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Sharing beliefs and the absence of betting in the Choquet expected utility model

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Abstract Choquet expected utility maximizers tend to behave in a more "cautious" way than Bayesian agents, i.e. expected utility maximizers. We illustrate this phenomenon in the particular case of betting behavior. Specifically, consider agents who are Choquet expected utility maximizers. Then, if the economy is large, Pareto optimal allocations provide full insurance if and only if the agents share at least one prior, *i.e.*, if the intersection of the core of the capacities representing their beliefs is non empty. In the expected utility case, this is true only if they have a common prior.

Keywords: Betting, Choquet expected utility, full insurance, Pareto optimality

1 Introduction

It is widely believed that the principal source of betting is differences in beliefs. As in the classical (theoretical) example of horse lotteries, people who do not agree on probability assessments will find it mutually beneficial to engage in uncertainty-generating trade. A less widely noted fact is

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that, in a Bayesian setting, any disagreement induces betting. Hence, in the Bayesian model of expected utility, any difference in beliefs leads agents to bet against each other. Optimality dictates either that there be no betting (in case beliefs are common to all agents) or that there be betting (in case of disagreement).

This, we argue, is difficult to reconcile with the fact that very little betting is observed on all the possible sources of uncertainty that we face. In this paper we argue that vagueness of the beliefs might be an explanation for this observed rarity of betting. While it is not our aim to explain the full complexity of betting behavior by the type of models we study here, we are led to ask, how much can be explained by these models if we relax some of the more demanding assumptions of the Bayesian model.

Specifically, consider Schmeidler's non-additive (Choquet) expected utility (Schmeidler [1989]) that capture Knightian uncertainty (Knight [1921]). Considering not necessarily convex capacities, we provide a characterization of capacities whose cores have a non-empty intersection. We show that if there is a prior that belongs to that intersection, then all optimal allocations provide full insurance. But it may be the case that the cores of the capacities do not intersect, yet some and even all optimal allocations provide full insurance. Yet, if the economy is "replicated", i.e., if we consider a continuum of agents of each type, the equivalence result is reinstated. Thus, for large economies commonality of beliefs is still necessary and sufficient for some, or all Pareto optimal allocations to entail full insurance.

In Billot, Chateauneuf, Gilboa and Tallon [2000] we treat the same problem in the multiple prior model of Gilboa and Schmeidler [1989], which includes the Choquet expected utility model with convex capacities. The results in the present paper cannot be deduced from results in the multiple prior model as no convexity assumptions on the capacities are made here.

The rest of this paper is organized as follows. Section 2 provides the set up of the model. In Section 3 we state the main result. Proofs are gathered in an appendix.

2 Set-up

The economy we consider is a two-period pure-exchange economy with uncertainty in the second period, and Choquet expected utility maximizers.

There are k possible states of the world in the second period, indexed by superscript j. Let S be the set of states of the world and $\mathcal{A}$ the set of subsets of S. There are n agents indexed by subscript i. We assume there is only one good, say money. C_i^j is the consumption by agent i in state j and $C_i = (C_i^1, \ldots, C_i^k)$. Initial endowments are denoted $w_i = (w_i^1, \ldots, w_i^k)$. Denote by $w^j = \sum_{i=1}^n w_i^j$ the aggregate endowment in state j. We make the assumption that there is no aggregate uncertainty throughout the paper. More formally, $w^j = w^{j'} \equiv w$ for all $j, j' \in S$. Thus trading an uncertain asset is interpreted as betting rather than as hedging.

An allocation $C = (C_1, \ldots, C_n)$ is feasible if $\sum_{i=1}^{n} C_i^j = w$ for all j. An allocation is interior if $C_i^j > 0$ for all i, all j. A feasible allocation is a full insurance allocation if $C_i^j = C_i^{j'}$ for all i and all j, j'. Finally, say that an allocation is Pareto optimal if it is not possible to improve an agent's welfare without decreasing another one's.

As we focus on Choquet expected utility agents, we assume the existence of a utility index $U_i : \mathbb{R}_+ \to \mathbb{R}$ that is cardinal, i.e., defined up to a positive affine transformation. Unless otherwise stated, U_i is taken to be differentiable, strictly increasing and strictly concave. Before formally defining Choquet expected utility model (CEU henceforth), we need to introduce the notion of a capacity and its core.

A capacity is a set function $\nu : \mathcal{A} \to [0,1]$ such that $\nu(\emptyset) = 0$, $\nu(S) = 1$, and, for all $A, B \in \mathcal{A}, A \subset B \Rightarrow \nu(A) \leq \nu(B)$. We will assume throughout that the capacities we deal with are such that $1 > \nu(A) > 0$ for all $A \in \mathcal{A}$, $A \neq S$, $A \neq \emptyset$. This technical restriction is needed in the proof of proposition 1, used in the proof of theorem 2, which relies on a result proved in Chateauneuf, Dana and Tallon [2000] requiring this restriction.

A capacity ν is convex if for all $A, B \in \mathcal{A}$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$.

The core of a capacity ν is defined as follows

$$\text{core}(\nu) = \left\{ \pi \in \mathbb{R}_+^k \mid \sum_j \pi^j = 1 \text{ and } \pi(A) \geq \nu(A), \ \forall A \in \mathcal{A} \right\}$$

where $\pi(A) = \sum_{j \in A} \pi^j$. Core($\nu$) is a compact, convex set which may be empty. Since $1 > \nu(A) > 0 \ \ \forall A \in \mathcal{A}$, $A \neq S$, $A \neq \emptyset$, any $\pi \in \text{core}(\nu)$ is such that $\pi \gg 0$.[1]

We now turn to the definition of the Choquet integral of $f \in \mathbb{R}^S$:

$$\int f d\nu \equiv E_\nu(f) = \int_{-\infty}^{0} (\nu(f \geq t) - 1)dt + \int_0^{\infty} \nu(f \geq t)dt$$

Hence, if $f^j = f(j)$ is such that $f^1 \leq f^2 \leq \ldots \leq f^k$:

$$\int f d\nu = \sum_{j=1}^{k-1} [\nu(\{j, \ldots, k\}) - \nu(\{j+1, \ldots, k\})] f^j + \nu(\{k\}) f^k$$

As a consequence, if we assume that an agent consumes C^j in state j, and that $C^1 \leq \ldots \leq C^k$, then her preferences are represented by:

$$\begin{aligned} V(C) = {} & [1 - \nu(\{2, .., k\})] \, U(C^1) + \ldots \\ & + [\nu(\{j, .., k\}) - \nu(\{j+1, .., k\})] \, U(C^j) + \ldots + \nu(\{k\}) U(C^k) \end{aligned}$$

It is well-known that when ν is convex, its core is non-empty (Shapley [1965]) and the Choquet integral of any random variable f is given by $\int f d\nu =$

[1] Say $\pi \gg 0$ if $\pi^j > 0$ for all j.

$\min_{\pi \in \mathrm{core}(\nu)} E_\pi f$ (Rosenmueller [1972] and Schmeidler [1986]). It is also well-known that when ν is not convex but has a non-empty core, $\int f d\nu \leq \min_{\pi \in \mathrm{core}(\nu)} E_\pi f$.

3 The main result

We show in this section that some commonality of beliefs is sufficient to ensure that the set of Pareto optimal allocations equals the set of full insurance allocations. The converse of this result is valid only if the economy is replicated an infinite number of times.

We start with a characterization of non-empty core intersection.

Theorem 1 *Let $\nu_1, \ldots, \nu_n$ be capacities. Then, the following are equivalent:*

(i) $\cap_i \mathrm{core}(\nu_i) \neq \emptyset$

(ii) For every collection of events and corresponding non-negative numbers,

$$\left(A_i^t\right)_{i=1,\ldots,n;t=1\ldots,t_i}, \ \left(\alpha_i^t\right)_{i=1,\ldots,n;t=1\ldots,t_i} \ \textit{satisfying} \ \sum_{i=1}^{n}\sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S,$$

it is true that $\sum_{i=1}^{n}\sum_{t=1}^{t_i} \alpha_i^t \nu_i\left(A_i^t\right) \leq 1.$

(iii) It is impossible to find events A_i^t, and numbers $\alpha_i^t \geq 0$ and $\beta_i^t \geq 0$, $i = 1, \ldots, n$, $t = 1 \ldots, t_i$ such that:

$$(\star) \begin{cases} \alpha_i^t \nu_i\left(A_i^t\right) \geq \beta_i^t \ \ \forall i, t \ \ \textit{and} > \textit{for some } i, t \\ \sum_{i=1}^{n}\sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S \\ \sum_{i=1}^{n}\sum_{t=1}^{t_i} \beta_i^t \geq 1 \end{cases}$$

The equivalence $(i) \Longleftrightarrow (ii)$ is a generalization of the Shapley-Bondareva (Bondareva [1963], Shapley [1967]) theorem characterizing non-emptiness of the core of a single capacity (transferable utility cooperative game in their context).[2] Part (iii) is merely a re-statement of part (ii) that helps elucidating its economic interpretation: assume that (iii) does not hold, that is, that there are A_i^t, α_i^t and β_i^t satisfying $(\star)$. The condition $\alpha_i^t \nu_i(A_i^t) \geq \beta_i^t$ says that an agent with beliefs ν_i and a linear utility function will be willing to trade a bet promising a payoff α_i^t if A_i^t occurs, for a sure payment of β_i^t. If this condition holds with strict inequality for some i, t, then for a

[2] Recall that a game (capacity) ν, has a non-empty core if and only if it is balanced, that is, for every collection of coalitions (events) and corresponding non-negative numbers, $\left(A^t\right)_{t=1\ldots,T}$, $\left(\alpha^t\right)_{t=1\ldots,t}$ satisfying $\sum_{t=1}^{T} \alpha^t \mathbf{1}_{A^t} = \mathbf{1}_S$, it is true that $\sum_{t=1}^{T} \alpha^t \nu\left(A^t\right) \leq 1$.

small enough sum the agent will prefer taking this bet to a full insurance allocation even for a strictly concave utility function.

The second condition in ($\star$) states that the bets α_i^t-on-A_i^t offered to the agents, add up to a risk-free investment that costs 1 unit at every state of the world. Finally, the third condition implies that the payments made by the agents for the various bets, β_i^t add up to 1 or more.

Taken together, it is tempting to endow part (*iii*) with a no-Dutch-book interpretation: should it fail to hold, a bookie could offer a bet α_i^t-on-A_i^t to individual i for a sum β_i^t. Since a strict inequality holds for some i, t, the corresponding β_i^t can be slightly increased. Such a bookie will make a profit with certainty. Alternatively, one can let the individuals trade bets among themselves and show that, if a bookie could make a sure profit, the allocation was not Pareto optimal to begin with.

However, this interpretation is inaccurate. The inequality $\alpha_i^t \nu_i(A_i^t) \geq \beta_i^t$ implies that an agent with beliefs ν_i, *holding a risk free allocation*, would prefer to bet on A_i^t at these odds. But it does not mean that such an individual would take this bet after having taken other bets of this nature. If theorem 1 were true with $t_i \leq 1$ for all i, the problem would not arise. Also, if all agents had convex ν_i, one can show that successive bets may be replaced by a single bet (with several payoff levels). However, for ν_i that are not necessarily convex, the interpretation of condition (*iii*) is more subtle. Specifically, assume that there is a continuum of agents of each type i, where types are defined by (U_i, ν_i). Then, one may re-interpret condition (*iii*) as suggested above, where no single individual is asked to bet on more than one event.

To sum, a non-empty intersection of the cores will suffice to make every full insurance allocation Pareto optimal, and vice versa (proposition 1 below). But if the cores do not intersect, the two sets of allocations need not be disjoint. Moreover, they can be identical (examples 1 and 2). Intuitively, this follows from the fact that a Choquet expected utility maximizer who is characterized by a non-convex capacity is even more uncertainty averse than might be suggested by the core of her capacity.[3] However, with a continuum of agents of each type, the set of Pareto optimal allocations and of full insurance allocations are identical if the cores of ν_i have a non-empty intersection, and they are disjoint if this intersection is empty (theorem 2).

Proposition 1 *Let agents be CEU decision makers. Assume that* $\cap_i core(\nu_i) \neq \emptyset$*, then an allocation is Pareto optimal if and only if it provides full insurance.*

The fact that Pareto optimal allocations provide full insurance is proposition 7.2 in Chateauneuf, Dana and Tallon [2000]. The proof of the converse can be found in Dana [1998].

The converse to this result is not true when there are a finite number of agents, as the two following examples show.

[3] This does not mean that such a decision maker is more uncertainty averse in any qualitative sense than is a CEU maximizer with a convex capacity.

Example 1 Consider an economy with two states α and β. There are two agents with utility indices U_1 and U_2 that satisfy all the maintained assumptions. Assume $\nu_1(\alpha) = \nu_1(\beta) = 2/3$ and $\nu_2(\alpha) = \nu_2(\beta) = 1/4$. Note that $\text{core}(\nu_1) = \emptyset$ and thus, trivially, $\cap_i \text{core}(\nu_i) = \emptyset$. However, Pareto optimal allocations are full insurance allocations, and vice versa.

Indeed, suppose not and assume, w.l.o.g. $C_1^\alpha < C_1^\beta$ and therefore $C_2^\alpha > C_2^\beta$. We have:

$$V_1\left(C_1^\alpha, C_1^\beta\right) = \tfrac{1}{3}U_1\left(C_1^\alpha\right) + \tfrac{2}{3}U_1\left(C_1^\beta\right)$$
$$V_2\left(C_2^\alpha, C_2^\beta\right) = \tfrac{1}{4}U_2\left(C_2^\alpha\right) + \tfrac{3}{4}U_2\left(C_2^\beta\right)$$

Let $\bar{C}_1 = \dfrac{1}{4}C_1^\alpha + \dfrac{3}{4}C_1^\beta$ and $\bar{C}_2 = \dfrac{1}{4}C_2^\alpha + \dfrac{3}{4}C_2^\beta$. The allocation giving $\bar{C}_i$ to agent i $(i = 1, 2)$ in both states is obviously feasible. Furthermore,

$$\begin{aligned} V_1\left(\bar{C}_1, \bar{C}_1\right) = U_1(\bar{C}_1) &> \frac{1}{4}U_1(C_1^\alpha) + \frac{3}{4}U_1(C_1^\beta) \\ &> \frac{1}{3}U_1(C_1^\alpha) + \frac{2}{3}U_1(C_1^\beta) = V_1\left(C_1^\alpha, C_1^\beta\right) \end{aligned}$$

Similarly,

$$V_2\left(\bar{C}_2, \bar{C}_2\right) = U_2(\bar{C}_2) > \frac{1}{4}U_2(C_2^\alpha) + \frac{3}{4}U_2(C_2^\beta) = V_2\left(C_2^\alpha, C_2^\beta\right)$$

Hence, giving $\bar{C}_i$ to agent i in both states Pareto dominates the allocation $\left((C_1^\alpha, C_1^\beta), (C_2^\alpha, C_2^\beta)\right)$. We can therefore conclude that, at a Pareto optimal allocation, agents are fully insured. Conversely, this also implies that any full insurance allocation is Pareto optimal. ◇

The next example does not rely on an agent's core being empty.

Example 2 Suppose there are three states and two agents. Agent 1's beliefs are represented by the following capacity: $\nu_1(j) = .1$, $j = 1, 2, 3$, $\nu_1(12) = .6$, $\nu_1(13) = \nu_1(23) = .7$ and $\nu_1(123) = 1$. Agent 2's beliefs are given by: $\nu_2(j) = .1$, $j = 1, 2, 3$, $\nu_2(23) = .6$, $\nu_2(12) = \nu_2(13) = .7$ and $\nu_2(123) = 1$.

It is easy to check that $\text{core}(\nu_1) = \{(.3, .3, .4)\}$ and $\text{core}(\nu_2) = \{(.4, .3, .3)\}$ and therefore they do not intersect.

It is also straightforward to see that these capacities are not convex. For instance, $\nu_1(13) + \nu_1(23) = 1.4 > \nu_1(123) + \nu_1(3) = 1$.

Now, start from a full-insurance allocation. To show that this is a Pareto optimal allocation, it is enough to show that there is no feasible trade, summing up to zero, that would improve both agents' welfare. This can be checked directly as follows. Let x^j be the amount received by agent 1 in state j. To ensure feasibility, the amount received by agent 2 must equal $-x^j$. A tedious but straightforward computation[4] shows that it is impossible to find

[4] One essentially has to check all six possible orders on (x^1, x^2, x^3).

triplets $x = (x^1, x^2, x^3)$ such that $E_{\nu_1} x \geq 0$ and $E_{\nu_2}(-x) \geq 0$, unless both inequalities are satisfied as equalities.

Similarly, one can prove that all Pareto optimal allocations are full insurance. Indeed, any allocation C is (weakly) preferred by both agents to its order permutation (satisfying $C_1^1 \leq C_1^2 \leq C_1^3$), and if $C_1^3 > C_1^1$, this allocation will be Pareto improved upon by smoothing consumption across states.

Observe finally that if the capacities were convex, with the same capacity for non-singleton sets, then the agents would give some weight to singletons. Agent 1, for instance, would have to give weight at least .4 to state 3, and this would allow a full insurance allocation to be improved upon. $\diamond$

The two examples above show that the converse to proposition 1 is not true. However, if we replicate the economy, non-empty core intersection becomes a necessary condition ensuring that Pareto optimal allocations are full insurance allocations.

For the next theorem we assume that there is a continuum $(i-1, i]$ of agents of type i, who share a capacity ν_i, a utility U_i and an initial endowment w_i. An allocation is a measurable function from the set of individuals $(0, n]$ to $\mathbb{R}_+$. For simplicity, we consider only allocations that assume finitely many values.

Theorem 2 *Under the above conditions, the following are equivalent:*

(i) There exists an interior Pareto optimal allocation C such that $C_i^j = C_i^{j'}$ for all i and all j, j'.

(ii) At any Pareto optimal allocation C, $C_i^j = C_i^{j'}$ for all i and all j, j'.

(iii) Every allocation C with $C_i^j = C_i^{j'}$ for all i and all j, j' is Pareto optimal

(iv) $\cap_i \mathit{core}(\nu_i) \neq \emptyset$

4 Concluding comments

In this paper, we have argued that vagueness of the beliefs can partially explain why we observe little betting on different sources of uncertainty, such as for instance the arrival time of a bus, the number of red cars and so on. Our result is based on the assumption that agents behave as Choquet expected utility maximizers, a generalization of the standard expected utility model. This model allows to represent agents' beliefs via a non-additive probability or capacity. We showed that if there exist additive beliefs compatible with all the agents' capacities, then the absence of betting is optimal. The converse is true under the assumption that there is a continuum of agents.

Appendix

Proof of theorem 1:
We first establish that $(i) \iff (ii)$.

Consider the program (in which α_i^t and γ are multipliers):

$$\begin{array}{ll} \text{Min} & 0.p \\ \text{s.t.} & \left\{ \begin{array}{lll} \sum_{j=1}^k p^j \delta_{j \in A_i^t} \geq \nu_i(A_i^t) & \forall A_i^t \in \mathcal{A} & (\alpha_i^t) \\ \sum_{j=1}^k p^j = 1 & j = 1, \ldots, k & (\gamma) \\ p^j \geq 0 & & \end{array} \right. \end{array}$$

Observe that this program is feasible if and only if $\cap_i \text{core}(\nu_i) \neq \emptyset$. Consider now its dual:

$$\begin{array}{ll} \text{Max}_{\alpha_i^t, \gamma} & \sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) + \gamma \\ \text{s.t.} & \left\{ \begin{array}{ll} \sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \delta_{j \in A_i^t} + \gamma \leq 0 \; j = 1, \ldots, k & (p^j) \\ \alpha_i^t \geq 0 & \end{array} \right. \end{array}$$

The primal is feasible if and only if the dual is bounded. We claim that the dual is bounded if and only if it is bounded by 0. Indeed, suppose it is not bounded by 0, i.e., there exists $\alpha_i^t \geq 0$ and γ such that $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) + \gamma > 0$ and $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} + \gamma \mathbf{1}_S \leq 0$. Then, $(\lambda \alpha_i^t, \lambda \gamma)$ is feasible and yields a larger amount if $\lambda > 1$, a contradiction.

Hence, the primal is feasible if and only if for all α_i^t and γ such that $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} + \gamma \mathbf{1}_S \leq 0$, one has $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) + \gamma \leq 0$.

One can normalize $\gamma = -1$ without loss of generality. Indeed, if $\gamma > 0$, the condition $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} + \gamma \mathbf{1}_S \leq 0$ cannot hold with $\alpha_i^t \geq 0$. If $\gamma = 0$, then $\alpha_i^t = 0$ for all i and t and the condition is trivially satisfied. If $\gamma < 0$, then the program is homogeneous in α and γ and there is no loss of generality in setting $\gamma = -1$.

Hence, this establishes that:

$$\cap_i \text{core}(\nu_i) \neq \emptyset \iff \Big[\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} \leq \mathbf{1}_S \implies \sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) \leq 1 \Big]$$

Finally, since $\alpha_i^t \geq 0$ and all events A^t are considered, the condition in the square brackets holds if and only if it holds whenever its antecedent holds as an equality, i.e.,

$$\cap_i \text{core}(\nu_i) \neq \emptyset \iff \Big[\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S \implies \sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) \leq 1 \Big]$$

We now establish that $(ii) \iff (iii)$.

First, suppose (ii) is true but there exists (A_i^t), (β_i^t), (α_i^t), $i = 1, \ldots, n$ and $t = 1 \ldots, t_i$ such that $(\star)$ holds. Then, $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S$ and $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) > \sum_{i=1}^n \sum_{t=1}^{t_i} \beta_i^t \geq 1$, a contradiction.

To prove the converse, assume (*ii*) is not true. Then, there exist (A_i^t), (α_i^t) such that $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S$ and $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) > 1$. Now choose, in violation of (*iii*), β_i^t such that:

$$\begin{cases} 0 \leq \beta_i^t < \alpha_i^t \nu_i(A_i^t) & \forall i, \forall t \\ \sum_{i=1}^n \sum_{t=1}^{t_i} \beta_i^t = 1 \end{cases}$$

Observe that it is possible to construct such β_i^t since $\sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \nu_i(A_i^t) > 1$. □

Proof of theorem 2: We prove that

$$(iv) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i) \Longrightarrow (iv)$$

The first three steps follow the logic of proposition 1. We are left with $(i) \Longrightarrow (iv)$.

Let $(C_i)_{i=1,\dots,n}$ be a Pareto optimal allocation, such that all agents of type i consume identical bundles and $C_i^j = C_i^{j'}$ for all j, j', i.e., $C_i = c_i \mathbf{1}_S$, with $c_i > 0$.

Assume that $\cap_{i \leq n} \text{core}(\nu_i) = \emptyset$. By theorem 1, there exist (A_i^t), (α_i^t) and (β_i^t), $i = 1, \dots, n$ and $t = 1, \dots, t_i$ such that:

$$\begin{cases} \alpha_i^t \nu_i(A_i^t) \geq \beta_i^t \quad (> \text{ for some } i, t) \\ \sum_{i=1}^n \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S \\ \sum_{i=1}^n \sum_{t=1}^{t_i} \beta_i^t = 1 \end{cases}$$

Since $\alpha_i^t \nu_i(A_i^t) > \beta_i^t$ for some i, t, we can find β_i^t as above that also satisfy this strict inequality for *all* i, t.

Construct now the following allocation. For all $i = 1, \dots, n$ and all $t = 1, \dots, t_i$, let a set of agents of type i, of measure α_i^t, consume

$$\widetilde{C}_i^t = \left[c_i - \varepsilon \frac{\beta_i^t}{\alpha_i^t} \right] \mathbf{1}_S + \varepsilon \mathbf{1}_{A_i^t}$$

where $\varepsilon > 0$. Let the rest of the agents of type i, of measure $[1 - \sum_{t=1}^{t_i} \alpha_i^t] \geq 0$ consume C_i. We now proceed to show that the allocation $\widetilde{C}$ Pareto dominates the allocation C. Consider an individual of type i whose consumption was changed to $\widetilde{C}_i^t$.

$$\begin{aligned} V_i\left(\widetilde{C}_i^t\right) &= (1 - \nu_i(A_i^t)) U_i\left(c_i - \varepsilon \frac{\beta_i^t}{\alpha_i^t}\right) + \nu_i(A_i^t) U_i\left(c_i - \varepsilon \frac{\beta_i^t}{\alpha_i^t} + \varepsilon\right) \\ &= U_i(c_i) - (1 - \nu_i(A_i^t)) U_i'(c_i) \varepsilon \frac{\beta_i^t}{\alpha_i^t} + \varepsilon o_1(\varepsilon) \\ &\quad + \nu_i(A_i^t) U_i'(c_i) \varepsilon \left[1 - \frac{\beta_i^t}{\alpha_i^t}\right] + \varepsilon o_2(\varepsilon) \\ &= U_i(c_i) + \varepsilon U_i'(c_i) \left[\nu_i(A_i^t) - \frac{\beta_i^t}{\alpha_i^t}\right] + \varepsilon o_3(\varepsilon) \end{aligned}$$

Now, $o_3(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\nu_i(A_i^t) - \frac{\beta_i^t}{\alpha_i^t} > 0$ for all i, t.

We still have to check that $\widetilde{C}$ is a feasible allocation. Define $\widetilde{C}_i$ to be total consumption of agents of type i:

$$\widetilde{C}_i = \sum_{t=1}^{t_i} \alpha_i^t \widetilde{C}_i^t + \left(1 - \sum_{t=1}^{t_i} \alpha_i^t\right) c_i \mathbf{1}_S$$

Then,

$$\widetilde{C}_i = c_i \mathbf{1}_S - \varepsilon \sum_{t=1}^{t_i} \beta_i^t \mathbf{1}_S + \varepsilon \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t}$$

Now,

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{C}_i &= \sum_{i=1}^{n} c_i \mathbf{1}_S + \varepsilon \left[\sum_{i=1}^{n} \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} - \sum_{i=1}^{n} \sum_{t=1}^{t_i} \beta_i^t \mathbf{1}_S\right] \\ &= \sum_{i=1}^{n} C_i\end{aligned}$$

since, by construction, $\sum_{i=1}^{n} \sum_{t=1}^{t_i} \alpha_i^t \mathbf{1}_{A_i^t} = \mathbf{1}_S$ and $\sum_{i=1}^{n} \sum_{t=1}^{t_i} \beta_i^t = 1$.

Hence, $\widetilde{C}$ Pareto dominates C, yielding the desired contradiction. □

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