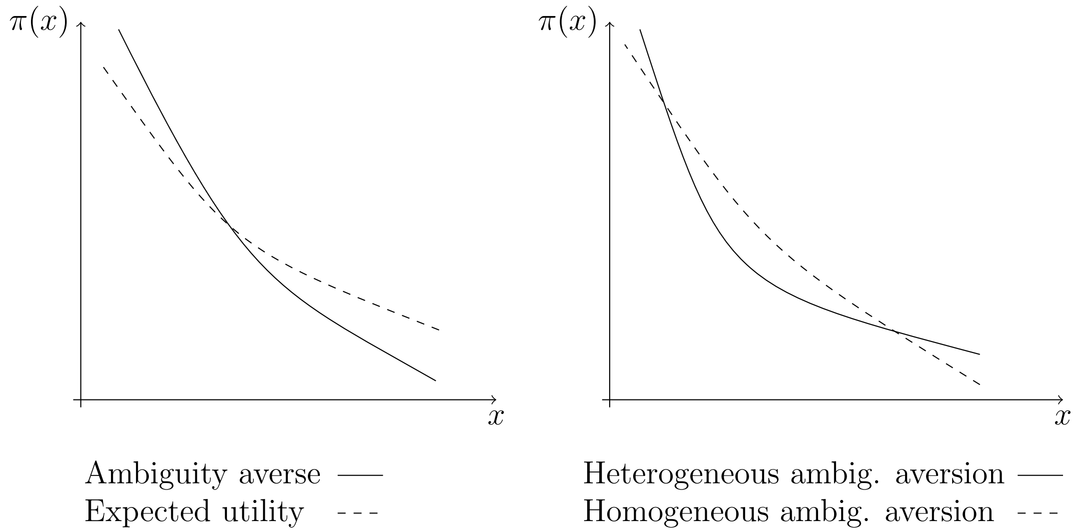

Figure 3: Pricing kernels

Proposition 6 *Suppose Assumption 3 holds and that $\phi''(.) < 0$. Then,*

1. *For every $x > 0$, $\varepsilon(x; \pi_{u,\phi}) \geq \varepsilon(x; \pi_{u,id})$, and every non-degenerate interval contains an x at which it holds as a strict inequality; and,*
2. $\dfrac{\sigma(\pi_{u,\phi})}{E(\pi_{u,\phi})} > \dfrac{\sigma(\pi_{u,id})}{E(\pi_{u,id})}$.

That $\phi''(.) < 0$ means that the representative consumer is strictly ambiguity averse. This would be true by Proposition 3 if there were at least one strictly ambiguity averse consumer in the economy. Let $w(.)$ be a Bernoulli utility such that $w'(x) = h(s, \mu)u'(x)$ where $s = \ln x$ and h is (as defined in equation (15)) a weighted average of $\phi'(E^P u(\bar{X}))$, weighted by the conditional probability $\mu(P|s) = \frac{\mu(P)P(s)}{P^\star(s)}$. Since $\mu(P|s)$ has an MLR property and ϕ' is strictly decreasing, h is strictly decreasing in s. Hence, w is a concave transform of u: ambiguity aversion reinforces the effect of risk aversion on the pricing kernel, a point observed earlier in Gollier (2011) and in the macro-finance literature. We further prove that the elasticity of the pricing kernel under ambiguity aversion is greater which, in turn, implies the relation between H-J bounds in the result. The relation between the kernels is depicted in the left panel of Figure 3.

4.3 The Gaussian case

We consider economies where Assumption 2 holds with $\zeta_i = 0$, and focus on how properties of the kernel pricing and thus H-J bounds differ across two such economies: one with homogeneous relative ambiguity aversion and the other with heterogeneous relative ambiguity aversion. We strengthen Assumption 3 to a Gaussian environment, which helps in obtaining an analytical characterization of the difference to the pricing kernel made by heterogeneity of ambiguity aversion. Now, the support of second order beliefs, $\mathcal{P}$, is the set of normal distributions parameterized by their means m and with common variance σ^2. Thus one may associate a model P with the mean m of a normal distribution.

Assumption 4

1. *$\mathcal{P}$ is the set of probability distributions $\mathcal{N}(m, \sigma^2)$, for every $m \in \mathbb{R}$;*
2. *The prior on the parameterized models $m \in \mathbb{R}$ is $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$;*

It is common in the macro-finance literature to assume that the aggregate consumption is log-normal, and part 1 of the above assumption follows that practice: s being normally distributed, $x = \exp(s)$ is log-normal. The parametrization in part 2 should make it more straightforward to calibrate consumers' belief (over models) to center on models that are empirically more plausible.

Proposition 7 *Suppose Assumption 4 holds, that u is CRRA and that $\phi''(.) < 0$.*

1. *If RRA_ϕ is constant and equal to $\gamma - \alpha$ (that is, v exhibits CRRA with coefficient γ), then:*

$$\varepsilon(x, \pi_{u;\phi}) = \frac{\sigma^2}{\sigma^2 + \hat{\sigma}^2}\alpha + \frac{\hat{\sigma}^2}{\sigma^2 + \hat{\sigma}^2}\gamma$$

   for every $x > 0$.

2. *If RRA'_ϕ is strictly negative (that is, the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x), then $\varepsilon(x; \pi_{u,\phi})$ is strictly decreasing in x.*

Hence, the elasticity pricing kernel is independent of, or strictly decreasing in, aggregate consumption if representative consumer's relative ambiguity aversion is constant, or strictly decreasing, respectively. The key argument in proving the proposition proceeds by establishing the direction of change of $h'(s,\mu)/h(s,\mu)$ as $s = \ln(x)$ increases.[24]

Consider an homogeneously ambiguity averse economy ($\underline{\gamma} = \bar{\gamma}$) and an heterogeneously ambiguity averse economy ($\underline{\gamma} < \bar{\gamma}$), underlying the representative consumer specifications in parts 1 and 2, respectively, of Proposition 7. Furthermore, suppose relative ambiguity aversion in the homogeneous economy lies strictly between $\bar{\gamma} - \alpha$ and $\underline{\gamma} - \alpha$, the maximum and minimum relative ambiguity aversion in the heterogeneously ambiguity averse economy. Normalize, so that two economies have the same risk free rate; hence, the pricing kernel from neither economy lies entirely above the other. Then, Proposition 7 implies that the kernels have exactly two points of intersection and are as in the right panel of Figure 3.[25]

Thus, the pricing kernels in the two economies are *qualitatively* different. For low values of (aggregate) consumption the kernel of the heterogeneously ambiguity averse economy will be more elastic and so, steeper than that of the homogeneously ambiguity averse economy. For high values of aggregate consumption the relation between the slopes of the two kernels is the other way round. Seemingly, under heterogeneous ambiguity aversion, unlike under homogenous ambiguity aversion, the pricing kernel is more variable in "bad times" compared to "good times". This suggests that the heterogeneity may cause the H-J bound and hence the market price of risk to be countercyclical.[26] The next proposition articulates this point more precisely in terms of FOSD shifts of the second order beliefs.

To state the next proposition, we write $\pi_{u,\phi}(x, \hat{m})$ instead of $\pi_{u,\phi}(x)$, and $E_{\hat{m}}$ and $\sigma_{\hat{m}}$ instead of E and σ, to make explicit the dependence of the pricing kernel, and the operators used to formulate the H-J bound, respectively, on $\hat{m}$, the mean of the second order beliefs. We formalize beliefs (in anticipation) of good times and bad times through shifts in $\hat{m}$.

[24]This requires applying a strict version of the Ahlswede-Daykin inequality by adapting techniques in Karlin and Rinott (1980) (see Appendix E).

[25]The details of the argument can be found in Proposition 9 in Appendix D.

[26]Rosenberg and Engle (2002) obtains a measure of "empirical risk aversion" using the risk aversion implied by the pricing kernel they estimate. They show that this risk aversion varies counter-cyclically, supporting earlier results of Fama and French (1989) who showed that risk premia are negatively correlated with the business cycle.

Proposition 8 *Suppose Assumption 4 holds, u is CRRA and $\phi''(.) < 0$. Then,*

1. *If RAA_ϕ is constant, then $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}$ is constant and equal to*

$$\left(\exp\left(\left(\frac{\sigma^2}{\sigma^2+\hat{\sigma}^2}\alpha+\frac{\hat{\sigma}^2}{\sigma^2+\hat{\sigma}^2}\gamma\right)^2\left(\sigma^2+\hat{\sigma}^2\right)\right)-1\right)^{1/2}$$

2. *If RAA_ϕ is strictly decreasing then $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}$ is strictly decreasing in $\hat{m}$.*

The formal argument for part 2 rests on showing that $h(s,\hat{m})$ is strictly log-supermodular[27] by applying a strict version of the Ahlswede-Daykin inequality by adapting techniques in Karlin and Rinott (1980) (see Appendix E).

To see the implications of the result for asset pricing, specifically for the H-J bound, consider two scenarios, 1 and 2, corresponding to $\hat{m}_i$, $i = 1, 2$, with $\hat{m}_2 > \hat{m}_1$. We interpret scenario 2 as one where a typical consumer views the immediate future as a boom relative to scenario 1. Under this interpretation, the result shows that the H-J bound, the market price of risk, is *counter-cyclical* if there is heterogeneity in the relative ambiguity aversion of the individual consumers of the underlying economy. Whereas, it is *constant* across the cycle if relative ambiguity aversion is homogeneous. Heterogeneity is only to be expected. But the macro-finance literature with ambiguity aversion, so far, did not have a foundation for the representative consumer, and could not take into account heterogeneity in ambiguity aversion. It has modeled representative consumer as having constant relative or constant absolute ambiguity aversion.[28] Our analysis, by adding foundation to the representative consumer, has shown that under the natural presumption of heterogeneity, ambiguity aversion has a more profound and qualitatively different implication for asset prices. In this case the market price of risk is

[27] $f : \mathbb{R}^N \to \mathbb{R}_{++}$ is strictly log-supermodular if $\ln f$ is strictly supermodular in the sense of Topkis (1998), Section 2.6.1.

[28] See, e.g., Ju and Miao (2012), Collard et al. (2018), Gallant et al. (2019),Thimme and Volkert (2015).

not just higher but also varies countercyclically. It so varies, not because individual consumers change their ambiguity aversion over the business cycle, but because an efficient market allocates resources in a way that makes the economy as a whole *behave as if* it were more relatively ambiguity averse in recessionary times.

4.4 Models ranked according to SOSD

Finally, if we were to relax the assumption of FOSD of models and allow, in particular, that the volatility associated with the model with the lower mean is greater, we show, by numerical examples, that the pricing kernel might have an upward sloping segment. Ambiguity aversion needs to be large enough to offset the decreasing pricing kernel in the expected utility case.

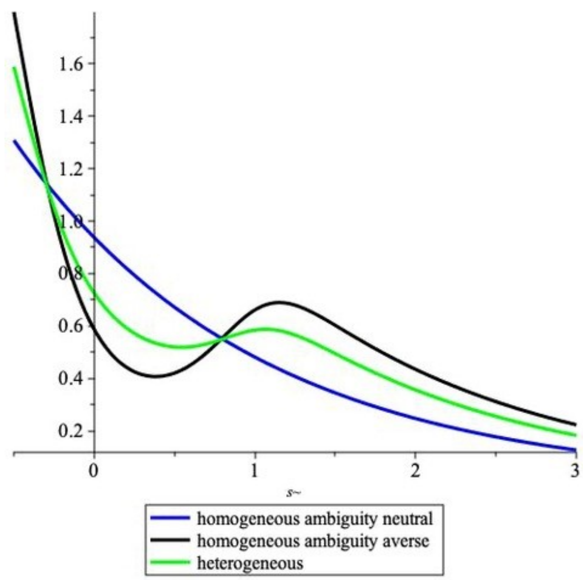

Figure 4: Kernel pricing with upward sloping parts

In the example, model uncertainty is generated by the presence of two regimes (see details in Appendix E), one in which the endowment is lognormally distributed with high mean and low variance ("booms") and one with low mean and high variance ("recessions"). This specification has the feature that for extremely high realizations of s, one is led to believe the bad regime more plausible, because of the different variances.[29] Figure 4 reports the benchmark expected utility (with CRRA utility function) pricing kernel, which is decreasing. It also reports the cases in which the representative consumer has the same CRRA utility function u and a CRRA v with high ambiguity aversion. It finally reports the case of an economy with one consumer with high ambiguity aversion and one with low ambiguity aversion, for which the kernel is in between the ones obtained for an economy with homogeneous ambiguity aversion. As can be seen, for high ambiguity aversion, the kernel has a positive slope for intermediate values of the endowments. This feature is qualitatively similar to the empirical findings of Rosenberg and Engle (2002).

This kind of non-monotonicity, also identified in Gollier (2011), can thus be used to provide a potential explanation to the so-called *pricing kernel puzzle,* discussed in Hens and Reichlin (2013) and Cuesdeanu and Jackwerth (2018). In a standard risk aversion expected utility economy the pricing kernel is downward sloping, as we have seen. However, there is strong empirical evidence that this decreasing relation is violated in reality: there is an interval usually in the area of zero returns where the pricing kernel is increasing. To see an intuition for this result, let us return to the paragraph following Proposition 6. There, the conditional probability $\mu(.|s)$ has a monotone likelihood property, thereby making the downward-sloping pricing kernel even more (not less) downward sloping. Here, in contrast, there is an interval of values of s (or x), at the lower end of the scale, where a greater value of s (or x) increases the conditional probability on models with higher variance, and therefore, with lower expected utility. The function h will increase on this interval and therefore, possibly, the kernel. Thus, the existence of uncertainty averse consumers fearing the occurrence of a model with high variance gives a potential for the resolution of the pricing kernel puzzle. Note, the higher the ambiguity aversion, the greater will be the effect because it will increase the "slant" of the conditional probability towards the worse models.

5 Concluding remarks

We studied, in this paper, the implications of efficient ambiguity sharing in a set up where model uncertainty is identifiable, embedded in the state space,

[29] VIX futures are traded and maybe used as claims contingent on volatility of aggregate equity.

aggregate endowment is ambiguous and consumers are heterogeneously ambiguity averse. When models rank aggregate endowments according to FOSD, we showed that efficiency was characterized by expected-utility comonotonicity: in better models, all consumers should get higher expected utility. This property still holds when models are not restricted but consumers' Bernoulli utilities satisfy LRT with common marginal risk tolerance. This class of expected utility preferences under risk has been extensively studied and ensures the existence of smooth ambiguity representative consumer,whose utility function can be used to "rank" models. In the case consumers' utilities satisfy LRT with common, non-zero, risk tolerance, the share that a consumer gets varies with models and is "single peaked". These peaks are ordered according to the relatively ambiguity aversion of the consumers, with the peak for the more relatively ambiguity averse consumer occurring at a worse model. Also, the effect of ambiguity aversion is robust to the presence of expected utility consumers.

We show that even when all agents have constant relative ambiguity aversion, the representative consumer has decreasing relative ambiguity aversion, if the ambiguity aversion is heterogeneous. In this case, the pricing kernel has a decreasing elasticity, whereas a constant relative ambiguity aversion representative consumer economy's pricing kernel exhibits constant elasticity. Finally, when the representative consumer has decreasing relative ambiguity aversion, the Hansen-Jannagathan bound or the market price of risk is countercyclical.

How might the market work to produce these phenomena? Suppose, as in the second example in the Introduction, consumers believe there are different growth distributions, depending on whether or not the economy is in a recession. Suppose too, that assets are available that are effectively claims contingent on events identifying a recession (or otherwise). What the foregoing analysis has shown is that in such a scenario, ambiguity averse agents will want to hedge not only against the poor endowment that comes with a recessionary outcome but also against the worse growth distribution that comes with it. Such insurance will allow relatively more ambiguity averse agents to command more of the economy's resources in more recessionary times, producing a counter-cyclical market price of risk. The conclusion does not require that events such as "recession" identify unique distributions. Even if they identify sets of distributions, as explained in Remark 2, a similar analysis could be done and the qualitative results we obtained would go through.

Appendices

A Ambiguity aversion and revealed beliefs

Denote by $b(\mathcal{E}_m)$ a bet that pays c^* on $\mathcal{E}_m$ and c_* off it, and by $b(\neg\mathcal{E}_m)$ the complementary bet. Note, $P_m(\mathcal{E}_m) = 1$. Normalize $u_i(c_*) = 0$ and $\phi_i(0) = 0$. Then consumer i evaluates these bets as:

$$U_i(b(\mathcal{E}_m)) = \mu(P_m)\,\phi_i(u_i(c^*)\,P_m(\mathcal{E}_m)) = \mu(P_m)\,\phi_i(u_i(c^*))$$

and $U_i(b(\neg\mathcal{E}_m)) = (1-\mu(P_m))\,\phi_i(u_i(c^*))$. Consider next a lottery ℓ^π which pays c^* with an objective probability π and c_* with probability $1-\pi$: hence, $U_i(\ell^\pi) = \phi_i(\pi u_i(c^*))$. If ϕ_i is strictly concave, then $U_i(b(\mathcal{E}_m)) < U_i(\ell^{\mu(P_m)})$ and $U_i(b(\neg\mathcal{E}_m)) < U_i(\ell^{1-\mu(P_m)})$. Define $\underline{\pi}, \bar{\pi} \in [0,1]$ to be s.th.

$$U_i(\ell^{\underline{\pi}}) = U_i(b(\mathcal{E}_m)) \quad \text{and} \quad U_i(\ell^{1-\bar{\pi}}) = U_i(b(\neg\mathcal{E}_m)).$$

Since ϕ is increasing, we get, $\underline{\pi} < \mu(P_m) < \bar{\pi}$.

Let $\bar{u} = u(c^*)$, $\underline{u} = u(c_*)$, and $\bar{u} - \underline{u} = h > 0$. To find $\underline{\pi}$ s.th., $U_i(\ell^{\underline{\pi}}) = U_i(b(\mathcal{E}_m))$, we solve:

$$\begin{aligned} \phi_i(\underline{\pi}(\underline{u}+h) + (1-\underline{\pi})\underline{u}) &= \mu(P_m)\,\phi_i(\underline{u}+h) + (1-\mu(P_m))\phi_i(\underline{u}+0) \\ \Leftrightarrow \underline{\pi}h + \underline{u} &= \phi_i^{-1}(\mu(P_m)\,\phi_i(\underline{u}+h) + (1-\mu(P_m))\phi_i(\underline{u}+0)) \end{aligned}$$

Therefore, applying the quadratic approximation in Cerreia-Vioglio et al. (2017),[30] we get,

$$\begin{aligned} \underline{\pi}h + \underline{u} &= \underline{u} + (\mu(P_m)\,h + (1-\mu(P_m))0) \\ &\quad - \frac{\lambda^{\phi_i}(\underline{u})}{2}\left[\mu(P_m)\,h^2 + (1-\mu(P_m))0^2 - (\mu(P_m)\,h)^2\right] + o\left(h^2\right) \\ \Leftrightarrow \quad & \underline{\pi} = \mu(P_m) - \frac{\lambda^{\phi_i}(\underline{u})}{2}\mu(P_m)\,(1-\mu(P_m))\,h + o(h) \end{aligned}$$

[30] Cerreia-Vioglio et al. (2017) show, under mild regularity conditions, if $\mathcal{M}$ is probabilistically sophisticated and twice differentiable at a constant q, then

$$\mathcal{M}(q+H) = q + \mathrm{E}[H] - \frac{\lambda(q)}{2}\mathrm{VAR}[H] + o\left(H^2\right) \tag{16}$$

where $\lambda(q)$ is a function of the first two derivatives of $\mathcal{M}$ at q. In the classical case, $\mathcal{M}(X) = v^{-1}\mathrm{E}[v(X)]$ with v is twice differentiable and strictly increasing, we have $\lambda(q) = -\frac{v''(q)}{v'(q)}$.

Similarly, we get, $\bar{\pi} = \mu(P_m) + \frac{\lambda^{\phi_i}(\underline{u})}{2}\mu(P_m)(1-\mu(P_m))h$. Hence, the size of the "probability matching" interval for $\mathcal{E}_m$ given by $[\underline{\pi}, \bar{\pi}]$ is increasing in λ^{ϕ_i}.

B Relative ambiguity aversion

Much of the literature, including KMM, studies absolute and relative ambiguity attitudes in terms of changes in utility rather than changes in wealth. Here we give a formulation a coefficient of relative ambiguity in terms wealth (which, in our time-less context, is the same as consumption). A singular exception in the literature is Cerreia-Vioglio et al. (2022) which studies relative ambiguity attitude in terms of wealth. Our formulation, below, is consistent with the notion they define.

Let h be an ambiguous prospect and w be initial wealth. Denote by $\pi \equiv \mu \otimes P$ the reduced measure, and by λ_f, the Arrow-Pratt index of f. The variance $\sigma^2_\mu(E_P(h))$ reflects the uncertainty on the expectation E^P given the uncertainty about P, and encapsulates ambiguity. To derive a measure of relative ambiguity aversion, $\lambda_\phi(u(w))u'(w)\,w$, in the Arrow-Pratt way, we derive an approximation of the ambiguity premium for a proportional ambiguous prospect wh :[31]

$$\begin{aligned} C(w+wh) &= w + E_Q(wh) - \frac{w^2}{2}\lambda_u(w)\,\sigma^2_Q(h) \\ &\quad -\frac{w^2}{2}\left(\lambda_v(w) - \lambda_u(w)\right)\sigma^2_\mu\left(E_P(h)\right) + o\left(\|h\|^2\right) \end{aligned}$$

Since, $\phi = v \circ u^{-1}$,

$$\begin{aligned} \lambda_\phi(u(w)) &= \frac{1}{u'(w)}\left(\lambda_v(w) - \lambda_u(w)\right) \\ &\Leftrightarrow \lambda_\phi(u(w))u'(w) = \lambda_v(w) - \lambda_u(w)\,. \end{aligned}$$

Thus, the ambiguity premium for wh, obtained by subtracting the risk premium from the overall uncertainty premium, is, as a proportion of wealth,

$$\left(\left(\lambda_v(w) - \lambda_u(w)\right)w\right) \times \frac{1}{2}\sigma^2_\mu\left(E_P(h)\right) = \lambda_\phi(u(w))u'(w)\,w \times \frac{1}{2}\sigma^2_\mu\left(E_P(h)\right).$$

[31]This is akin to the quadratic approximation of certainty equivalent obtained by Maccheroni et al. (2013): $C(w+h) = w + E_\pi(h) - \frac{1}{2}\lambda_u(w)\,\sigma^2_\pi(h) - \frac{1}{2}\left(\lambda_v(w) - \lambda_u(w)\right)\sigma^2_\mu\left(E^P(h)\right) + o\left(\|h\|^2\right)$.

Analogous to the original Arrow-Pratt argument, we may therefore think of $\lambda_{\phi}\left(u(w)\right)u'\left(w\right)w$ as representing relative ambiguity aversion, relative to wealth. In the LRT specification we consider we may take wealth to equal effective consumption $(x-\zeta)$. Now, by differentiating $v=\phi\circ u$,

$$-\frac{v''(x)}{v'(x)}=-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)-\frac{u''(x)}{u'(x)}. \tag{17}$$

Multiplying both sides by $x-\zeta$, we obtain

$$-\frac{v''(x)(x-\zeta)}{v'(x)}=-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)(x-\zeta)-\frac{u''(x)(x-\zeta)}{u'(x)}. \tag{18}$$

Hence, if u exhibits "shifted" constant relative risk aversion, then, $x\mapsto\lambda_v(x)\left(x-\zeta\right)$ is monotonic with

$$x\mapsto\lambda_{\phi}\left(u(x)\right)u'\left(x\right)\left(x-\zeta\right).$$

Note, (18) reduces to

$$-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)(x-\zeta)=\gamma-\alpha \tag{19}$$

when u and v are CRRA with $-\frac{v''(x)(x-\zeta)}{v'(x)}=\gamma$ and $-\frac{u''(x)(x-\zeta)}{u'(x)}=\alpha$, an expression we'll use extensively in Section 3.

C Proofs for Sections 2 and 3

Proof of Corollary 1 Parts 1 and 2 come from the fact that efficient allocations are conditionally efficient and therefore, efficient allocations of each P-conditional EU; there are thus comonotone (conditionally on each P). Furthermore, it is well-known that the set of efficient allocations in an EU economy with common beliefs does not depend on the beliefs. Part 3. is a consequence of Corollary 6 in Strzalecki and Werner (2011). $\square$

Proof of Proposition 2 Denote the vector of prices of the (state, model)-contingent commodities by $(\psi^P)_{P\in\mathcal{P}}$ with $\psi^P=(\psi^P(s))_{s\in S}$ as defined in (6).

Since μ is common, for every i and every s,

$$\frac{\psi^Q(s)}{\psi^P(s)}=\frac{\phi_i'\left(E^Q u_i\left(X_i^Q\right)\right)u_i'\left(X_i^Q(s)\right)}{\phi_i'\left(E^P u_i\left(X_i^P\right)\right)u_i'\left(X_i^P(s)\right)}. \tag{20}$$

Let κ be s.th.

$$\frac{\phi_\kappa'\left(E^Q u_\kappa\left(X_\kappa^Q\right)\right)}{\phi_\kappa'\left(E^P u_\kappa\left(X_\kappa^P\right)\right)} \geq \frac{\phi_i'\left(E^Q u_i\left(X_i^Q\right)\right)}{\phi_i'\left(E^P u_i\left(X_i^P\right)\right)} \quad \forall i = 1\ldots, I.$$

To simplify exposition, we let $\kappa = 1$. By (20),

$$\frac{u_1'\left(X_1^Q(s)\right)}{u_1'\left(X_1^P(s)\right)} \leq \frac{u_i'\left(X_i^Q(s)\right)}{u_i'\left(X_i^P(s)\right)}$$

for every i. If $X_1^P(s) > X_1^Q(s)$, then the l.h.s. is strictly greater than one. Hence $X_i^P(s) > X_i^Q(s)$ for every i, a contradiction, since feasibility and model-independence imply that $\sum_i X_i^P(s) = \bar{X}(s) = \sum_i X_i^Q(s)$. Hence, $X_1^P(s) \leq X_1^Q(s)$ for every s.

Since $u_i' > 0$, $E^P u_i\left(X_1^P\right) \leq E^P u_i\left(X_1^Q\right)$. Since X_1^Q is a strictly monotone function of $\bar{X}$ when they are regarded as functions on S, and since $P \circ \bar{X}^{-1}$ is FOS dominated by $Q \circ \bar{X}^{-1}$, $P \circ \left(X_1^Q\right)^{-1}$ is FOS dominated by $Q \circ \left(X_1^Q\right)^{-1}$. Thus, $E^P u_1\left(X_1^Q\right) \leq E^Q u_1\left(X_1^Q\right)$. Hence, given that $E^P u_i\left(X_1^P\right) \leq E^P u_i\left(X_1^Q\right)$, we obtain $E^P u_1\left(X_1^P\right) \leq E^Q u_1\left(X_1^Q\right)$. Thus,

$$\frac{\phi_i'\left(E^Q u_i\left(X_i^Q\right)\right)}{\phi_i'\left(E^P u_i\left(X_i^P\right)\right)} \leq \frac{\phi_1'\left(E^Q u_1\left(X_1^Q\right)\right)}{\phi_1'\left(E^P u_1\left(X_1^P\right)\right)} \leq 1 \quad \forall i = 1, \ldots, I. \tag{21}$$

Since $X_1^P \leq X_1^Q$ and $u_1'' < 0$,

$$\frac{u_1'\left(X_1^Q(s)\right)}{u_1'\left(X_1^P(s)\right)} \leq 1 \tag{22}$$

for every s. By (20), (21), (22),

$$\frac{\psi^Q(s)}{\psi^P(s)} = \frac{\phi_1'\left(E^Q u_1\left(X_1^Q\right)\right) u_1'\left(X_1^Q(s)\right)}{\phi_1'\left(E^P u_1\left(X_1^P\right)\right) u_1'\left(X_1^P(s)\right)} \leq 1 \quad \forall i. \tag{23}$$

To show that $E^P u_i\left(X_i^P\right) \leq E^Q u_i\left(X_i^Q\right)$, we consider two cases according to whether $X_i^P \leq X_i^Q$ or not.

If $X_i^P \leq X_i^Q$, then, as we have shown for $i = 1$, we can show that $E^P u_i\left(X_i^P\right) \leq E^P u_i\left(X_i^Q\right) \leq E^Q u_i\left(X_i^Q\right)$. If it is false that $X_i^P \leq X_i^Q$, there is an s s.th. $X_i^P(s) > X_i^Q(s)$. For such an s,

$$\frac{u_i'\left(X_i^Q(s)\right)}{u_i'\left(X_i^P(s)\right)} > 1. \tag{24}$$

By (20), (23), and (24),

$$\frac{\phi_i'\left(E^Q u_i\left(X_i^Q\right)\right)}{\phi_i'\left(E^P u_i\left(X_i^P\right)\right)} < 1.$$

Since $\phi_i'' < 0$, $E^P u_i\left(X_i^P\right) < E^Q u_i\left(X_i^Q\right)$. □

Proof of Lemma 1

Focus first on a vNM economy where consumers have CARA utility functions with parameter α_i. We know from the literature that the representative consumer has a CARA utility function as well (with parameter α where $\sum_i(\alpha/\alpha_i) = 1$) and that the sharing rule takes the form $X_i = \alpha/\alpha_i \bar{X} + \tau_i$ where $\sum_i \tau_i = 0$. Direct computation yields that the linearity of the sharing rule translates to a similar relationship between certainty equivalents:

$$\begin{aligned} u_i^{-1}\left(Eu_i\left(X_i\right)\right) &= -\frac{1}{\alpha_i}\ln\left(E[\exp(-\alpha\bar{X})]\exp\left(-\alpha_i\tau_i\right)\right) \\ &= \frac{\alpha}{\alpha_i}\left(-\frac{1}{\alpha}\ln\left(E[\exp(-\alpha\bar{X})]\right)\right) + \tau_i \\ &= \frac{\alpha}{\alpha_i}u^{-1}\left(Eu\left(\bar{X}\right)\right) + \tau_i. \end{aligned} \tag{25}$$

Hence, $\sum_i u_i^{-1}\left(Eu_i(X_i)\right) = u^{-1}\left(Eu(\bar{X})\right)$.

Consider next the case where consumers have non zero common marginal risk tolerance,

$$u_i(X_i) = \frac{\alpha}{1-\alpha}\left(\frac{X_i - \zeta_i}{\alpha}\right)^{1-\alpha}$$

for $\alpha \neq 0$ and $\alpha \neq 1$.[32] We then have $u_i^{-1}(z) = \alpha\left(\frac{1-\alpha}{\alpha}z\right)^{1/(1-\alpha)} + \zeta_i$. The representative consumer has utility $u(X) = \frac{\alpha}{1-\alpha}\left(\frac{X-\zeta}{\alpha}\right)^{1-\alpha}$, where $\zeta = \sum_i \zeta_i$.

[32]The complete LRT with common marginal risk tolerance family also includes $u_i(X_i) = \ln\left(X_i - \zeta_i\right)$. This class of Bernoulli utilities are commonly called *shifted power* utility. See Back (2017) section 1.3.

The sharing rule takes the form $X_i = \theta_i(\bar{X} - \zeta) + \zeta_i$ where $\sum_i \theta_i = 1$. We then get,

$$\begin{aligned}
u_i^{-1}\left(Eu_i(X_i)\right) &= u_i^{-1}\left(Eu_i(\theta_i(X - \zeta) + \zeta_i)\right) \\
&= u_i^{-1}\left(E\frac{\alpha}{1-\alpha}\left(\frac{\theta_i(X-\zeta)}{\alpha}\right)^{1-\alpha}\right) \\
&= u_i^{-1}\left(\theta_i^{1-\alpha}Eu(X)\right) = \alpha\left(\frac{1-\alpha}{\alpha}\theta_i^{1-\alpha}Eu(X)\right)^{1/(1-\alpha)} + \zeta_i \\
&= \theta_i\left(u^{-1}\left(Eu(X)\right) - \zeta\right) + \zeta_i \qquad (26)
\end{aligned}$$

which leads to $\sum_i u_i^{-1}\left(Eu_i\left(X_i\right)\right) = u^{-1}\left(Eu\left(\bar{X}\right)\right)$. □

Proof of Proposition 3

We prove here that if there exists i s.th. $\phi_i'' < 0$ then $\phi'' < 0$ as well. It is well-known (see, e.g., Wilson (1968)) that the risk tolerance of the representative consumer is the sum of the risk tolerance of the consumers. Call the (absolute) risk tolerance of u at x, $ART(x;u)$. Then, $ART(x,u) = \sum_i ART(g_i(x), u_i)$ where $g_i(x)$ is a solution to program (8) (with $x = X^P(s)$). Similarly, $ART(x,v) = \sum_i ART(f_i(x), v_i)$, where x is now understood as certainty equivalents and $f_i(x)$ is the solution to program (9). Note, and $\sum_i g_i(x) = x \sum_i f_i(x)$ As u_is are LRT with common marginal risk tolerance,

$$\begin{aligned}
ART(x;u) &= \sum_i ART(g_i(x), u_i) = \sum_i a_i + b\sum_i g(x_i) = \sum_i a_i + bx \\
&= \sum_i a_i + b\sum_i f(x_i) = \sum_i a_i + \sum_i ART(f_i(x), u_i)
\end{aligned}$$

Since v_i is more concave than u_i for all i and strictly more concave for at least one i, $\sum_i a_i + \sum_i ART(f_i(x), u_i) > \sum_i a_i + \sum_i ART(f_i(x), v_i)$. Hence, $ART(x;u) > ART(x;v)$ for all x, that is v is more concave than u or, said differently, $\phi'' < 0$.

EU-comonotonicity comes from the standard comonotonocity result of efficient allocation in vNM economies applied to program (9). It can be also be separately derived from Proposition 4 by direct computation of $E^P u_i(X_i^P)$ at optimal allocations and from the proof of Proposition 5 where it is implied by the fact that $f_i'(z) > 0$.

□

Proof of Proposition 4
1. See Section 3.6 in Back (2017).
2. As explained right after introducing the program (9), each efficient allocation can be obtained by solving that program, for some choice of $(\lambda_i)_i$, to allocate certainty equivalents c_i^P under each model P. For each $c \in \mathbb{R}$, let $(f_i(c))_i$ be the solution to this program. Since v_i exhibits constant absolute risk aversion with coefficient γ_i, there is a $(\kappa_i)_i \in \mathbb{R}^I$ such that $\sum_i \kappa_i = 0$ and $f_i(c) = (\gamma/\gamma_i)c + \kappa_i$ for every i and every c. We now prove that if we define a function τ_i using this κ_i, then $\tau_i^P = \tau_i(c^P)$ for every P. By Part 1 of this proposition, the certainty equivalent of X_i^P is equal to $(\alpha/\alpha_i)c^P + \tau_i^P$. Since $\left(X_i^P\right)_{P,i}$ is efficient,

$$\frac{\gamma}{\gamma_i}c^P + \kappa_i = \frac{\alpha}{\alpha_i}c^P + \tau_i^P$$

for every i and every P. Hence,

$$\tau_i^P = \left(\frac{\gamma}{\gamma_i} - \frac{\alpha}{\alpha_i}\right)c^P + \kappa_i = \frac{\gamma}{\gamma_i}\left(1 - \frac{\gamma_i/\alpha_i}{\gamma/\alpha}\right)c^P + \kappa_i = \tau_i(c^P).$$

3. Since v_i exhibits constant absolute risk aversion with coefficient γ_i, it follows from Table III of Wilson (1968) that the value function v of the program (9) exhibits constant absolute risk aversion with coefficient γ. Hence, $\phi(t) \propto -(-t^{\gamma/\alpha})$.

□

Proof of Proposition 5
1. See Section 3.6 in Back (2017).
2. As explained right after introducing the program (9), each efficient allocation can be obtained by solving that program, for some choice of $(\lambda_i)_i$, to allocate certainty equivalents c_i^P under each model P. For each $c > \zeta$, let $(\hat{f}_i(c))_i$ be the solution to this program. Then, $\hat{f}_i(c^P) = u_i^{-1}\left(E^P u_i\left(X_i^P\right)\right)$. Then, for each $z > 0$, define $f_i(z) = \hat{f}_i(z+\zeta) - \zeta_i$. Then define a function θ_i by $\theta_i(z) = f_i(z)/z$. In words, the θ_i's represent an efficient allocation of certainty equivalents in terms of fractions of individual consumption levels, relative to aggregate consumption levels, in excess of the minimum consumption levels ζ_i and ζ. Then $u_i^{-1}\left(E^P u_i\left(X_i^P\right)\right) = \theta_i(c^P-\zeta)(c^P-\zeta)+\zeta_i$. We now prove that $\theta_i(c^P) = u_i^{-1}\left(E^P u_i\left(X_i^P\right)\right)$ for every P and every i. By Part 1 of this proposition, the certainty equivalent of X_i^P is equal to $\theta_i^P(c^P-\zeta)+\zeta_i$.

Since $\left(X_i^P\right)_{P,i}$ is efficient,

$$\theta_i(c^P-\zeta)(c^P-\zeta)+\zeta_i=\theta_i^P(c^P-\zeta)+\zeta_i$$

for every i and every P. Hence, $\theta_i(c^P-\zeta)=\theta_i^P$.

Part of the subsequent argument follows those of Hara et al. (2007). The F.O.C. to the problem (9) is that for every $z>0$, there is a $\psi_z>0$ s.th. $\lambda_i v_i'(f_i(z)+\zeta_i)=\psi_z$ for every i. Since v_i is infinitely differentiable and satisfies $v_i''<0<v_i'$ for every i, the implicit function theorem implies that the f_i are infinitely differentiable as well. Thus v is also infinitely differentiable. By the envelop theorem, $\psi_z=v'(z+\zeta)$ for every $z>0$. Hence $v'(z+\zeta)>0$ and

$$\lambda_i v_i'(f_i(z)+\zeta_i)=v'(z+\zeta)$$

for every i and every $z>0$. Differentiating w.r.t. z, we obtain

$$\lambda_i v_i''(f_i(x)+\zeta_i)f_i'(z)=v''(z+\zeta)$$

for every i and every $z>0$. By dividing each side of the second equality by the corresponding side of the first equality, we obtain

$$\frac{v_i''(f_i(z)+\zeta_i)f_i'(z)}{v_i'(f_i(z)+\zeta_i)}=\frac{v''(z+\zeta)}{v'(z+\zeta)},$$

which can be rewritten as

$$-\frac{v_i''(f_i(z)+\zeta_i)f_i(z)}{v_i'(f_i(z)+\zeta_i)}\frac{f_i'(z)}{f_i(z)}+\frac{v''(z+\zeta)}{v'(z+\zeta)}=0$$

for every i and $z>0$. Since v_i exhibits LRT with parameters (γ_i,ζ_i), this can be further rewritten as

$$\gamma_i\frac{f_i'(z)}{f_i(z)}+\frac{v''(z+\zeta)}{v'(z+\zeta)}=0. \tag{27}$$

Since $\sum_i f_i(z)=z$ there is an i s.th. $f_i'(z)>0$. Thus, $v''(z+\zeta)<0$. Hence, $f_i'(z)>0$ for every i. Moreover,

$$\frac{d}{dz}\ln\frac{f_j(z)}{f_i(z)}=\frac{f_j'(z)}{f_j(z)}-\frac{f_i'(z)}{f_i(z)}=-\frac{v''(z+\zeta)}{v'(z+\zeta)}\left(\frac{1}{\gamma_j}-\frac{1}{\gamma_j}\right)\gtreqless 0$$

if and only if $\gamma_i \gtreqless \gamma_j$. Since $f_j(z)/f_i(z) = \theta_j(z)/\theta_i(z)$, the limit behavior in (a) of part 2 is proved.

Note that the left-hand side of (27) is equal to the derivative of the logarithm of the function $z \mapsto (f_i(z))^{\gamma_i}\, v'(z+\zeta)$. Hence this function is, in fact, constant. Thus, if there is an i s.th. $f_i(z)$ is bounded from above, then $v'(z)$ is bounded away from zero. Then, in fact, $f_i(z)$ is bounded from above for every i. But, it would contradict the assumption that $\sum_i f_i(z) = z$ for a sufficiently large $z > 0$. Hence, for every i, $f_i(z) \to \infty$ as $z \to \infty$. We can analogously show that for every i, $f_i(z) \to 0$ as $z \to 0$. This also shows that $v'(x) \to \infty$ as $x \to \zeta$ and $v'(x) \to 0$ as $x \to \infty$.

Denote the constant value of $(f_i(z))^{\gamma_i}\, v'(z+\zeta)$ by κ_i. Then, for every i and j,

$$0 < \theta_i(z) = \frac{f_i(z)}{z} < \frac{f_i(z)}{f_j(z)} = \frac{\left(\dfrac{\kappa_i}{v'(z+\zeta)}\right)^{1/\gamma_i}}{\left(\dfrac{\kappa_j}{v'(z+\zeta)}\right)^{1/\gamma_j}} = \frac{\kappa_i^{1/\gamma_i}}{\kappa_j^{1/\gamma_j}} \left(v'(z+\zeta)\right)^{1/\gamma_j - 1/\gamma_i}.$$

If $\gamma_i < \bar{\gamma} = \gamma_j$, then $1/\gamma_j - 1/\gamma_i < 0$. Since $v'(z+\zeta) \to \infty$ as $z \to 0$, the far right-hand side of the above equality converges to 0 as $z \to 0$. Hence $\theta_i(z) \to 0$ as $z \to 0$. We can analogously show that for every i, if $\gamma_i > \underline{\gamma}$, then $\theta_i(z) \to 0$ as $z \to \infty$. The proof of part 2 is now completed.

For every i and every $z > 0$, we next prove that

$$\frac{1}{\gamma_i} = \frac{f_i'(z) z}{f_i(z)} \frac{1}{b(z)}, \tag{28}$$

$$\sum_i \theta_i(z) \frac{1}{\gamma_i} = \frac{1}{b(z)}. \tag{29}$$

where $b(z) = -\frac{v''(z+\zeta)z}{v'(z+\zeta)}$. By rearranging (27), we obtain (28). It can be further rewritten as $\theta_i(z)/\gamma_i = b(z) f_i'(z)$. Since $\sum_i f_i'(z) = 1$, by summing both sides over i, we obtain the equality in (a) of part 3. The limiting behavior in (a) of part 3 follows from the equality in (b) of part 2. This completes the proof of (a) of part 3.

Since

$$\theta_i'(z) = \frac{d}{dz}\frac{f_i(z)}{z} = \frac{f_i(z)}{z^2}\left(\frac{f_i'(z) z}{f_i(z)} - 1\right) = \frac{f_i(z)}{z^2}\left(\frac{b(z)}{\gamma_i} - 1\right) \tag{30}$$

for every i and every $z > 0$, $1/\gamma_i \lesseqgtr 1/b(z)$ if and only if $\theta_i'(z) \lesseqgtr 0$. This proves (c) of part 3.

It remains to prove (b) of part 3. Since $\underline{\gamma} < \bar{\gamma}$, by $\sum_i \theta_i(z) = 1$ and (29), $\sum_i \theta_i'(z) = 0$ and

$$\sum_i \theta_i'(z) \frac{1}{\gamma_i} = -\frac{b'(z)}{(b(z))^2}.$$

Thus,

$$\sum_i \theta_i'(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right) = -\frac{b'(z)}{(b(z))^2}.$$

By (30),

$$\sum_i \theta_i'(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right) = \frac{b(z)}{z} \sum_i \theta_i(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right)^2.$$

Since there is an i s.th. $1/\gamma_i < 1/b(z)$, and there is another i s.th. $1/\gamma_i > 1/b(z)$, (29) implies that this is strictly positive. Hence $b'(z) < 0$. This completes the proof.

□

D Proofs for Section 4

First, we give general results on the comparison of the Hansen-Jagannathan bounds of two pricing kernels, and also of a pricing kernel under two probabilities

Lemma 2 *1. Let P be any non-degenerate probability on $\mathbb{R}_{++}$. For each $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be continuous. Assume that π_2 is non-increasing and π_2/π_1 is strictly increasing. Then, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$, where E and σ are the mean and standard deviation under P.*

2. For each $n = 1, 2$, let P_n be any non-degenerate probability on $\mathbb{R}_{++}$. Assume that P_n has a probability density function g_n and that there is a $k > 1$ such that $g_1(x) = kg_2(kx)$ for every $x > 0$. Let $\pi : \mathbb{R}_{++} \to \mathbb{R}_{++}$

be differentiable. Assume that $\pi' < 0$ and $-\pi'(x)x/\pi(x)$ is strictly decreasing in x. Then $\sigma^{P_1}(\pi)/E^{P_1}(\pi) > \sigma^{P_2}(\pi)/E^{P_2}(\pi)$, where, for each n, E^{P_1} and σ^{P_1} are the mean and standard deviation under P_n.

Proof of Lemma 2
1. For each n, the integral of the function $x \mapsto (E(\pi_n))^{-1}\pi_n(x)$ under P is equal to one. Since it is continuous, (the graphs of) these two functions $n = 1, 2$ cross at least once. Since π_2/π_1 is strictly increasing, they cross exactly once. Let x^* be such that $\pi_1(x^*)/E(\pi_1) = \pi_2(x^*)/E(\pi_2)$ and denote this value by z^*. Then $\pi_1(x)/E(\pi_1) \gtreqqless \pi_2(x)/E(\pi_2)$ if and only if $x \lesseqqgtr x^*$. Since π_2 is non-increasing,

$$\frac{\pi_1(x)}{E(\pi_1)} \gtreqqless \frac{\pi_2(x)}{E(\pi_2)} \gtreqqless z^*$$

if and only if $x \lesseqqgtr x^*$. Thus, for every $x \neq x^*$,

$$\left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 > \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2.$$

If $x = x^*$, then the above inequality would hold as an equality. Since P is not degenerate,

$$\int \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2 P(dx) > \int \left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 P(dx).$$

Note that, for each $n = 1, 2$,

$$\begin{aligned}
\frac{\sigma(\pi_n)^2}{E(\pi_n)^2} &= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - 1\right)^2 P(dx) \\
&= \int \left(\left(\frac{\pi_n(x)}{E(\pi_n)} - z^*\right) + (z^* - 1)\right)^2 P(dx) \\
&= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - z^*\right)^2 P(dx) - (z^* - 1)^2.
\end{aligned}$$

Thus, $\sigma(\pi_1)^2/E(\pi_1)^2 > \sigma(\pi_2)^2/E(\pi_2)^2$. Thus, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$.

2. We prove this part by applying part 1. To do so, write P for P_1, g for g_1, and π_1 for π. Define π_2 by letting $\pi_2(x) = \pi_1(kx)$ for every $x \in I$. We now

show that $E^P(\pi_2) = E^{P_2}(\pi)$ and $\sigma^P(\pi_2) = \sigma^{P_2}(\pi)$. Since $g_1(x) = kg_2(kx)$, the change-of-variable formula implies that

$$E^P(\pi_2) = \int \pi_2(x)g_1(x)dx = \int \pi_2(kx)kg_2(kx)dx = \int \pi(x)f_2(x)dx = E^{P_2}(\pi).$$

By this equality and the change-of-variables formula,

$$\begin{aligned}\sigma^P(\pi_2) &= \left(\int \left(\pi_2(x) - E^P(\pi_2)\right)^2 g_1(x)dx\right)^{1/2} \\ &= \left(\int \left(\pi(kx) - E^{P_2}(\pi)\right)^2 kg_2(kx)dx\right)^{1/2} \\ &= \left(\int \left(\pi(x) - E^{P_2}(\pi)\right)^2 g_2(x)dx\right)^{1/2} = \sigma^{P_2}(\pi).\end{aligned}$$

Thus, $\sigma^P(\pi_2)/E^P(\pi_2) = \sigma^{P_2}(\pi)/E^{P_2}(\pi)$. It, thus, suffices to prove that $\sigma^P(\pi_1)/E^P(\pi_1) > \sigma^P(\pi_2)/E^P(\pi_2)$. By part 1, it suffices to prove that π_2/π_1 is strictly increasing.

Differentiate both sides of $\pi_2(x) = \pi_1(kx)$ with respect to x, we obtain $\pi_2'(x) = \pi_1'(kx)k$. Thus,

$$-\frac{\pi_2'(x)x}{\pi_2(x)} = -\frac{\pi_1'(kx)kx}{\pi_1(kx)}.$$

Since $k > 1$ and $-\pi_1'(x)x/\pi_1(x)$ is a strictly decreasing function of x,

$$-\frac{\pi_1'(kx)kx}{\pi_1(kx)} < -\frac{\pi_1'(x)x}{\pi_1(x)}.$$

Thus, $-\pi_2'(x)x/\pi_2(x) < -\pi_1'(x)x/\pi_1(x)$, that is, $-\pi_2'(x)/\pi_2(x) < -\pi_1'(x)/\pi_1(x)$ for every x. This is equivalent to $(\pi_2/\pi_1)' > 0$. The proof is thus completed.

□

Proof of Proposition 6 :

1. For each s, the Radon-Nikodym derivative of the conditional probability of the second-order belief μ given state s, with respect to μ itself, is equal to the function $P \mapsto P(s)/P^*(s)$, where $P^*(s) = \int_{\mathcal{P}} Q(s)\mu(dQ)$, the reduced probability of state s. For two states s_1 and s_2 with $s_1 < s_2$, the ratio of the two Radon-Nikodym derivatives at each $P \in \mathcal{P}$ is equal to

$$\frac{P(s_2)/P^*(s_2)}{P(s_1)/P^*(s_1)}.$$

For two probabilities P_1 and P_2 in $\mathcal{P}$, if P_1 is strictly dominated by P_2 with respect to the monotone likelihood ratio, (that is, $P_2(s)/P_1(s)$ is strictly increasing in s), then $P_1(s_2)/P_1(s_1) < P_2(s_2)/P_2(s_1)$ and, hence,

$$\frac{P_1(s_2)/P^*(s_2)}{P_1(s_1)/P^*(s_1)} < \frac{P_2(s_2)/P^*(s_2)}{P_2(s_1)/P^*(s_1)}.$$

That is, the conditional second-order belief given s_1 is strictly dominated by the conditional second-order belief given s_2 with respect to the monotone likelihood ratio, where the monotonicity is defined in terms of the monotone likelihood ratio of probabilities in $\mathcal{P}$.

Since $\phi'(E^P u(\bar{X}))$ is strictly decreasing in P with respect to the monotone likelihood ratio of probabilities in $\mathcal{P}$, this implies that

$$\int_{\mathcal{P}} \phi'(E^P u(\bar{X})) \frac{P(s)}{P^*(s)} \mu(dP)$$

is strictly decreasing in s. That is, h is a strictly decreasing function. Thus $h'(s) \leq 0$ for every s, and every non-degenerate interval contains a state s at which $h'(s) < 0$. By differentiating the logarithm of (14) and multiplying -1, we obtain

$$-\frac{\tilde{\pi}'_{u,\phi}(s)}{\tilde{\pi}_{u,\phi}(s)} = -\frac{u''(\bar{X}(s))\bar{X}'(s)}{u'(\bar{X}(s))} - \frac{\dfrac{\partial h}{\partial s}(s,\mu)}{h(s,\mu)}$$

for every s. Since $\bar{X}'(s) = \bar{X}(s)$ and $\pi'_{u,\phi}(x) = \tilde{\pi}'_{u,\phi}(\ln x)/x$,

$$\varepsilon(x;\pi_{u,\phi}) = -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)} = -\frac{u''(x)x}{u'(x)} - \frac{\dfrac{\partial h}{\partial s}(\ln x,\mu)}{h(\ln x,\mu)} \tag{31}$$

for every $x > 0$. In the case of ambiguity neutrality, $\varepsilon(x;\pi_{u,id}) = -u''(x)x/u'(x)$ for every $x > 0$. Thus,

$$\varepsilon(x;\pi_{u,\phi}) - \varepsilon(x;\pi_{u,id}) = -\frac{\dfrac{\partial h}{\partial s}(\ln x,\mu)}{h(\ln x,\mu)}$$

for every $x > 0$. Hence, $\varepsilon(x;\pi_{u,\phi}) \geq \varepsilon(x;\pi_{u,id})$ for every x, and every non-degenerate interval contains an x at which $\varepsilon(x;\pi_{u,\phi}) > \varepsilon(x;\pi_{u,id})$. This proves part 1.

2. By definition,

$$\frac{d}{dx}\ln\frac{\pi_{u,id}(x)}{\pi_{u,\phi}(x)} = \frac{\varepsilon(x;\pi_{u,\phi}) - \varepsilon(x;\pi_{u,id})}{x}.$$

By part 1, the right-hand side of this equality is non-negative for every x and every non-degenerate interval contains an x at which it is strictly positive. Thus, $\pi_{u,id}(x)/\pi_{u,\phi}(x)$ is strictly increasing in x. By apply part 1 of Lemma 2 to $\pi_1 = \pi_{u,\phi}$ and $\pi_2 = \pi_{u,id}$, we complete the proof. $\square$

We now proceed to prove Propositions 7 and 8 in the Gaussian case, on Assumption 4 is imposed. In this case, it is convenient to work on density functions (with respect to the Lebesgue measure) of the parameters m that designates first-order beliefs. Denote the probability density functions of the second-order belief $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$ and a first-order belief $\mathcal{N}(m, \sigma^2)$ by p_M and $p_{S|M}(\,\cdot\,|\,m)$. It follows from Bayes' formula that the conditional second-order belief given a state s is

$$\mathcal{N}\left(\frac{\hat{\sigma}^2 s + \sigma^2\hat{m}}{\sigma^2+\hat{\sigma}^2}, \frac{\hat{\sigma}^2\sigma^2}{\sigma^2+\hat{\sigma}^2}\right). \tag{32}$$

Denote its probability density function by $p_{M|S}(\,\cdot\,|\,s)$. Then, (15) can be rewritten as

$$h(s,\hat{m}) = \int \frac{v'(c(m))}{u'(c(m))} p_{M|S}(m\,|\,s)dm, \tag{33}$$

where $c(m) = u^{-1}\left(E^m u(\bar{X})\right)$ and E^m is the expectation under $\mathcal{N}(m,\sigma^2)$. The relation (14) can be rewritten as $\tilde{\pi}_{u,\phi}(s,\hat{m}) = \lambda(\hat{m})u'(\bar{X}(s))h(s,\hat{m})$.

Write $r = \hat{\sigma}^2/(\sigma^2+\hat{\sigma}^2)$, then $0 < r < 1$. Denote by q the probability density function of

$$\mathcal{N}\left(0, \frac{\hat{\sigma}^2\sigma^2}{\sigma^2+\hat{\sigma}^2}\right).$$

Then, the probability density function of (32) coincides with the function $s \mapsto q(m - (rs + (1-r)\hat{\sigma}))$. Then, (33) can be rewritten as

$$h(s,\hat{m}) = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} q(m - (rs + (1-r)\hat{m}))\,dm.$$

The following two lemmas are consequences of Proposition 11 in Appendix E, which is a general result on strict log-supermodularity (SLSPM for short).

The first one, Lemma 3, will be used to prove Proposition 8, which is on the countercyclicality of Hansen-Jagannathan bounds, and the second one, Lemma 4, will be used to prove Proposition 7. which is on the decreasing elasticity of a pricing kernel.

Lemma 3 *Suppose that Assumption 4 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, h is strictly log-supermodular, that is,*

$$h(s_1,\hat{m}_1)h(s_2,\hat{m}_2) < h(\max\{s_1,s_2\},\max\{\hat{m}_1,\hat{m}_2\})h(\min\{s_1,s_2\},\min\{\hat{m}_1,\hat{m}_2\})$$

for all $(s_1,\hat{m}_1)$ and $(s_2,\hat{m}_2)$, unless $(s_1,\hat{m}_1) \leq (s_2,\hat{m}_2)$ or $(s_1,\hat{m}_1) \geq (s_2,\hat{m}_2)$.

Proof of Lemma 3 By part 1 of Assumption 4,

$$c(m) = \exp\left(m + \frac{\sigma^2}{2}(1-\alpha)\right).$$

Define $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}_{++}$ by

$$f(s,\hat{m},m) = \frac{v'(c(m+rs+(1-r)\hat{m}))}{u'(c(m+rs+(1-r)\hat{m}))}q(m).$$

Since $c'(m+rs) = c(m+rs)$,

$$\begin{aligned}
\frac{\partial}{\partial s}\ln f(s,\hat{m},m) &= \frac{\mathrm{d}}{\mathrm{d}s}\ln v'(c(m+rs+(1-r)\hat{m}))) - \frac{\mathrm{d}}{\mathrm{d}s}\ln u'(c(m+rs+(1-r)\hat{m})))\\
&= \frac{v''(c(m+rs+(1-r)\hat{m})))}{v'(c(m+rs+(1-r)\hat{m})))}c'(m+rs+(1-r)\hat{m}))r -\\
&\quad -\frac{u''(c(m+rs+(1-r)\hat{m})))}{u'(c(m+rs+(1-r)\hat{m})))}c'(m+rs+(1-r)\hat{m}))r\\
&= \left(\frac{v''(c(m+rs+(1-r)\hat{m})))c(m+rs+(1-r)\hat{m}))}{v'(c(m+rs+(1-r)\hat{m})))} -\right.\\
&\quad \left.-\frac{u''(c(m+rs+(1-r)\hat{m})))c(m+rs+(1-r)\hat{m}))}{u'(c(m+rs+(1-r)\hat{m})))}\right)r\\
&= \left(\frac{v''(c(m+rs+(1-r)\hat{m})))c(m+rs+(1-r)\hat{m}))}{v'(c(m+rs+(1-r)\hat{m})))} - \alpha\right)r. \qquad (34)
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial \hat{m}}\ln f(s,\hat{m},m) = \left(\frac{v''(c(m+rs+(1-r)\hat{m})))c(m+rs+(1-r)\hat{m}))}{v'(c(m+rs+(1-r)\hat{m})))} - \alpha\right)(1-r).$$

Thus,

$$\frac{\partial^2}{\partial s \partial \hat{m}} \ln f(s,\hat{m},m) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{v''(x)x}{v'(x)}\right|_{x=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r(1-r) > 0,$$

by the differentiably strictly decreasing relative risk aversion of v. Similarly,

$$\frac{\partial^2}{\partial s \partial m} \ln f(s,\hat{m},m) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{v''(x)x}{v'(x)}\right|_{x=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r > 0,$$

$$\frac{\partial^2}{\partial \hat{m} \partial m} \ln f(s,\hat{m},m) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{v''(x)x}{v'(x)}\right|_{x=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})(1-r) > 0.$$

Thus, by Proposition 11, the function $(s,\hat{m}) \mapsto \int_{-\infty}^{\infty} f(s,\hat{m},m)\,\mathrm{d}m$ has SLSPM. By the change of variable,

$$\int_{-\infty}^{\infty} f(s,\hat{m},m)\,\mathrm{d}m = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} b(m-(rs+(1-r)\hat{m}))\,\mathrm{d}m = h(s,\hat{m}). \tag{35}$$

This completes the proof. □

Lemma 4 *Suppose that Assumption 4 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, for every $\hat{m} \in \mathbb{R}$,*

$$\frac{\frac{\partial h}{\partial s}(s,\hat{m})}{h(s,\hat{m})}$$

is strictly increasing in $s \in \mathbb{R}$.

Proof of Lemma 4 Let $\hat{m} \in \mathbb{R}$. Let f be as in the proof of Lemma 3. Define $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{++}$ by $k(s,\varepsilon,m) = f(s+\varepsilon,\hat{m},m)$. By (34),

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial \varepsilon} \ln k(s,\varepsilon,m) &= \frac{\partial^2}{\partial s^2} \ln f(s,\hat{m},m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{v''(x)x}{v'(x)}\right|_{x=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r^2 > 0, \\
\frac{\partial^2}{\partial m \partial \varepsilon} \ln k(s,\varepsilon,m) &= \frac{\partial^2}{\partial m \partial s} \ln k(s,\varepsilon,m) \\
&= \frac{\partial^2}{\partial s \partial m} \ln f(s,\hat{m},m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{v''(x)x}{v'(x)}\right|_{x=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r > 0.
\end{aligned}$$

By Proposition 11, the function $(s, \varepsilon) \mapsto \int_{-\infty}^{\infty} k(s, \varepsilon, m) \, \mathrm{d}m$ has SLSPM. Since $k(s, \varepsilon, m) = f(s + \varepsilon, \hat{m}, m)$, by (35), this function is equal to $(s, \varepsilon) \mapsto h(s + \varepsilon, \hat{m})$. Since it has SLSPM, if $s_1 < s_2$ and $\varepsilon > 0$, then

$$\frac{h(s_1 + \varepsilon, \hat{m})}{h(s_1, \hat{m})} < \frac{h(s_2 + \varepsilon, \hat{m})}{h(_2, \hat{m})}.$$

This means that $h(s + \varepsilon, \hat{m})/h(s, \hat{m})$ is a strictly increasing function of s. Since

$$\frac{d}{ds} \ln \frac{h(s + \varepsilon, \hat{m})}{h(s, \hat{m})} = \frac{\frac{\partial h}{\partial s}(s + \varepsilon, \hat{m})}{h(s + \varepsilon, \hat{m})} - \frac{\frac{\partial h}{\partial s}(s, \hat{m})}{h(s, \hat{m})},$$

and the left-hand side is nonnegative, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is non-decreasing in s. To prove that it is, in fact, strictly increasing, suppose not. Then, there is an interval, say $(\underline{s}, \overline{s})$, over which it is constant. Take a small $\varepsilon > 0$. Then, over an interval of s with $\underline{s} < s < s + \varepsilon < \overline{s}$, the right-hand side is constantly equal to 0. Hence, $h(s + \varepsilon, \hat{m})/h(s, \hat{m})$ is constant. But this is contradiction. Thus, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is strictly increasing in s. □

Proof of Proposition 7

1. This follows from direct calculation.

2. Since u exhibits constant relative risk aversion, the first fraction on the right-hand side of (31) (where μ is replaced by $\hat{m}$) is independent of x. By Lemma 4, the second fraction is strictly increasing in x. Thus, $\varepsilon(x; \pi_{u,\phi})$ is strictly decreasing in x.

□

Proof of Proposition 8

1. This follows from part 1 of Proposition 7 via direct calculation.

2. By Lemma 3, $h(s, \hat{m}_2)/h(s, \hat{m}_1)$ is strictly increasing in s. Thus, by (14), where μ is replaced by $\hat{m}_1$ and $\hat{m}_2$, $\tilde{\pi}_{u,\phi}(s; \hat{m}_2)/\tilde{\pi}_{u,\phi}(s; \hat{m}_1)$ is strictly increasing in s; and so is $\pi_{u,\phi}(x; \hat{m}_2)/\pi_{u,\phi}(x; \hat{m}_1)$ in x. Thus, by part 1 of Lemma 2,

$$\frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_1)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_1)\right)} > \frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_2)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_2)\right)}. \tag{36}$$

For each n, under the second-order belief is $\mathcal{N}(\hat{m}_n, \hat{\sigma}^2)$, the reduced probability over states (marginal distribution on S) coincides with $\mathcal{N}(\hat{m}_n, \sigma^2 + \hat{\sigma}^2)$. Since $\bar{X}(s) = \exp s$, the reduced probability over consumption levels coincides with the log-normal distribution $\mathcal{LN}(\hat{m}_n, \hat{\sigma}^2 + \sigma^2)$. Let g_n be the probability density function of this distribution and $k = \exp(\hat{m}_2 - \hat{m}_1)$, then $k > 1$ and $g_1(x) = kg_2(kx)$ for every $x > 0$. Thus, by part 2 of Lemma 2,

$$\frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)} > \frac{\sigma^{\hat{m}_2}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}{E^{\hat{m}_2}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}. \tag{37}$$

By (36) and (37), the proof is completed.

□

By applying the following Proposition to the case where π_1 is the pricing kernel of a homogeneous economy, which exhibits constant relative ambiguity aversion, and π_2 is the pricing kernel of a heterogeneous economy, which exhibits strictly decreasing relative ambiguity aversion, we can give the argument behind the right panel of Figure 3, i.e., that the kernels cross exactly twice.

Proposition 9 *For each $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable and suppose that $\pi_n' < 0$. Suppose, moreover, that $\varepsilon(x;\pi_1)$ is independent of x, $\varepsilon(x;\pi_2)$ is strictly decreasing in x, and the value of the former is contained in the range of the latter. Suppose, furthermore, that there is a non-degenerate probability P on $\mathbb{R}_{++}$ s.th. $\int \pi_1(x)P(dx) = \int \pi_2(x)P(dx)$. Then, there are an $x_* \in \mathbb{R}_{++}$ and an $x^* \in \mathbb{R}_{++}$ with $x_* < x^*$ s.th. $\pi_1(x) < \pi_2(x)$ if $x < x_*$ or $x > x^*$; $\pi_1(x) > \pi_2(x)$ if $x_* < x < x^*$; and $\pi_1(x) = \pi_2(x)$ if $x = x_*$ or $x = x^*$.*

The equality $\int \pi_1(x)P(dx) = \int \pi_2(x)P(dx)$ means that the two pricing kernels give the same price for the risk-free bond.

Proof of Proposition 9 Define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = \ln \pi_2(\exp z) - \ln \pi_1(\exp z)$. Then,

$$g'(z) = \frac{\pi_2'(\exp z)\exp z}{\pi_2(\exp z)} - \frac{\pi_1'(\exp z)\exp z}{\pi_1(\exp z)}.$$

Thus, g' is strictly increasing, and there are a $\underline{z}$ and a $\overline{z}$ s.th. $g'(\underline{z}) < 0 < g'(\overline{z})$. Then, $g'(z) \leq g'(\underline{z})$ for every $z \leq \underline{z}$ and $g'(z) \geq g'(\overline{z})$ for every $z \geq \overline{z}$. By applying the mean-value theorem to g on the interval $[z, \underline{z}]$ and the strict increasingness of g', we obtain $g(\underline{z}) \leq g'(\underline{z})(\underline{z} - z) + g(z)$, that is, $g(z) \geq -g'(\underline{z})(\underline{z} - z) + g(\underline{z})$ for every $z < \underline{z}$. As $z \to -\infty$, the right-hand side diverges to ∞. Similarly, $g(z) \geq g'(\overline{z})(z - \overline{z}) + g(\overline{z})$ for every $z > \overline{z}$. As $z \to \infty$, the right-hand side diverges to ∞. Thus, g attains its minimum (over the entire $\mathbb{R}$). Denote by $\hat{z}$ a point at which the minimum is attained. Then, $g'(\hat{z}) = 0$ by the first-order condition. Since g' is strictly increasing, $g'(z) < 0$ for every $z < \hat{z}$, and $g'(z) > 0$ for every $z > \hat{z}$. Thus, g is strictly decreasing on $(-\infty, \hat{z})$ and strictly increasing on $(\hat{z}, -\infty)$.

If $g(\hat{z}) \geq 0$, then $g(z) \geq 0$ for every z, with a strict inequality possibly except at $z = \hat{z}$. Thus, $\pi_2(x) \geq \pi_1(x)$ for every x, with a strict inequality possibly except for $x = \exp \hat{z}$, and the integral assumption is violated. Thus, $g(\hat{z}) < 0$. By the intermediate value theorem, there is a unique $z_* < \hat{z}$ s.th. $g(z_*) = 0$; and there is a unique $z^* > \hat{z}$ s.th. $g(z^*) = 0$. Let $x_* = \exp z_*$ and $x^* = \exp z^*$, to complete the proof. □

E Parameters for Figure 4

Assume that $\bar{X}(s) = \exp(s)$ is lognormally distributed with mean m and variance σ^2, but the distribution is unknown as both n and σ^2 are unknown.

The consumers believe two regimes are possible (and equiprobable), one with $(m_1, \sigma_1^2) = (.15, .1)$, the other with $(m_2, \sigma_2^2) = (-.15, .5)$

The three economies considered are:

-one in which there is an EU representative consumer with a CRRA utility function with relative risk aversion equal to 2/3.

-one in which there is a smooth ambiguity representative consumer with CRRA u with relative risk aversion equal to 2/3 and CRRA v with ambiguity aversion 64/3.

-one in which there is a consumer of each of the two types described above.

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Online Appendix 1: EU comonotonicity

We saw in Proposition 2 that the comonotonicity is a necessary condition for efficiency if the support of the second-order belief (which is assumed to be common across consumers) is totally ordered by the first-order stochastic dominance with regards to the aggregate endowment $\bar{X}$. In this appendix, we show that the comononicity is also sufficient for efficiency; and, thus, that coupled with conditional efficiency, it exhausts all the implications of efficiency. Specifically, for any given profile $(u_i)_{i\in I}$ of Bernoulli utility functions for pure risk, any second-order belief μ, and any conditionally efficient allocation $(X_i)_{i\in I}$ that is also expected-utility-comonotone, we find a profile $(\phi_i)_{i\in I}$ of functions of ambiguity attitudes such that $(X_i)_{i\in I}$ is an efficient allocation of the economy of KMM utility functions $(u_i,\phi_i,\mu)_{i\in I}$. To do so, we assume that S is finite but the support $\operatorname{supp}\mu$ can be parameterized by a scalar, and exploit some (weak) differentiability assumptions on how the P-efficient allocation $(X_i^P)_{i\in I}$ depends on the parameter of P.

Let $u_i : \mathbb{R}_{++} \to \mathbb{R}$ satisfy $u_i'' < 0 < u_i'$. Let $\mu \in \Delta(\Delta(S))$. We also assume that S is finite and $P(s) > 0$ for all $P \in \operatorname{supp}\mu$ and $s \in S$. Let $\bar{X}$ be aggregate consumption that is model-independent but not state-independent.

Given $(u_i)_{i\in I}$ and $\bar{X}$, it is possible to parameterize the set of efficient allocations of the EU economy under any common (first-order) belief.

Lemma 5 *There are an open set T in $\mathbb{R}^{I-1}$ and a C^1-diffeomorphism $Z^\cdot = (Z_i^\cdot)_i : T \to \left(\mathbb{R}^S_{++}\right)^I$ onto the set of all efficient allocations of the EU economy of $(u_i)_{i\in I}$ (under any common belief on S).*

The proof of this lemma can be adopted from the proof of in Mas-Colell (1985, Proposition 4.6.10). This lemma, in short, claims that the set of all efficient allocations of the EU economy (under any common belief on S) is an $(I-1)$-dimensional C^1 manifold.

The following assumption gives a parametrization of the second-order belief.

Assumption 5 *There are an open interval M in $\mathbb{R}$ and a bijection $P^\cdot : M \to \operatorname{supp}\mu$ s.th. for every $s \in S$, the function $P^\cdot(s) : m \mapsto P^m \equiv (P^m(s))_s$ is continuously differentiable when $\operatorname{supp}\mu$ is regarded as a subset of $\mathbb{R}^S$.*

For each i, define $b_i : T \times M \to \mathbb{R}$ by $b_i(t,m) = E^{P^m} u_i\left(Z_i^t\right)$, then $b_i(t,m) = \sum_s P^m(s) u_i\left(Z_i^t(s)\right)$. Thus, under Assumption 5, b_i is continuously differentiable. In addition, we impose the following assumption on its partial derivative with respect to m.

Assumption 6 *For all $(t,m) \in T \times M$ and $i \in I$, $\dfrac{\partial b_i}{\partial m}(t,m) > 0$.*

It is a differential version of the assumption that $\operatorname{supp}\mu$ is totally ordered by the first-order stochastic dominance relation with regards to the aggregate endowments $\bar{X}$ (because, for each t, the conditional allocation $(Z_i^t)_i$ is comonotone along with $\bar{X}$, in terms of consumption levels), and can also accommodate other types of ordering on $\operatorname{supp}\mu$ as long as the consumers unanimously agree on the ordering.

Each conditionally efficient allocation can be identified with a mapping from $\operatorname{supp}\mu$ to the set of all efficient allocations in the EU economy. By the parameterizations of Lemma 5 and Assumption 5, it can be identified with a function $a : M \to T$ via $X_i^{P^m} = Z_i^{a(m)}$. For each i, define $r_i : M \to \mathbb{R}$ by $r_i(m) = b_i(a(m), m)$. Then, $r_i(m)$ the expected utility level that consumer i enjoys under the first-order belief P^m at the conditional efficient allocation identified with a. If a is continuously differentiable, then so is r_i and, by the chain rule differentiation,

$$\nabla r_i(m) = \nabla_t b_i(a(m), m) Da(m) + \frac{\partial b_i}{\partial m}(a(m), m),$$

where $\nabla_t b_i(a(m), m) Da(m)$ is the product of an $(I-1)$-dimensional row (partial gradient) vector $\nabla_t b_i(a(m), m)$ and an $(I-1) \times 1$ Jacobian matrix $Da(m)$, and $\dfrac{\partial b_i}{\partial m}(a(m), m)$ is the partial derivative with respect to the parameter m.

Proposition 10 *Suppose that for each i, b_i satisfies Assumptions 5 and 6. Let $a : M \to T$ be continuously differentiable, and suppose that $r_i'(m) > 0$ for all i and m, where $r_i(m) = b_i(a(m), m)$. Then, for each i, there is a twice continuously differentiable function $\phi_i : r_i(M) \to \mathbb{R}$ s.th. $\phi_i' > 0$, $\phi_i'' \leq 0$, and the allocation $\left(X_i^{P^m}\right)_{m,i}$ defined by $X_i^{P^m} = Z_i^{a(m)}$ for all m and i is an efficient allocation in the KMM economy $(u_i, \phi_i, \mu)_{i \in I}$.*

In this proposition, the function ϕ_i is defined on the range $r_i(M) = \{r_i(m) \mid m \in M\}$, which is a strict subset of the range of u_i, $u_i(\mathbb{R}_{++})$. After the proof, we give a sufficient condition under which ϕ_i can be extended to

$u_i(\mathbb{R}_{++})$. We will also explain how the conclusion $\phi_i'' \leq 0$ can be strengthened to $\phi_i'' < 0$, which guarantees strict ambiguity aversion of all consumers.

Proof of Proposition 10 Let $s^0 \in S$ and define $W : M \to \mathbb{R}$ by

$$W(m) = \min_i \frac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m).$$

For each i, define $h_i : M \to \mathbb{R}$ by

$$h_i(m) = \frac{\dfrac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m) - W(m)}{r_i'(m)}$$

Then h_i is continuous and nonnegative-valued.

Since M is open and r_i is a continuously differentiable with strictly positive derivatives, $r_i(M)$ is an open interval in $\mathbb{R}$. Denote by $r_i^{-1} : r_i(M) \to M$ the inverse function of r_i. By the inverse function theorem, r_i^{-1} is continuously differentiable with strictly positive derivatives. Let $\phi_i : r_i(M) \to \mathbb{R}$ be s.th. $\phi_i'(y_i) > 0$, $\phi_i''(y_i) \leq 0$, and $-\phi_i''(y_i)/\phi_i'(y_i) = h_i\left(r_i^{-1}(y_i)\right)$ for every $y_i \in r_i(M)$. Such a ϕ_i indeed exists. We can let $y_i^0 \in r_i(M)$ and define $\phi_i'(y_i) = \exp\left(-\displaystyle\int_{y_i^0}^{y_i} h_i\left(r_i^{-1}(z_i)\right) dz_i\right)$.

By (6), it suffices to prove that for all m and s

$$\frac{\phi_i'\left(E^{P^m} u_i\left(X_i^{P^m}\right)\right) u_i'\left(X_i^{P^m}(s)\right)}{\phi_i'\left(E^{P^{m^0}} u_i\left(X_i^{P^{m^0}}\right)\right) u_i'\left(X_i^{P^{m^0}}(s^0)\right)} \tag{38}$$

is independent of i. Since $\left(X_i^{P^m}\right)_{m,i}$ is conditionally efficient, for each m,

$$\frac{u_i'\left(X_i^{P^m}(s)\right)}{u_i'\left(X_i^{P^m}(s^0)\right)} \tag{39}$$

is independent of i. By dividing (38) by (39), we see that it is sufficient to show that (38) is independent of i when $s = s^0$. By noting that when $s = s^0$, the numerator of (38) can be written as $\phi_i'\left(r_i(m)\right) u_i'\left(Z_i^{a(m)}(s^0)\right)$, and taking the logarithm and differentiate with respect to m, we obtain

$$\frac{\phi_i''(r_i(m))}{\phi_i'(r_i(m))} r_i'(m) + \frac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m).$$

It is sufficient to show that it is a function of m that is independent of i. By definition, for each i, this is equal to

$$\begin{aligned}
& - h_i(m) r_i'(m) + \frac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m) \\
= & - \frac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m) + W(m) + \frac{u_i''\left(Z_i^{a(m)}(s^0)\right)}{u_i'\left(Z_i^{a(m)}(s^0)\right)} \nabla_t Z_i^{a(m)}(s^0) Da(m) \\
= & W(m),
\end{aligned}$$

which is independent of i. $\square$

We make three remarks on the above proof. First, the function ϕ_i was defined on $r_i(M)$, which is a strict subset of the range of u_i, $u_i(\mathbb{R}_{++})$, because the range of $Z_i^{\cdot}$, $\{Z_i^{a(m)} \in \mathbb{R}_{++}^S \mid m \in M\}$, is bounded from above by $\bar{X}$. Thus, we should extend the domain of ϕ_i to $u_i(\mathbb{R}_{++})$, while maintaining its continuous differentiability, strict increasingness, and (strict) concavity. This is possible if the ranges $\phi_i'(r_i(M))$ and $\phi_i''(r_i(M)))$ are bounded. These conditions will be met if $\inf\{Z_i^{a(m)}(s) \mid m \in M\} > 0$ for all i and s. (If $\inf\{Z_i^{a(m)}(s) \mid m \in M\} = 0$, then $\inf r_i(M) = \inf u_i(\mathbb{R}_{++})$ and there will be no need to extend the domain of ϕ_i downwards.) Second, h_i is nonnegative valued. But, by considering, for example, $h_i(m)+1$ in place of $h_i(m)$, we can make it strictly-positive-valued, and this would not require any modification in the subsequent argument. Thus, we can guarantee that $\phi_i'' < 0$. Third, the construction of ϕ_i does not depend on the common second-order belief μ, except that every element P of $\operatorname{supp}\mu$ has a full support (that is, $P(s) > 0$ for all $P \in \operatorname{supp}\mu$ and s). This is consistent with an easy-to-prove fact that every efficient allocation under a common second-order belief is also an efficient under another common second-order belief as long as both second-order beliefs satisfy the full-support condition.

Online Appendix 2: Strict log-supermodularity

In this Appendix, we give a general result on strict log-supermodularity (SLSPM for short) from which part 2 of Proposition 7 and part 2 of Proposition 8 can be derived.

Let N be a positive integer. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$ and each $y = (y_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x \geq y$ when $x_n \geq y_n$ for every n. We also write $x \vee y = (\max\{x_n, y_n\})_{n=1,2,\ldots,N}$ and $x \wedge y = (\min\{x_n, y_n\})_{n=1,2,\ldots,N}$. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x_{-N} = (x_n)_{n=1,2,\ldots,N-1} \in \mathbb{R}^{N-1}$. By a slight abuse of notation, we use $\geq$, $\leq$, $\vee$, and $\wedge$ for vectors in $\mathbb{R}^{N-1}$ as well.

Let $f : \mathbb{R}^N \to \mathbb{R}_+$. We say that f is *strictly log-supermodular* (SLSPM for short) if

$$f(x)f(y) < f(x \vee y)f(x \wedge y)$$

for every $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ unless $x \leq y$ or $x \geq y$. That is, the strict log-supermodularity is a stronger property than the log-supermodularity (LSPM) in that the left-hand side is strictly smaller than the right-hand side. If $x \leq y$ or $x \geq y$, then $\{x, y\} = \{x \vee y, x \wedge y\}$ and the left- and right-hand sides would necessarily be equal. The constraint that neither should hold is needed to exclude this case. If $f(x) > 0$ for every $x \in \mathbb{R}^N$, then f is SLSPM if and only if $\ln f$ is strictly supermodular in the sense of Topkis (1998, Section 2.6.1).

Throughout this Appendix, we assume, for every $f : \mathbb{R}^N \to \mathbb{R}_+$ under consideration, that f is differentiable and $f(x) > 0$ for every $x \in \mathbb{R}^N$.

The first part of the following result is stated in Topkis (1998, Section 2.6.1). The second part can be proved in an analogues manner. The proof is omitted.

Lemma 6 *1. f is LSPM if and only if, for all n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a nondecreasing function of x_m.*

2. f is SLSPM if, for every n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a strictly increasing function of x_m.

The following proposition is the main result of this Appendix. It underlies the two propositions on the pricing kernel when the representative consumer exhibits strictly decreasing relative ambiguity aversion.

Proposition 11 *Suppose that for all $m < N$ and n, $\partial \ln f(x)/\partial x_m$ is non-decreasing in x_n, and strictly increasing in x_n if $n = N$. Define $g : \mathbb{R}^{N-1} \to$*

$\mathbb{R}_{++}$ *by* $g(x_{-N}) = \int_{\mathbb{R}} f(x_{-N}, x_N)\,\mathrm{d}x_N$ *for every* $x_{-N} \in \mathbb{R}^{N-1}$. *Then* g *is SLSPM.*

The assumptions of this proposition imply that f is LSPM but not that f is SLSPM. In fact, they can be met even when f is not SLSPM. The proposition, thus, implies that g can be SLSPM even when f is not. For a twice continuously differentiable f, they are satisfied if, for every $x \in \mathbb{R}^N$,

$$\frac{\partial^2}{\partial x_m \partial x_N} \ln f(x) > 0$$

for every $m < N$, and

$$\frac{\partial^2}{\partial x_m \partial x_n} \ln f(x) \geq 0$$

for all $m < N$ and $n \neq m$.

The following proof method is essentially due to Karlin and Rinott (1980, Theorem 2.1). We only need to take special care of preserving strict inequalities under integration.

Proof of Proposition 11 By Fubini's theorem,

$$\begin{aligned}
&g(x_{-N})g(y_{-N}) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}w\mathrm{d}z \\
&= \int_{\mathbb{R}\times\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}(z, w) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z=w\}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}(z, w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z<w\}} (f(x_{-N}, z) f(y_{-N}, w) + f(y_{-N}, w) f(x_{-N}, z))\,\mathrm{d}(z, w).
\end{aligned} \tag{40}$$

We can similarly show that

$$\begin{aligned}
&g(x_{-N} \vee y_{-N})g(x_{-N} \wedge y_{-N}) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}|z=w\}} f(x_{-N} \vee y_{-N}, z)f(x_{-N} \wedge y_{-N}, w)\,\mathrm{d}(z,w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}|z<w\}} (f(x_{-N} \vee y_{-N}, z)f(y_{-N} \wedge y_{-N}, w) \\
&\qquad\qquad + f(x_{-N} \vee y_{-N}, w)f(x_{-N} \wedge y_{-N}, z))\,\mathrm{d}(z,w). \qquad (41)
\end{aligned}$$

When $z = w$, $(x_{-N}, z) \vee (y_{-N}, w) = (x_{-N} \vee y_{-N}, z)$ and $(x_{-N}, z) \wedge (y_{-N}, w) = (x_{-N} \wedge y_{-N}, w)$. Since f is LSPM,

$$f(x_{-N}, z)f(y_{-N}, w) \leq f(x_{-N} \vee y_{-N}, z)f(x_{-N} \wedge y_{-N}, w).$$

Thus, the first term of the right-hand side of (40) is less than or equal to that of (41).

To compare that second terms, assume that $z < w$ and that it is false that $x_{-N} \leq y_{-N}$. Write

$$\begin{aligned}
A(z,w) &= f(x_{-N}, z)f(y_{-N}, w), \\
B(z,w) &= f(x_{-N}, w)f(y_{-N}, z), \\
C(z,w) &= f(x_{-N} \vee y_{-N}, z)f(y_{-N} \wedge y_{-N}, w), \\
D(z,w) &= f(x_{-N} \vee y_{-N}, w)f(x_{-N} \wedge y_{-N}, z).
\end{aligned}$$

Note first that

$$\begin{aligned}
A(z,w)B(z,w) &= (f(x_{-N}, z)f(y_{-N}, z))\,(f(x_{-N}, w)f(y_{-N}, w)) \\
&\leq (f(x_{-N} \vee y_{-N}, z)f(x_{-N} \wedge y_{-N}, z))\,(f(x_{-N} \vee y_{-N}, w)f(y_{-N} \wedge y_{-N}, w)) \\
&= C(z,w)D(z,w).
\end{aligned}$$

Next, without loss of generality, we can assume that there is an M with $1 \leq M < N$ s.th. $x_n > y_n$ if and only if $n \leq M$. Then,

$$\begin{aligned}
x_{-N} \vee y_{-N} &= (x_1, \ldots, x_M, y_{M+1}, \ldots, y_{N-1}), \\
x_{-N} \wedge y_{-N} &= (y_1, \ldots, y_M, x_{M+1}, \ldots, x_{N-1}).
\end{aligned}$$

Moreover,

$$x_{-N} - x_{-N} \wedge y_{-N} = x_{-N} \vee y_{-N} - y_{-N} = (x_1 - y_1, \ldots, x_M - y_M, 0, \ldots, 0).$$

Denote this by v. For each $m \leq M$, write

$$v^m = (x_1 - y_1, \ldots, x_m - y_m, 0, \ldots, 0).$$

Then $v^M = v$, $v^0 = 0$, and

$$v^m - v^{m-1} = (0, \ldots, 0, x_m - y_m, 0, \ldots, 0).$$

Write $h = \ln f$. Then, for every $m \leq M$

$$\begin{aligned}
&h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) \\
&= \int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, x_{M+1} \ldots, x_{N-1}, z)\,\mathrm{d}r, \\
&h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w) \\
&= \int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, y_{M+1} \ldots, y_{N-1}, w)\,\mathrm{d}r.
\end{aligned}$$

Since $\partial h / \partial x_m$ is nondecreasing in x_n with $n = M+1, \ldots, N-1$ and strictly increasing in x_N,

$$\begin{aligned}
&\frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, x_{M+1} \ldots, x_{N-1}, z) \\
&< \frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, y_{M+1} \ldots, y_{N-1}, w)
\end{aligned}$$

for every r. Thus,

$$h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) < h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w).$$

Since $x_{-N} \wedge y_{-N} + v^M = x_{-N}$ and $y_{-N} + v^M = x_{-N} \vee y_{-N}$, by taking the summation of each side over $m \leq M$, we obtain

$$h(x_{-N}, z) - h(x_{-N} \wedge y_{-N}, z) < h(x_{-N} \vee y_{-N}, w) - h(y_{-N}, w).$$

That is, $A(z, w) < D(z, w)$.

By swapping the roles of x_{-N} and y_{-N} (while maintaining the assumption that $z < w$), we can show that $B(z, w) < D(z, w)$.

Since $A(z, w)B(z, w) \leq C(z, w)D(z, w)$, $A(z, w) < D(z, w)$, $B(z, w) < D(z, w)$, and

$$\begin{aligned}
&(C(z, w) + D(z, w)) - (A(z, w) + B(z, w)) \\
&= \frac{1}{D(z, w)} \left((C(z, w)D(z, w) - A(z, w)B(z, w)) + (D(z, w) - A(z, w))(D(z, w) - B(z, w)) \right),
\end{aligned}$$

we have $A(z,w)+B(z,w)<C(z,w)+D(z,w)$. Since the second term of the right-hand side of (40) is nothing but the integral of $A(z,w)+B(z,w)$ on $\{(z,w)\in\mathbb{R}\times\mathbb{R}\mid z<w\}$ and that of (41) is nothing but the integral of $C(z,w)+D(z,w)$ on the same domain, this completes the proof. $\square$

This proposition can be extended to the case in which the domain of the function is $X_1\times X_2\times\cdots\times X_N$, where X_n is an interval in $\mathbb{R}$ for every n.

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