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NOTES, COMMENTS, AND LETTERS TO THE EDITOR

Beliefs and Pareto Efficient Sets: A Remark[1]

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We show that, in a two-period economy with uncertainty in the second period, if an allocation is Pareto optimal for a given set of beliefs and remains optimal when these beliefs are changed, then the set of optimal allocations of the two economies must actually coincide. We identify equivalence classes of beliefs, giving rise to the same set of Pareto optimal allocations. *Journal of Economic Literature* Classification Numbers: D51, D61.
Key Words: beliefs; Pareto optimality.

1. INTRODUCTION

In this Note, we seek to answer a very simple question: what can we learn about agents' beliefs by the sole knowledge that a given allocation is Pareto optimal? More specifically, consider a multiple-goods, two-period

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economy with uncertainty in the second period and agents that are subjective expected utility maximizers. Take a Pareto optimal allocation of this economy. Is it possible that this allocation still be Pareto optimal in an economy in which agents' beliefs have changed?

We answer this question affirmatively and actually identify the exact change of beliefs needed. The result we obtain is actually stronger: if agent h's subjective probability of state s divided by that of state s' in the second economy (i.e., the economy after beliefs have changed) is proportional (with the same coefficient of proportionality for all the agents) to the same ratio in the initial economy, then, the *set* of Pareto optimal allocations is the same in those two economies. We furthermore show that this is equivalent to the two sets of Pareto optimal allocations having one (interior) point in common. Hence, two contract curves associated to two economies with different beliefs are either equal or disjoint.

To the best of our knowledge, this point, as simple as it seems, has not been studied in the literature. In a sense, the class of probabilities we identify is similar to what Radner [2] called "confounding" probabilities in a (rational expectations) equilibrium set-up, since in our set-up, two such sets of beliefs lead to the same Pareto optimal set.

2. THE SET UP AND MAIN RESULT

We consider a standard two-period economy with uncertainty in the second period. There are H agents, $h = 1, \ldots, H$ and C commodities, $c = 1, \ldots, C$, in each spot market. Without loss of generality we assume that there is no consumption in the first period. Uncertainty is represented by a state space $\mathscr{S} = \{1, \ldots, S\}$, with $s \in \mathscr{S}$ a state of nature. Total contingent endowments are given by $e = (e(1), \ldots, e(S)) \in \mathbb{R}^{CS}_{++}$.

Agents are subjective expected utility maximizers with beliefs $\pi_h = (\pi_h(1), \ldots, \pi_h(S))$. It is assumed that $\pi_h(s) > 0$ $\forall s \in \{1, \ldots, S\}$ and, naturally that $\sum_s \pi_h(s) = 1$ for all h. Agent h has consumption set $\mathbb{R}^{CS}_{++}$, certainty preferences represented by the von Neumann–Morgenstern utility index $u_h\colon \mathbb{R}^C_{++} \to \mathbb{R}$. u_h is assumed to be twice continuously differentiable, differentiably strictly increasing (i.e., $\nabla u_h(x) \gg 0$ for $x \gg 0$) and differentiably strictly concave (i.e., $\Delta x^t\, \nabla^2 u_h(x)\, \Delta x < 0$ for $x \gg 0$, $\Delta x \neq 0$), and to have indifference surfaces with closures in $\mathbb{R}^C_{++}$. Finally, the household evaluates its contingent consumption plan, represented by the vector $x_h = (x_h(1), \ldots, x_h(S)) \in \mathbb{R}^{CS}_{++}$, according to the von Neumann–Morgenstern functional $V_h(x_h(1), \ldots, x_h(S)) = \sum_s \pi_h(s)\, u_h(x_h(s))$.

An allocation $x = (x_1, \ldots, x_H)$ is feasible if $x_h(s) \gg 0$ for all h and all s and $\sum_{h=1}^H x_h(s) = e(s)$ for all s. An allocation x is Pareto optimal if there is no other feasible allocation x' such that $V_h(x'_h) \geqslant V_h(x_h)$ for all h and $V_h(x'_h) > V_h(x_h)$ for some h.

In this note, we take the von Neumann–Morgenstern utility indices and total endowments to be fixed and allow changes in agents' beliefs. Let $P(\pi)$ be the set of Pareto optima of the economy where agents have beliefs $\pi = (\pi_1, ..., \pi_H)$. Recall that in our simple setup[3] an allocation x is a Pareto optimal allocation if and only if there exists a vector of weights $\lambda = (\lambda_1, ..., \lambda_H) \gg 0$ such that x is a solution to the problem $\max \sum_{h=1}^{H} \lambda_h \sum_{s=1}^{S} \pi_h(s)\, u_h(x_h(s))$ s.t. $\sum_{h=1}^{H} x_h(s) = e(s)$ for all s and $x_h \gg 0$ for all h.

The main result of this note is to compare the set of Pareto optimal allocations in two economies differing only by the agents' beliefs.

PROPOSITION 1. *The following three assertions are equivalent:*

(i) $P(\pi) = P(\hat{\pi})$

(ii) $P(\pi) \cap P(\hat{\pi}) \neq \varnothing$

(iii) $\forall h, h', \forall s, s', \frac{\pi_h(s)/\pi_h(s')}{\pi_{h'}(s)/\pi_{h'}(s')} = \frac{\hat{\pi}_h(s)/\hat{\pi}_h(s')}{\hat{\pi}_{h'}(s)/\hat{\pi}_{h'}(s')}$

Proof. Recall first the following lemma (see, e.g., [1]):

LEMMA. *A feasible allocation x is Pareto optimal if and only if there exist positive weights, $\lambda_h > 0$, all h, and strictly positive contingent goods prices (multipliers) for each state, $\mu(s) \gg 0$, all s, such that*

$$\lambda_h \pi_h(s)\, \nabla u_h(x_h(s)) = \mu(s), \qquad \textit{all } h, s.$$

Let us now prove our result.

That (i) implies (ii) is trivial.

(ii) $\Rightarrow$ (iii). Assume that $P(\pi) \cap P(\hat{\pi}) \neq \varnothing$ and pick a feasible allocation x in $P(\pi) \cap P(\hat{\pi})$. Then, there exist $\lambda = (\lambda_1, ..., \lambda_H) \gg 0$ and $\hat{\lambda} = (\hat{\lambda}_1, ..., \hat{\lambda}_H) \gg 0$ as well as $\mu = (\mu(1), ..., \mu(S))$ and $\hat{\mu} = (\hat{\mu}(1), ..., \hat{\mu}(S))$ such that, for all h, h' and all s:

$$\lambda_h \pi_h(s)\, \nabla u_h(x_h(s)) = \lambda_{h'} \pi_{h'}(s)\, \nabla u_{h'}(x_{h'}(s)) = \mu(s)$$

$$\hat{\lambda}_h \hat{\pi}_h(s)\, \nabla u_h(x_h(s)) = \hat{\lambda}_{h'} \hat{\pi}_{h'}(s)\, \nabla u_{h'}(x_{h'}(s)) = \hat{\mu}(s).$$

Hence,

$$\frac{\lambda_h \pi_h(s)}{\lambda_{h'} \pi_{h'}(s)} = \frac{\hat{\lambda}_h \hat{\pi}_h(s)}{\hat{\lambda}_{h'} \hat{\pi}_{h'}(s)}, \qquad \forall h, h', s.$$

[3] See for instance [1].

Therefore, for all s, s', h and h',

$$\frac{\pi_h(s)}{\pi_{h'}(s)}\frac{\hat{\pi}_{h'}(s)}{\hat{\pi}_h(s)}=\frac{\pi_h(s')}{\pi_{h'}(s')}\frac{\hat{\pi}_{h'}(s')}{\hat{\pi}_h(s')}$$

proving (iii).

(iii) ⇒ (i). Let $x \in P(\pi)$. Then, by the lemma, there exists a vector of weights $\lambda=(\lambda_1, ..., \lambda_H) \gg 0$ and multipliers (contingent goods prices) $\mu=(\mu(1), ..., \mu(S)) \in \mathbb{R}^{CS}_{++}$ such that, for all h and all s,

$$\lambda_h \pi_h(s)\, \nabla u_h(x_h(s))=\mu(s). \tag{1}$$

Now, by assumption,

$$\frac{\pi_h(s)/\pi_h(1)}{\pi_1(s)/\pi_1(1)}=\frac{\hat{\pi}_h(s)/\hat{\pi}_h(1)}{\hat{\pi}_1(s)/\hat{\pi}_1(1)}$$

for all h and all s. Hence, (1) is equivalent to

$$\lambda_h \frac{\pi_h(1)}{\hat{\pi}_h(1)}\hat{\pi}_h(s)\, \nabla u_h(x_h(s))=\frac{\pi_1(1)}{\pi_1(s)}\frac{\hat{\pi}_1(s)}{\hat{\pi}_1(1)}\mu(s)$$

for all h and all s. Therefore, defining $\hat{\lambda}_h=\lambda_h \frac{\pi_h(1)}{\hat{\pi}_h(1)}$ and $\hat{\mu}(s)=\frac{\pi_1(1)}{\pi_1(s)}\frac{\hat{\pi}_1(s)}{\hat{\pi}_1(1)}\mu(s)$, we get that

$$\hat{\lambda}_h \hat{\pi}_h(s)\, \nabla u_h(x_h(s))=\hat{\mu}(s)$$

for all h and all s. Since $\hat{\lambda}_h>0$ and $\hat{\mu}(s)>0$, this establishes (by the lemma above) that $x \in P(\hat{\pi})$. Therefore, $P(\pi) \subseteq P(\hat{\pi})$. The converse inclusion also holds by a symmetric argument. Hence $P(\pi)=P(\hat{\pi})$. ∎

Observe that condition (iii) in the proposition does not imply that $\pi_h=\hat{\pi}_h$ for all h, as shown by the following example: $H=2$, $S=2$, and $\pi_1(1)=\frac{1}{4}$, $\pi_2(1)=\frac{1}{3}$, $\hat{\pi}_1(1)=\frac{13}{16}$ and $\hat{\pi}_2(1)=\frac{13}{15}$.

To interpret condition (iii), observe that the ratio $\frac{\pi_h(s)}{\pi_h(s')}$ is simply the marginal rate of substitution, say of good 1, between state s and s' when agent h is risk neutral (i.e., has a linear utility index). Alternatively, it is the marginal rate of substitution between state s and s' at points where the consumer is fully insured, i.e., consumes the same bundle in each of these states.

Remark 1. If we were to take $\mathbb{R}^{CS}_{+}$ rather than $\mathbb{R}^{CS}_{++}$ as households' consumption set and extend the domain of the utility function accordingly, the same result would continue to hold, in which condition (ii) is replaced by $P(\pi) \cap P(\hat{\pi}) \cap \mathbb{R}^{CSH}_{++} \neq \varnothing$.

Remark 2. The framework developed can be reinterpreted in an intertemporal setting, with time-separable, time-independent preferences. Indeed, interpreting s as a time index and writing $\pi_h(s) = (\beta_h)^s$ where β_h is h's stationary discount factor, our result says that if the discount factor changes but the ratios of the discount factors for any two agents remain the same, then the two economies have the same Pareto optima.

REFERENCES

1. D. Cass, G. Chichilnisky, and H.-M. Wu, Individual risk and mutual insurance, *Econometrica* **64** (1996), 333–341.
2. R. Radner, Rational expectations equilibrium: Generic existence and the information revealed by prices, *Econometrica* **47** (1979), 655–678.