A Comment on "Ellsberg's two-color experiment, portfolio inertia and ambiguity"

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July 5, 2007

Abstract

The final step in the proof of Proposition 1 (p.311) of Mukerji and Tallon (2003) may not hold in general because $\varepsilon > 0$ in the proof cannot be chosen independently of w, z. We point out by a counterexample that the axioms they impose are too weak for Proposition 1. We introduce a modified set of axioms and re-establish the proposition.

JEL classification: D81

1. Introduction

Say that a decision maker (DM) exhibits portfolio inertia for an asset if she strictly prefers a zero position to taking a short or long position under a non-degenerate price interval. Portfolio inertia has been attributed to ambiguity by several literature, which rely on particular functional forms, for example, maximin expected utilities. In a non-parametric setting, Proposition 1 in Mukerji and Tallon (2003) claims that Ellsberg-type ambiguity averse behavior implies portfolio inertia.

As is known, however, *smooth ambiguity preferences* can accommodate the Ellsberg Paradox.[1] This smooth model suggests that ambiguity aversion does not necessarily imply portfolio inertia, because portfolio inertia means that preference has a kink at the origin (zero position).

In this note, we provide as a counterexample to Proposition 1 in Mukerji and Tallon (2003) a smooth ambiguity preference satisfying all the assumptions, but exhibiting no portfolio inertia. To reestablish the proposition, we introduce three additional axioms and show that, under those axioms, their original assumptions and axioms are sufficient for subjective portfolio inertia. The additional axioms do not a priori preclude most of smooth, convex, and monotonic preferences. The counterexample turns out to be a knife-edge case. The key axiom introduced to re-establish

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[1] An axiomatic foundation of the smooth ambiguity model is provided by Klibanoff, Marinacci and Mukerji (2003).

the proposition is Axiom 4, which we have called *Persistence*. The axiom allows the DM to persist in prefering a risky act to an ambiguous act even as payoffs (passing to the limit) vanish to zero. The axiom is arguably of interest in of itself, as a property of preferences that accomodate ambiguity sensitive behavior. It provides a simple test, based on verifiable/observable behavior between two prominent classes of preferences that accomodate ambiguity: the MEU/CEU class of preferences and smooth ambiguity preferences. The final section of the note expands on these points.

2. Model

The following is a brief summary of the setting of Mukerji and Tallon (2003). Consider two urns. Urn 1 consists of red and black balls with known 50-50 proportion, while urn 2 consists of red and black balls with unknown proportion. The state space S is defined by

$$S \equiv \{s_1 \equiv (R,R), s_2 \equiv (R,B), s_3 \equiv (B,R), s_4 \equiv (B,B)\},$$

where (R,B) means a red ball is drawn from urn 1 and a black ball is drawn from urn 2, and so on. Let $\mathcal{F} \equiv \mathbb{R}^4$ be the set of all real-valued Savage acts, $f : S \to \mathbb{R}$. Preference $\succsim$ is defined on $\mathcal{F}$.

Mukerji and Tallon (2003) consider the following axioms on $\succsim$ and claim in Proposition 1 that those axioms and some other assumptions are sufficient for subjective portfolio inertia.

Axiom 1 (Local Invariance) *For all $f, g \in \mathcal{F}$, $c \in \mathbb{R}$, and $\bar{\lambda} > 0$, if $\lambda(f + c) \succ \lambda(g + c)$ for any $\lambda \in (0, \bar{\lambda})$, then there exists $\bar{\lambda}' > 0$ such that $\lambda f \succ \lambda g$ for any $\lambda \in (0, \bar{\lambda}')$.*

A1 Let $w, z \in X \subseteq \mathbb{R}$, $w \neq z$. Then, $\begin{pmatrix} w \\ w \\ z \\ z \end{pmatrix} \succ \begin{pmatrix} w \\ z \\ w \\ z \end{pmatrix}, \begin{pmatrix} z \\ z \\ w \\ w \end{pmatrix} \succ \begin{pmatrix} z \\ w \\ z \\ w \end{pmatrix}.$

A2 Let $w, z \in X \subseteq \mathbb{R}$, $w \neq z$. Then, $\begin{pmatrix} w \\ w \\ z \\ z \end{pmatrix} \sim \begin{pmatrix} z \\ z \\ w \\ w \end{pmatrix}.$

A1 captures ambiguity averse behavior in the Ellsberg experiment. **A2** reveals that the decision maker considers the events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are equally likely.

Proposition 1. *Assume that events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous events and that $\succsim$ is weakly risk averse on the unambiguous acts $\mathcal{F}^{ua}$. Suppose $\succsim$ satisfies **A1** and **A2** for $X = \mathbb{R}$. Then, for any $w, z \in \mathbb{R}$ with $w \neq z$, there exists an ε-neighborhood of $\overline{p} \equiv (w+z)/2$, $N_\varepsilon(\overline{p})$, such that, for any $p \in N_\varepsilon(\overline{p})$ and $\lambda > 0$,*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} p-w \\ p-z \\ p-w \\ p-z \end{pmatrix}.$$

3. A Counterexample

The final step in the proof of Proposition 1 (p.311) of Mukerji and Tallon (2003) may not hold in general because $\varepsilon > 0$ in the proof cannot be chosen independently of w, z. More explicitly, we provide the following counterexample of the proposition: Consider following spreference $\succsim$ on $\mathcal{F}$ represented with a second-order probability a:

$$U(f) = \int_0^1 \varphi\left(\frac{1}{4}f(s_1) + \frac{1}{4}f(s_2) + \frac{q}{2}f(s_3) + \frac{1-q}{2}f(s_4)\right) \mathrm{d}q, \tag{1}$$

where $\varphi(x) = -e^{-x}$. Since U is smooth, it does not exhibit portfolio inertia. As shown below, however, U satisfies all the assumptions and axioms of Proposition 1.

- *Probabilistic Sophistication on the set of unambiguous acts* $\mathcal{F}^{ua}$: we can verify that

  $$\mathcal{A} = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_3, s_4\}, \{s_1, s_3, s_4\}, \{s_2, s_3, s_4\}, S\}$$

  is the set of unambiguous events and that $\succsim$ is probabilistically sophisticated on $\mathcal{F}^{ua}$ with $\pi(\{s_1\}) = \pi(\{s_2\}) = 1/4$, $\pi(\{s_1, s_2\}) = \pi(\{s_3, s_4\}) = 1/2$, and $\pi(\{s_1, s_3, s_4\}) = \pi(\{s_2, s_3, s_4\}) = 3/4$.

  In order to verify $\{s_1, s_3\}$ is an ambiguous event, let $A = \{s_2\}$, $B = \{s_4\}$, $T^c \setminus (A \cup B) = \emptyset$, $T = \{s_1, s_3\}$, $x = 100$, $x^* = 0$, $z = 0$, and $z' = 100$ in the definition of unambiguous events. Since φ is strictly concave, this combination violates the definition. Similarly, in order to verify $\{s_3\}$ is an ambiguous event, let $A = \{s_2\}$, $B = \{s_4\}$, $T^c \setminus (A \cup B) = \{s_1\}$, $T = \{s_3\}$, $h = 100$, $x = 100$, $x^* = 0$, $z = 0$, and $z' = 100$. The symmetric argument works for the other cases.

- *Weak Risk Aversion on* $\mathcal{F}^{ua}$: for any $f \in \mathcal{F}^{ua}$, let $\pi \circ f^{-1}$ denote the distribution of π induced by f. Since $U(f) = \varphi(\mathbb{E}[\pi \circ f^{-1}])$, $\mathbb{E}[\pi \circ f^{-1}] \sim f$.

- *Axiom 1 (Local Invariance)*: since

  $$\begin{aligned} U(\lambda(f+c)) &= \int_0^1 -e^{-\left(\frac{1}{4}\lambda(f(s_1)+c)+\frac{1}{4}\lambda(f(s_2)+c)+\frac{q}{2}\lambda(f(s_3)+c)+\frac{1-q}{2}\lambda(f(s_4)+c)\right)}\mathrm{d}q \\ &= e^{-\lambda c}\int_0^1 -e^{-\left(\frac{1}{4}\lambda f(s_1)+\frac{1}{4}\lambda f(s_2)+\frac{q}{2}\lambda f(s_3)+\frac{1-q}{2}\lambda f(s_4)\right)}\mathrm{d}q \\ &= e^{-\lambda c}U(\lambda f), \end{aligned}$$

  we have $U(\lambda(f+c)) > U(\lambda(g+c)) \Leftrightarrow e^{-\lambda c}U(\lambda f) > e^{-\lambda c}U(\lambda g) \Leftrightarrow U(\lambda f) > U(\lambda g)$, for any $\lambda \in (0, \bar{\lambda})$. Hence, the same $\bar{\lambda}$ works.

- *A1 for* $X = \mathbb{R}$: let $f' \equiv \begin{pmatrix} w \\ w \\ z \\ z \end{pmatrix}$ and $f \equiv \begin{pmatrix} w \\ z \\ w \\ z \end{pmatrix}$. Since

  $$\int_0^1 \left(\frac{1}{4}w + \frac{1}{4}z + \frac{q}{2}w + \frac{1-q}{2}z\right) \mathrm{d}q = \frac{1}{2}w + \frac{1}{2}z,$$

strict concavity of φ implies

$$U(f') = \varphi\left(\frac{1}{2}w + \frac{1}{2}z\right) > \int_0^1 \varphi\left(\frac{1}{4}w + \frac{1}{4}z + \frac{q}{2}w + \frac{1-q}{2}z\right) \mathrm{d}q = U(f).$$

- *A2 for $X = \mathbb{R}$*: take $f' \equiv \begin{pmatrix} w \\ w \\ z \\ z \end{pmatrix}$ and $f \equiv \begin{pmatrix} z \\ z \\ w \\ w \end{pmatrix}$. Then, $U(f') = \varphi\left(\frac{1}{2}w + \frac{1}{2}z\right) = U(f)$.

4. Modification

In order to reestablish Proposition 1, we introduce additional axioms as follows:

Axiom 2 (Local Convexity) *If $f \succsim (\mathbf{0})$, there exists $\lambda' > 0$ such that $\lambda f \succsim (\mathbf{0})$ for any $\lambda \in (0, \lambda')$.*

Axiom 2 always holds if the upper contour set of $\succsim$ at $\mathbf{0}$ is convex.

Axiom 3 (Monotonicity) *For all $f, g \in \mathcal{F}$, if $f(s) \geq g(s)$ for all s, then $f \succsim g$.*

The next axiom roughly says that in a neighborhood at $\mathbf{0}$ the valuation of s_2 is not identical with one of s_3.

Axiom 4 (Persistence) *There exist $w \neq 0$ and $\varepsilon > 0$ such that, for all $\lambda \in (0, 1]$,*

$$\lambda \begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix} \succ \lambda \begin{pmatrix} w+\varepsilon \\ -w+\varepsilon \\ w+\varepsilon \\ -w+\varepsilon \end{pmatrix}.$$

Notice that Axiom 4 does not a priori exclude most of smooth preferences. Any utility representation $U : \mathcal{F} \to \mathbb{R}$ which is smooth at $\mathbf{0}$ satisfies Axiom 4 as long as $U_2(\mathbf{0})$ does not coincide with $U_3(\mathbf{0})$, where $U_i(\mathbf{0})$ is the partial derivative at $\mathbf{0}$ with respect to coordinate i.[2] Hence, any SEU decision maker with a smooth utility u and a subjective probability π such that $\pi_2 \neq \pi_3$ satisfies Persistence. CEU decision makers might or not satisfy the axiom, depending on the capacity. The axiom is discussed at length in the next section.

[2]Let $f_\lambda = \lambda \begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix}$ and $g_\lambda = \lambda \begin{pmatrix} w+\varepsilon \\ -w+\varepsilon \\ w+\varepsilon \\ -w+\varepsilon \end{pmatrix}$. By the Taylor expansion at $\mathbf{0}$, $U(f_\lambda) - U(g_\lambda) = \lambda(2w(U_2(\mathbf{0}) - U_3(\mathbf{0})) - \varepsilon \sum_i U_i(\mathbf{0})) + o(f_\lambda) - o(g_\lambda)$, where $o(f_\lambda)$ and $o(g_\lambda)$ are residual functions. As long as $U_2(\mathbf{0}) \neq U_3(\mathbf{0})$, we can choose $w \neq 0$ so as to make the term $2w(U_2(\mathbf{0}) - U_3(\mathbf{0}))$ positive. Thus, there exist $\varepsilon > 0$ and $\bar{\lambda} > 0$ small enough such that $U(f_\lambda) > U(g_\lambda)$ for any $\lambda \in (0, \bar{\lambda})$. Thus, redefining w, ε if necessary, any smooth representation with $U_2(\mathbf{0}) \neq U_3(\mathbf{0})$ satisfies Local Disagreement.

The function (1) satisfies Local Convexity and Monotonicity, but violates Persistence. Indeed, taking into account $U_i(\mathbf{0}) = \varphi'(0)/4$ for all i, the same argument in footnote 2 implies, for any $w \neq 0$ and $\varepsilon > 0$,

$$\begin{aligned} U(f_\lambda) - U(g_\lambda) &= \lambda \left(2w(U_2(\mathbf{0}) - U_3(\mathbf{0})) - \varepsilon \sum U_i(\mathbf{0})\right) + o(f_\lambda) - o(g_\lambda) \\ &= -\lambda\varepsilon\varphi'(0) + o(f_\lambda) - o(g_\lambda). \end{aligned}$$

For all $\lambda > 0$ small enough, $U(f_\lambda) < U(g_\lambda)$, hence U violates Persistence.

Under these new axioms, we can reestablish Proposition 1.

Proposition 1′. *Consider preference $\succsim$ satisfying Local Convexity, Monotonicity, and Persistence. Assume events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous events and that $\succsim$ is weakly risk averse on $\mathcal{F}^{ua}$. Suppose $\succsim$ satisfies **A1** and **A2** for $X = \mathbb{R}$. Then, for all w, z with $w \neq z$, there exists a subjective portfolio inertia interval $N_{\bar{\varepsilon}}(p^*) \equiv (p^* - \bar{\varepsilon}, p^* + \bar{\varepsilon}), \bar{\varepsilon} > 0$, such that for all $p \in N_{\bar{\varepsilon}}(p^*)$ and all $\lambda > 0$,*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} p-w \\ p-z \\ p-w \\ p-z \end{pmatrix}.$$

Two remarks are in order regarding Proposition 1′. First, p^* in the subjective portfolio inertia interval $N_{\bar{\varepsilon}}(p^*)$ may not be the mean price $\overline{p} \equiv (w+z)/2$ unlike the original proposition.

Second, Proposition 1′ implies that most of smooth preferences, that is, representations U with $U_2(\mathbf{0}) \neq U_3(\mathbf{0})$, which satisfy Local Convexity, Monotonicity and **A2**, are inconsistent with the Ellsberg-type behavior **A1**. The counterexample (1) turns out to be a "knife-edge" case. Indeed, once U is slightly perturbed in terms of the second-order probability, it must violate **A1**. For example, consider an ε-perturbed representation

$$U^\varepsilon(x) \equiv \int_0^1 \varphi\left(\left(\frac{1}{4} + \varepsilon\right) f(s_1) + \left(\frac{1}{4} - \varepsilon\right) f(s_2) + \frac{q}{2} f(s_3) + \frac{1-q}{2} f(s_4)\right) \mathrm{d}q,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly increasing, and strictly concave function. Now U^ε satisfies Persistence because $U_2(\mathbf{0}) = \varphi'(0)(1/4 - \varepsilon) \neq \varphi'(0)/4 = U_3(\mathbf{0})$. It satisfies also Local Convexity, Monotonicity and **A2**, and hence, from Proposition 1′, violates **A1**.

5. More on Axiom 4

In this section we discuss Axiom 4 in more details. In particular, we show how Axiom 4 provides a neat dividing line between two classes of ambiguity sensitive preferences, the smooth ambiguity preferences and MEU/CEU preferences. To interpret the axiom, we consider preferences satisfying **A1**: hence, SEU decision makers are excluded because they are inconsistent with **A1**. But, a class of smooth ambiguity preferences (second-order probability model) is still consistent with **A1**.

Assume that $\succsim$ satisfies **A1** for $X = \mathbb{R}$.

Axiom 4 (Persistence): There exist $w \neq 0$ and $\varepsilon > 0$ such that, for all $\lambda \in (0, 1]$,

$$\lambda \begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix} \succ \lambda \begin{pmatrix} w+\varepsilon \\ -w+\varepsilon \\ w+\varepsilon \\ -w+\varepsilon \end{pmatrix}.$$

To understand the spirit of the axiom (and, indeed, why we call it "persistence") take the negation of Axiom 4.

Axiom 4' (Preference Reversal for Small Payoffs): For all $w \neq 0$ and $\varepsilon > 0$, there exists $\lambda \in (0, 1)$ such that

$$\lambda \begin{pmatrix} w+\varepsilon \\ -w+\varepsilon \\ w+\varepsilon \\ -w+\varepsilon \end{pmatrix} \succsim \lambda \begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix}. \tag{2}$$

Take any $w \neq 0$. **A1** implies $\begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix} \succ \begin{pmatrix} w \\ -w \\ w \\ -w \end{pmatrix}$. As long as preference satisfies continuity, there exists a small $\varepsilon > 0$ such that

$$\begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix} \succ \begin{pmatrix} w+\varepsilon \\ -w+\varepsilon \\ w+\varepsilon \\ -w+\varepsilon \end{pmatrix}. \tag{3}$$

This ranking means that certain payoff $\varepsilon > 0$ is not enough for the DM to undertake the ambiguous act. Axiom 4' requires that, if all payoffs are diminishing proportionally to zero, ranking (3) is always reversed. That is, any small payoff ε is a sufficient compensation for the ambiguous act if all payoffs are close to zero. Presumably, this is because the DM perceives that the numbers of red and black balls in urn II are identical, that is, $\{s_1, s_3\}$ and $\{s_2, s_4\}$ are equally likely events when payoffs are negligible. In this case, the expected payoff of the first act in (2) is $\lambda\varepsilon$, while that of the second act is zero. That is why ranking (3) is reversed if λ is sufficiently small.

Notice that **A1** and Axiom 4' capture the characteristic of the DM with smoothly ambiguous preferences. She may exhibit ambiguity averse behavior if two acts are sufficiently different, but not when payoffs are arbitrarily close to zero, making the two acts "virtually" the same. In other words, she is "locally" an SEU decision maker, that is, she simply calculates the expected utility with respect to a base prior (mean prior) to evaluate the acts whose consequences are sufficiently close to zero.

Now the meaning of Axiom 4 is clear. It requires that there exists $w \neq 0$ and $\varepsilon > 0$ such that ranking (3) is not reversed no matter how the payoffs get small. In other words, for some $w \neq 0$ and $\varepsilon > 0$, certain payoff $\lambda\varepsilon$ is not enough for the DM to take the ambiguous act. She persists in exhibiting ambiguity aversion, i.e., a preference for the risky act as compared to the ambiguous act. (A risky (respectively, ambiguous) act is an act measurable with respect to a partition of the state space consisting of unambiguous (ambigous) events.) Hence, while ***A1*** captures ambiguity aversion, *together with* ***A1***, ***Axiom 4*** captures how strongly the DM exhibits ambiguity aversion. For some $w \neq 0$ and a certainty payoff $\varepsilon > 0$, the DM persistently prefers the risky act to the ambiguous act even when all payoffs are negligibly small.

Next we show that there are MEU preferences that satisfy both A1 and Axiom 4.

Let $\Delta \subset \mathbb{R}^4$ be the 3-dimensional unit simplex. Since there are identical numbers of red and black balls in urn I, we pay attention to MEU representations with sets of priors

$$M \subset \Delta_I \equiv \{\pi \in \Delta \,|\, \pi_1 + \pi_2 = \pi_3 + \pi_4 = 1/2\}. \tag{4}$$

Thus $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are equally-likely unambiguous events. Let

$$\Delta_I^{II} \equiv \{\pi \in \Delta_I \,|\, \pi_1 + \pi_3 = \pi_2 + \pi_4 = 1/2\}.$$

If $M \subset \Delta_I^{II}$, $\{s_1, s_3\}$ and $\{s_2, s_4\}$ are also unambiguous events for any MEU decision maker with (u, M). Notice that any $\pi \in \Delta_I^{II}$ can be rewritten as $(1/4-\delta, 1/4+\delta, 1/4+\delta, 1/4-\delta)$ for some $\delta \in \mathbb{R}$.

Claim 1. *Assume that a payoff function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and smooth at zero. An MEU representation (u, M) with $M \subset \Delta_I$ satisfies Axiom 4 if and only if $M \not\subset \Delta_I^{II}$.*

Proposition 3 of Mukerji and Tallon (2003) shows that, Under **A1**, $\{s_1, s_3\}$ and $\{s_2, s_4\}$ are ambiguous events whenever $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous events. Thus, Claim 1 implies that any MEU preferences satisfying **A1** and the natural restriction (4) automatically satisfies Axiom 4. In other words, Axiom 4 is innocuous for this class of preferences.

A CEU decision maker with the capacity

$$\nu(s_1, s_2) = \nu(s_3, s_4) = 1/2,\ \nu(s_1, s_3) = \nu(s_2, s_4) = \nu(s_i) = 1/8,\ i = 1, \cdots, 4,$$

satisfies Axiom 4. To see this, notice that the same preference admits the MEU representation with the set of priors

$$\text{core}(\nu) \equiv \{\pi \in \Delta | \pi(E) \geq \nu(E),\ \text{for all event } E\}.$$

Any $\pi \in \text{core}(\nu)$ satisfies $\pi_1 + \pi_2 = \pi_3 + \pi_4 = 1/2$. Moreover, the belief $(\pi_1^*, \pi_2^*, \pi_3^*, \pi_4^*) = (3/8, 1/8, 3/8, 1/8) \in \text{core}(\nu)$ satisfies $\pi_1^* + \pi_3^* \neq \pi_2^* + \pi_4^*$. From Theorem 1, this MEU representation satisfies Axiom 4.

For a smooth preference, the gradient at 0 can be interpreted as the DM's probabilistic belief. Since urn I contains identical numbers of red and black balls, we assume that $U_1(\mathbf{0}) + U_2(\mathbf{0}) = U_3(\mathbf{0}) + U_4(\mathbf{0})$.

Claim 2. *Assume that $U : \mathbb{R}^4 \to \mathbb{R}$ is strictly increasing and smooth at zero, and $U_1(\mathbf{0}) + U_2(\mathbf{0}) = U_3(\mathbf{0}) + U_4(\mathbf{0})$. Then, U satisfies Axiom 4 if and only if $U_1(\mathbf{0}) + U_3(\mathbf{0}) \neq U_2(\mathbf{0}) + U_4(\mathbf{0})$.*

That is, the DM with a smooth preference satisfies Axiom 4 if and only if $\{s_1, s_3\}$ and $\{s_2, s_4\}$ are not equally likely in terms of the belief at 0.

Corollary 1. *Assume that a payoff function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and smooth at zero. An SEU representation (u, π) with $\pi_1 + \pi_2 = \pi_3 + \pi_4$ satisfies Axiom 4 if and only if $\pi_1 + \pi_3 \neq \pi_2 + \pi_4$.*

4. Conclusion

Ambiguity aversion can be explained by several models. Especially, there exist two different models, MEU model and second-order probability model. A question is how those two models are distinguished by observable behavior.[3]

In the model of Ellsberg's two-color experiment, the following restrictions seem plausible:

(i) MEU Model: (1) any $\pi \in M$ satisfies $\pi_1 + \pi_2 = \pi_3 + \pi_4$, and (2) there exists $\pi^* \in M$ such that $\pi_1^* + \pi_3^* \neq \pi_2^* + \pi_4^*$ (because urn II is ambiguous).

(ii) A Smooth Ambiguity Preference Model: the mean prior $\bar{\pi}$ of the second-order probability μ satisfies $\bar{\pi}_i = 1/4$ for all i (because of the symmetric nature of the model).

Suppose that $\succsim$ satisfies **A1**. This decision maker could be an MEU agent or a second-order probability agent (or something else). From Theorem 1 and Theorem 2, we conclude that

$$\begin{aligned} \text{MEU} &\Rightarrow \textbf{Axiom 4} \\ \text{2nd-order Prob.} &\Rightarrow \textbf{Axiom 4'} \end{aligned}$$

That is, to distinguish MEU from smooth ambiguity preference, we can check whether the decision maker exhibits preference reversal for small payoffs. Moreover, together with the modified version of Proposition 1,

$$\begin{aligned} \text{MEU} &\Rightarrow \text{portfolio inertia} \\ \text{2nd-order Prob.} &\Rightarrow \text{no portfolio inertia} \end{aligned}$$

This is a testable implication.

A Proofs

Proof of Proposition 1′. Fix $w \neq z \in \mathbb{R}$. Assume $w > z$.[4] Since **A1** holds for $X = \mathbb{R}$, we have, for all $p > 0$ and the given w and z:

$$\begin{pmatrix} w-p \\ w-p \\ z-p \\ z-p \end{pmatrix} \succ \begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \text{ and } \begin{pmatrix} z-p \\ z-p \\ w-p \\ w-p \end{pmatrix} \succ \begin{pmatrix} z-p \\ w-p \\ z-p \\ w-p \end{pmatrix}. \tag{5}$$

Take $\bar{p} = \frac{w+z}{2}$; then (5) implies,

$$\begin{pmatrix} \frac{w-z}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{z-w}{2} \end{pmatrix} \succ \begin{pmatrix} \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{z-w}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{w-z}{2} \end{pmatrix} \succ \begin{pmatrix} \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \end{pmatrix}. \tag{6}$$

[3] Axiomatic foundations of those models would make this separation possible. Though the second-order probability model is axiomatized by KMM, their primitives are different from those of the standard Savage or Anscombe-Aumann setting. The comparison does not seem immediate.

[4] The symmetric argument works even when $w < z$.

Now, $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous events and therefore $\begin{pmatrix} \frac{w-z}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{z-w}{2} \end{pmatrix}$ and $\begin{pmatrix} \frac{z-w}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{w-z}{2} \end{pmatrix}$ are unambiguous acts. Let π be a probability measure such that $\succsim$ is probabilistically sophisticated with π on the set of unambiguous acts. Then **A2** implies $\pi(\{s_1, s_2\}) = \pi(\{s_3, s_4\}) = 1/2$, since $\{s_1, s_2\} \cup \{s_3, s_4\} = \Omega$. Therefore these two acts have zero expected *value* under the probability π. Hence, by (weak) risk aversion, it is the case that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succsim \begin{pmatrix} \frac{w-z}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{z-w}{2} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succsim \begin{pmatrix} \frac{z-w}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{w-z}{2} \end{pmatrix}. \tag{7}$$

Thus, taking (6) and (7) together,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \begin{pmatrix} \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \begin{pmatrix} \frac{z-w}{2} \\ \frac{w-z}{2} \\ \frac{z-w}{2} \\ \frac{w-z}{2} \end{pmatrix}.$$

Up to here it has simply been the old proof. The following is the reconstructed argument. The reasoning led above is true for all w, z and hence we do know for $\bar{p} = \left(\frac{w+z}{2}\right)$ that the following is true for all $\lambda > 0$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} w - \bar{p} \\ z - \bar{p} \\ w - \bar{p} \\ z - \bar{p} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} \bar{p} - w \\ \bar{p} - z \\ \bar{p} - w \\ \bar{p} - z \end{pmatrix}.$$

From Persistence, there exist $w^* \neq 0$ and $\varepsilon > 0$ such that $\lambda' \begin{pmatrix} w^* \\ w^* \\ -w^* \\ -w^* \end{pmatrix} \succ \lambda' \begin{pmatrix} w^* + \varepsilon \\ -w^* + \varepsilon \\ w^* + \varepsilon \\ -w^* + \varepsilon \end{pmatrix}$ for all $\lambda' \in (0, 1]$. First consider the case of $w^* > 0$. Weak risk aversion for unambiguous acts implies $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succsim \lambda' \begin{pmatrix} w^* \\ w^* \\ -w^* \\ -w^* \end{pmatrix}$, and hence $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda' \begin{pmatrix} w^* + \varepsilon \\ -w^* + \varepsilon \\ w^* + \varepsilon \\ -w^* + \varepsilon \end{pmatrix}$. Let $\lambda^* \equiv \frac{2w^*}{w-z} > 0$. Then, $\begin{pmatrix} w^* \\ -w^* \\ w^* \\ -w^* \end{pmatrix} = \lambda^* \begin{pmatrix} w - \bar{p} \\ z - \bar{p} \\ w - \bar{p} \\ z - \bar{p} \end{pmatrix}$. Hence, for all $\lambda \in (0, \lambda^*]$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} w - \bar{p} + \varepsilon' \\ z - \bar{p} + \varepsilon' \\ w - \bar{p} + \varepsilon' \\ z - \bar{p} + \varepsilon' \end{pmatrix}$, where $\varepsilon' \equiv \frac{\varepsilon}{\lambda^*}$. Let $\bar{\varepsilon} \equiv \frac{\varepsilon'}{2}$ and $p^* \equiv \bar{p} - \bar{\varepsilon}$. By Monotonicity, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda \begin{pmatrix} w - p \\ z - p \\ w - p \\ z - p \end{pmatrix}$ for all $p \in N_{\bar{\varepsilon}}(p^*)$ and $\lambda \in (0, \lambda^*]$.

In order to show the same ranking for all $\lambda > 0$, suppose otherwise. Then, there exist $\bar{\lambda} > 0$ and $p \in N_{\bar{\varepsilon}}(p^*)$ such that $\bar{\lambda}\begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \succsim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. By Local Convexity, there exists a non-degenerate interval $(0, \nu')$, such that for any $\nu \in (0, \nu')$, $\nu\bar{\lambda}\begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \succsim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. For ν small enough, $\nu\bar{\lambda} < \lambda^*$ and hence this would contradict the fact that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix}$ for all $\lambda \in (0, \lambda^*]$.

On the other hand, by Monotonicity, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} p-w \\ p-z \\ p-w \\ p-z \end{pmatrix}$ for all $p \in N_{\bar{\varepsilon}}(p^*)$ and $\lambda > 0$. Hence, for all $w \neq z \in \mathbb{R}$, it follows that for all $\lambda > 0$, and for all $p \in N_{\bar{\varepsilon}}(p^*)$,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} p-w \\ p-z \\ p-w \\ p-z \end{pmatrix}.$$

If $w^* < 0$, let $\lambda^* \equiv \frac{2w^*}{z-w} > 0$. The similar argument implies $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} \overline{p}-w+\varepsilon \\ \overline{p}-z+\varepsilon \\ \overline{p}-w+\varepsilon \\ \overline{p}-z+\varepsilon \end{pmatrix}$ for all $\lambda \in (0, \lambda^*]$. Let $\bar{\varepsilon} \equiv \frac{\varepsilon}{2}$ and $p^* \equiv \bar{p} + \bar{\varepsilon}$. By Monotonicity, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} p-w \\ p-z \\ p-w \\ p-z \end{pmatrix}$ for all $p \in N_{\bar{\varepsilon}}(p^*)$ and $\lambda \in (0, \lambda^*]$. By the same argument above, this ranking holds for all $\lambda > 0$. Finally, Monotonicity implies $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \succ \lambda\begin{pmatrix} w-p \\ z-p \\ w-p \\ z-p \end{pmatrix}$ for all $p \in N_{\bar{\varepsilon}}(p^*)$ and $\lambda > 0$. ■

Proof of Claim 1:

(if part) For $w \neq 0$, $\lambda > 0$ and $\varepsilon > 0$, let $f_\lambda = \lambda(w, w, -w, -w)$ and $g_\lambda = \lambda(w+\varepsilon, -w+\varepsilon, w+\varepsilon, -w+\varepsilon)$. We want to show that there exist $w \neq 0$ and $\varepsilon > 0$ such that $U(f_\lambda) > U(g_\lambda)$ for all $\lambda \in (0, 1]$. By assumption, there exists a belief $\pi^* \in M$ such that $\pi_1^* + \pi_3^* \neq \pi_2^* + \pi_4^*$.

For all $w \neq 0$, $\lambda > 0$, and $\varepsilon > 0$, notice that

$$U(f_\lambda) = \min_{\pi \in M} \sum \pi_i u(f_\lambda(i)) = \frac{1}{2}u(\lambda w) + \frac{1}{2}u(-\lambda w), \tag{8}$$

because $\pi_1 + \pi_2 = \pi_3 + \pi_4 = 1/2$ for any $\pi \in M$. On the other hand, by definition,

$$(\pi_1^* + \pi_3^*)u(\lambda(w+\varepsilon)) + (\pi_2^* + \pi_4^*)u(\lambda(-w+\varepsilon)) \geq \min_{\pi \in M} \sum \pi_i u(g_\lambda(i)) = U(g_\lambda). \tag{9}$$

From (8) and (9), it is enough to show that there exist $w, \bar{\lambda}$, ε such that, for all $\lambda \in (0, \bar{\lambda}]$,

$$\frac{1}{2}u(\lambda w) + \frac{1}{2}u(-\lambda w) > (\pi_1^* + \pi_3^*)u(\lambda(w+\varepsilon)) + (\pi_2^* + \pi_4^*)u(\lambda(-w+\varepsilon)). \tag{10}$$

By the Talor expansion at 0,

$$\begin{aligned}
& \frac{1}{2}u(\lambda w) + \frac{1}{2}u(-\lambda w) - ((\pi_1^* + \pi_3^*)u(\lambda(w+\varepsilon)) + (\pi_2^* + \pi_4^*)u(\lambda(-w+\varepsilon))) \\
= & \frac{1}{2}u(0) + \frac{1}{2}\lambda w u'(0) + \frac{1}{2}u(0) - \frac{1}{2}\lambda w u'(0) \\
& -\Big((\pi_1^* + \pi_3^*)u(0) + (\pi_1^* + \pi_3^*)\lambda(w+\varepsilon)u'(0) + (\pi_2^* + \pi_4^*)u(0) + (\pi_2^* + \pi_4^*)\lambda(-w+\varepsilon)u'(0)\Big) \\
& +o(\lambda) \\
= & -\lambda u'(0)\Big(w((\pi_1^* + \pi_3^*) - (\pi_2^* + \pi_4^*)) + \varepsilon\Big) + o(\lambda), \qquad (11)
\end{aligned}$$

where $o(\lambda)$ is the residual function. Since $\pi_1^* + \pi_3^* \neq \pi_2^* + \pi_4^*$, we can find $w \neq 0$ and $\varepsilon > 0$ satisfying $w((\pi_1^* + \pi_3^*) - (\pi_2^* + \pi_4^*)) + \varepsilon < 0$. Since $u'(0) > 0$ and $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$, there exists $\bar{\lambda}$ such that (11) is positive for any $\lambda \in (0, \bar{\lambda}]$. That is, we can find $w \neq 0, \bar{\lambda} > 0, \varepsilon > 0$ such that (10) holds for any $\lambda \in (0, \bar{\lambda}]$. Finally, let $w^* \equiv \bar{\lambda} w$ and $\varepsilon^* \equiv \bar{\lambda}\varepsilon$. The pair (w^*, ε^*) is the required object.

(only if part) We show that U violates Axiom 4 whenever $M \subset \Delta_I^2$. For all $w \neq 0$, $\lambda > 0$, and $\varepsilon > 0$,

$$U(g_\lambda) = \min_{\pi \in M} \sum \pi_i u(g_\lambda(i)) = \frac{1}{2}u(\lambda(w+\varepsilon)) + \frac{1}{2}u(\lambda(-w+\varepsilon)), \tag{12}$$

because $\pi_1 + \pi_3 = \pi_2 + \pi_4 = 1/2$ for any $\pi \in M$. From (8) and (12), it is enough to show that, for any $w \neq 0$ and $\varepsilon > 0$, there exists $\lambda \in (0, 1)$ such that

$$\frac{1}{2}u(\lambda w) + \frac{1}{2}u(-\lambda w) < \left(\frac{1}{2}u(\lambda(w+\varepsilon)) + \frac{1}{2}u(\lambda(-w+\varepsilon)\right). \tag{13}$$

By the Talor expansion at 0,

$$\begin{aligned}
& \frac{1}{2}u(\lambda w) + \frac{1}{2}u(-\lambda w) - \left(\frac{1}{2}u(\lambda(w+\varepsilon)) + \frac{1}{2}u(\lambda(-w+\varepsilon)\right) \\
= & \frac{1}{2}u(0) + \frac{1}{2}\lambda w u'(0) + \frac{1}{2}u(0) - \frac{1}{2}\lambda w u'(0) \\
& -\Big(\frac{1}{2}u(0) + \frac{1}{2}\lambda(w+\varepsilon)u'(0) + \frac{1}{2}u(0) + \frac{1}{2}\lambda(-w+\varepsilon)u'(0)\Big) + o(\lambda) \\
= & -\lambda u'(0)\varepsilon + o(\lambda), \qquad (14)
\end{aligned}$$

where $o(\lambda)$ is the residual function. Since $u'(0) > 0$ and $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$, there exists $\lambda \in (0, 1)$ such that (14) is negative. That is, (13) holds. ■

Proof of Claim 2:

(if part) For $w \neq 0$, $\lambda > 0$ and $\varepsilon > 0$, let $f_\lambda = \lambda(w, w, -w, -w)$ and $g_\lambda = \lambda(w+\varepsilon, -w+\varepsilon, w+\varepsilon, -w+\varepsilon)$. We want to show that there exist $w \neq 0$ and $\varepsilon > 0$ such that $U(f_\lambda) > U(g_\lambda)$ for all $\lambda \in (0, 1]$. By the Talor expansion at 0,

$$
\begin{aligned}
& U(f_\lambda) - U(g_\lambda) \\
= \; & U(\mathbf{0}) + (U_1(\mathbf{0}) + U_2(\mathbf{0}))\lambda w - (U_3(\mathbf{0}) + U_4(\mathbf{0}))\lambda w \\
& -\Big(U(\mathbf{0}) + (U_1(\mathbf{0}) + U_3(\mathbf{0}))\lambda(w+\varepsilon) + (U_2(\mathbf{0}) + U_4(\mathbf{0}))\lambda(-w+\varepsilon)\Big) + o(\lambda) \\
= \; & -\lambda\left(\Big((U_1(\mathbf{0}) + U_3(\mathbf{0})) - (U_2(\mathbf{0}) + U_4(\mathbf{0}))\Big)w + \varepsilon\sum_i U_i(\mathbf{0})\right) + o(\lambda), \qquad (15)
\end{aligned}
$$

where $o(\lambda)$ is the residual function. Since $U_1(\mathbf{0}) + U_3(\mathbf{0}) \neq U_2(\mathbf{0}) + U_4(\mathbf{0})$, we can find $w \neq 0$ and $\varepsilon > 0$ such that the first term of (15) is positive. Since $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$, there exists $\bar\lambda$ such that, for any $\lambda \in (0, \bar\lambda)$, $U(f_\lambda) > U(g_\lambda)$. Finally, let $w^* \equiv \bar\lambda w$ and $\varepsilon^* \equiv \bar\lambda\varepsilon$. Then, the pair (w^*, ε^*) satisfies Axiom 4.

(only if part) We show that U violates Axiom 4 if $U_1(\mathbf{0}) + U_3(\mathbf{0}) = U_2(\mathbf{0}) + U_4(\mathbf{0})$. It is enough to show that, for any $w \neq 0$ and $\varepsilon > 0$, there exists $\lambda \in (0, 1)$ such that $U(f_\lambda) < U(g_\lambda)$. By the Talor expansion at 0,

$$
\begin{aligned}
& U(f_\lambda) - U(g_\lambda) \\
= \; & U(\mathbf{0}) + (U_1(\mathbf{0}) + U_2(\mathbf{0}))\lambda w - (U_3(\mathbf{0}) + U_4(\mathbf{0}))\lambda w \\
& -\Big(U(\mathbf{0}) + (U_1(\mathbf{0}) + U_3(\mathbf{0}))\lambda(w+\varepsilon) + (U_2(\mathbf{0}) + U_4(\mathbf{0}))\lambda(-w+\varepsilon)\Big) + o(\lambda) \\
= \; & -\lambda\varepsilon\sum_i U_i(\mathbf{0}) + o(\lambda), \qquad (16)
\end{aligned}
$$

where $o(\lambda)$ is the residual function, which satisfies $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$. Since $\varepsilon > 0$ and $\sum_i U_i(\mathbf{0}) > 0$, (16) is negative for all λ sufficiently small. Thus, there exists $\lambda \in (0, 1)$ such that $U(f_\lambda) < U(g_\lambda)$. ■

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