

## NOTES AND COMMENTS

### SHARING BELIEFS: BETWEEN AGREEING AND DISAGREEING

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#### 1. INTRODUCTION

WHEN IS IT PARETO OPTIMAL for risk averse agents to take bets? Under what conditions do they choose to introduce uncertainty into an otherwise certain economic environment? One obvious case is where they do not share beliefs. As in the classical (theoretical) example of horse lotteries, people who do not agree on probability assessments do find it mutually beneficial to engage in uncertainty-generating trade.

If the agents involved are Bayesian expected utility maximizers and strictly risk averse, it is not hard to see that disagreement on probabilities is the only way that betting, understood as trade of an uncertain asset, may be Pareto improving when starting from a full insurance allocation. On the other hand, *any* such disagreement induces betting. Put differently, Pareto optimality dictates either that there be no betting (in case beliefs are common to all agents) or that there be betting (in case of disagreement). This is somewhat puzzling, because there is no lack of allocation-neutral, “sunspot” sources of uncertainty in the world around us. If every disagreement on probabilities of states of the world suggests a Pareto improving trade, one might have expected to see much more betting taking place.

Rather than believing that people who do not bet necessarily share probabilistic beliefs about anything they do not bet on (or, to be precise, share these beliefs up to some slack allowed by transaction costs), we tend to take the relative rarity of bets as a piece of empirical evidence against the Bayesian model. It seems that often people do not bet because they are uncertainty averse, and they therefore tend to avoid uncertainty that they know little about. It follows that a person’s willingness to bet will increase with her subjective confidence in her information and in her likelihood assessments. It is worth emphasizing that Bewley’s (1986) motivation for his work on Knightian decision theory was partly this absence of observed widespread betting.

While we do not attempt to argue that the full complexity of betting behavior can be explained by the type of models we study here,<sup>2</sup> we are led to ask, how much can be explained by these models if we relax some of the more demanding assumptions of the Bayesian model. Specifically, we consider maxmin expected utility with a nonunique prior (Gilboa and Schmeidler (1989)) that captures Knightian uncertainty (Knight (1921)). Assume that such uncertainty averse agents who are also risk averse, give rise to an economy in which there is no aggregate risk. When does there exist full insurance, i.e., no-bet allocations that are also Pareto optimal? When is it the case that *all* Pareto

<sup>1</sup> We thank participants of the Erasmus conference at Tilburg University and two referees for useful comments.

<sup>2</sup> In particular, we ignore the social aspects of betting as well as the strategic ones (see, e.g., Milgrom and Stokey (1982)).

optimal allocations are full insurance? Is any betting due to different beliefs, and, conversely, does a difference in beliefs always trigger some betting?

In the multiple prior model an individual is characterized by a utility function and a nonempty, closed, and convex set of probability measures. The individual evaluates every act by its expected utility according to each possible probability measure, and chooses an act whose minimal expected utility is the highest. The family of preference relations described by this model strictly contains the relations described by Choquet expected utility with a convex capacity (see Schmeidler (1989)).

Consider now a pair of agents conforming to the multiple prior model. It is an easy extension of the expected utility analysis to show that these agents will not bet against one another if they share at least one prior. Moreover in a general framework with more than two agents and complex bets possibly involving several of them, it is easy to show, following Dow and Werlang (1992) early intuition, that Pareto optimal allocations are indeed full insurance allocations whenever agents' sets of priors have a nonempty intersection (see, e.g., Tallon (1998), Dana (1998)).

The question of whether the converse to this result holds arises naturally: is commonality of beliefs, in the sense of agents sharing a prior in common, exactly what is needed to explain, within the framework of the multiple prior model, the absence of betting on the many possible sources of "extrinsic" uncertainty? Differently put, is the observation of a Pareto optimal allocation that is immune to sunspots enough to tell us something about the intersection of agents' sets of priors?

It turns out that we can answer this question affirmatively and that the result in the Bayesian model has a conceptually identical counterpart in the multiple prior model. Under the same nontriviality conditions, there exists a Pareto optimal full insurance allocation if and only if *all* Pareto optimal allocations provide full insurance, and this holds if and only if all agents share a prior probability on the states of the world. In other words, commonality of beliefs is the necessary and sufficient condition to explain the absence of betting. Whereas in the Bayesian model "sharing a prior" could only mean "having an identical prior," in the multiple prior model this phrase may be read as "having at least one prior in common." With this grammatical convention in place, the result holds verbatim.

Bayesian agents either agree on probability assessments, or disagree enough to bet against each other. By contrast, uncertainty averse agents can be in a "grey area" between agreeing and disagreeing: they may not agree in the sense of having the same set of possible priors, yet not disagree in the sense of being willing to bet against each other.

Finally, we emphasize another contribution of this note. In showing that commonality of beliefs is the minimal assumption explaining the absence of bets, we prove a separation theorem for  $n$  convex sets that might be of interest on its own.

The rest of this paper is organized as follows. Section 2 provides the set up of the model. In Section 3 we state the main result and the separation theorem. Proofs are relegated to an Appendix.

## 2. SET-UP

The economy we consider is a standard two-period pure-exchange economy with uncertainty in the second period, but for agents' preferences. The state space is  $S$ , and  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $S$ , so that  $(S, \Sigma)$  is a measurable state space. There are  $n$

agents indexed by subscript  $i$ . We assume (i) that there is only one good, which can be interpreted as income or money; and (ii) that there is no aggregate uncertainty. Trading an uncertain asset is thus interpreted as betting rather than as hedging. Let  $B(S, \Sigma)$  be the Banach space of real-valued, bounded, and measurable functions on  $S$ , endowed with the sup-norm. Let  $ba(S, \Sigma)$  be the space of bounded finitely additive measures on  $(S, \Sigma)$  endowed with the weak\*-topology. Agent  $i$ 's consumption  $C_i$ , is a positive element of  $B(S, \Sigma)$ , that is,  $C_i(s)$  is the consumption of agent  $i$  in state  $s$ . Denote by  $w \in B(S, \Sigma)$  the constant-across-states aggregate endowment, and assume that  $w > 0$ . An allocation  $C = (C_1, \dots, C_n)$  is feasible if  $\sum_{i=1}^n C_i = w$ . An allocation is interior if  $C_i(s) > 0$  for all  $i$ , for all  $s$ .

In the multiple-prior approach, each agent  $i$  is endowed with a utility index  $U_i: \mathbb{R}_+ \rightarrow \mathbb{R}$  and a set  $\mathcal{P}_i$  of probability distributions over  $S$ .  $U_i$  is defined up to a positive affine transformation, and is taken to be differentiable, strictly increasing, and strictly concave.  $\mathcal{P}_i$  is a convex and closed set of  $ba(S, \Sigma)$ . We assume that all priors in  $\mathcal{P}_i$  are  $\sigma$ -additive.<sup>3</sup> Note that  $\mathcal{P}_i$  is compact in the weak\*-topology since it is a weak\*-closed subset of the set of finitely-additive probability measures on  $\Sigma$ , which is compact in the weak\*-topology (see, e.g., Dunford and Schwarz (1958)). The norm-dual of  $B(S, \Sigma)$  which is isometrically isomorphic to  $ba(S, \Sigma)$  will be denoted  $B^*(S, \Sigma)$ .

The overall utility function  $V_i$  defined over  $B(S, \Sigma)$  then takes the following form:

$$V_i(C_i) = \min_{\pi \in \mathcal{P}_i} E_{\pi} U_i(C_i).$$

We assume throughout that:

$$\forall A \in \Sigma, \forall i, j, \forall \pi_i \in \mathcal{P}_i, \forall \pi_j \in \mathcal{P}_j, \pi_i(A) = 0 \Leftrightarrow \pi_j(A) = 0.$$

This assumption essentially says that all agents agree on “null events.”

The last definition we need is that of a full insurance allocation. An allocation  $C$  is said to be full insurance if it is constant apart from a set  $A \in \Sigma$  that has  $\pi_i(A) = 0$  for some (and therefore, by the assumption of mutual absolute continuity, for all)  $\pi_i \in \mathcal{P}_i$  and  $i$ .<sup>4</sup>

### 3. THE MAIN RESULT

The following theorem states that the set of Pareto optimal allocations and the set of full insurance allocations are either identical or disjoint. Moreover, they are identical if and only if the agents share at least one prior.

<sup>3</sup> Note that the axiomatization of Gilboa and Schmeidler (1989) delivers only finitely additive probability distributions.

<sup>4</sup> It is straightforward to check that  $C$  is of full-insurance if and only if  $\forall i, C_i$  is constant apart from a set  $A_i \in \Sigma$  that has  $\pi_i(A_i) = 0$  for some (and therefore, by assumption of mutual absolute continuity, for all)  $\pi_i \in \mathcal{P}_i$ .

THEOREM 1: *Under the maintained assumptions, the following assertions are equivalent:*

- (i) *There exists an interior full insurance Pareto optimal allocation.*
- (ii) *Any Pareto optimal allocation is a full insurance allocation.*
- (iii) *Every full insurance allocation is Pareto optimal.*
- (iv)  $\bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset$ .

The intuition for the proof (and the role of some assumptions) is as follows. We prove that (iv)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (iv). If there is a common prior (iv), one can use strict concavity to show that a risk bearing allocation is Pareto dominated by the full insurance allocation that equals its expectation at every state, proving (ii).<sup>5</sup> This step uses the mutual absolute continuity assumption, as well as the assumption that the probability measures we deal with are  $\sigma$ -additive (rather than only finitely additive). Observe that with finitely additive measures the implication (iv)  $\Rightarrow$  (ii) does not hold, even in a Bayesian set-up. This is so because the integral of a function with respect to a finitely additive measure may be strictly smaller than each of the values the function assumes. Therefore individuals who hold assets that they view as uncertain may not benefit from smoothing them across states. If every Pareto improving allocation provides full insurance (ii), the converse (iii) also holds, since no two full insurance allocations can be Pareto ranked,<sup>6</sup> and it follows trivially that there is at least one such allocation (i). Finally, the crucial step and the main contribution of the theorem is that the existence of a full insurance Pareto optimal allocation (i) implies that there is a common prior (iv). This step does not require concavity of the utility index.<sup>7</sup> In proving this last part we make use of the following theorem, which generalizes the standard separating hyperplane theorem, and may be of interest on its own. In the Appendix we also comment on the geometric interpretation of this result, which may be viewed as a separation theorem among  $n$  convex sets.

THEOREM 2: *Let  $X$  be a locally convex linear topological space and let  $\mathcal{P}_i \subseteq X$ ,  $1 \leq i \leq n$ , be convex, nonempty, and compact. Then, the following are equivalent:*

- (i)  $\bigcap_{i=1}^n \mathcal{P}_i = \emptyset$ .
- (ii) *There exist  $I \subseteq \{1, \dots, n\}$ ,  $I \neq \emptyset$  and  $p \in \text{co}(\bigcup_{i \in I} \mathcal{P}_i)$  and for each  $i \in I$ , there exists a continuous linear functional  $h_i: X \rightarrow \mathbb{R}$  such that:*

- (a)  $\forall i \in I, \quad h_i(q - p) > 0 \quad \text{for all } q \in \mathcal{P}_i,$
- (b)  $\sum_{i \in I} h_i = 0.$

An immediate corollary of Theorem 2 is that, under the same assumption, if  $\bigcap_{i=1}^n \mathcal{P}_i = \emptyset$ , there exist continuous linear functionals  $h_i$ ,  $i = 1, \dots, n$ , and a point  $p$  such that (a')  $h_i(q - p) \geq 0$  for all  $q \in \mathcal{P}_i$ , for all  $i$ , (b')  $\sum_{i=1}^n h_i = 0$ , and (c') there exist  $i, i'$  such that the inequality in (a') is strict.

It is worthy of note that a similar result, developed independently and with a rather different motivation, is to be found in Samet (1998), for subsets of a finite dimensional

<sup>5</sup> This implication follows the logic of similar results for Choquet expected utility in Chateauneuf, Dana, and Tallon (1998).

<sup>6</sup> The fact that (iv) implies (ii) and (iii) also appears in Dana (1998) but in a finite set-up.

<sup>7</sup> Dana (1998) shows that if there is a full insurance competitive equilibrium in this economy with finitely many states, then agents share a prior in common. Her proof, however, uses the concavity of the utility index and relies on the existence of a competitive equilibrium.

simplex. Samet's result is weaker in the sense that it guarantees the existence of linear functionals as in our case, but does not guarantee that the separating hyperplanes will intersect at one point  $p$  in the convex hull of the sets, and therefore does not yield itself to a straightforward geometric interpretation. Further, Samet's result can be easily derived from the corollary above specialized to subsets of the simplex. It does not appear that Samet's argument could easily be amended to get ours.

Theorem 1 has two immediate corollaries. First, in the Choquet expected utility model with convex capacities, nonempty core intersection is equivalent to some, or all, Pareto optimal allocations being full insurance. Second, in the expected utility case, where the sets of priors are reduced to one point, some, or all, Pareto optimal allocations are full insurance allocations if and only if agents have the same beliefs (i.e., the same prior).

Note that even though we cast the argument in the multiple prior model, it should be clear from the proof that a similar result holds for the Bewley (1986) approach. In Bewley's approach, agents are also endowed with a set of priors and move away from a (exogenously defined) status quo situation only if the new situation is better than the status quo for all the probability distributions in their set of priors. While Bewley characterizes a partial order over acts, a proposed bet will be preferred to a certain status quo if and only if this preference holds in the multiple prior model of Gilboa and Schmeidler.<sup>8</sup>

Our analysis is conducted for an economy with one good. However, the only use we make of this assumption is in arguing that all full insurance allocations are Pareto optimal. Indeed, one can generalize our results to an economy with  $m$  goods, with the slight modification that full insurance allocations that are considered for optimality be assumed Pareto optimal in each state.

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## APPENDIX

PROOF OF THEOREM 1: We first prove (iv)  $\Rightarrow$  (ii) Assume to the contrary that there exists an agent, say, agent 1, such that for every  $\pi_1 \in \mathcal{F}_1$  and every  $c \in \mathbb{R}_+$ ,  $\pi_1(\{s|C_1(s) < c\}) + \pi_1(\{s|C_1(s) > c\}) > 0$ .

<sup>8</sup> Bewley (1989) contains a similar no-trade result for agents whose preferences are given by partial orders as in Bewley (1986). His proof is very similar to Samet's, and his result is weaker than Theorem 2 in the same sense that Samet's is.

Let  $\pi \in \cap_i \mathcal{P}_i$  and define  $\bar{C}_i = E_\pi C_i$  for all  $i$ . Abusing notation, let  $\bar{C}_i$  also denote the constant allocation giving  $\bar{C}_i$  to agent  $i$  in all states.  $\bar{C} = (\bar{C}_i)_i$  is a feasible allocation since  $\sum_i \bar{C}_i = \sum_i E_\pi C_i = E_\pi(\sum_i C_i) = E_\pi w \mathbf{1}_S = w$ . Now,

$$V_i(C_i) = \min_{\varphi \in \mathcal{P}_i} E_\varphi U_i(C_i) \leq E_\pi U_i(C_i).$$

Furthermore,

$$E_\pi U_i(C_i) \leq U_i(E_\pi(C_i)) = U_i(\bar{C}_i) = V_i(\bar{C}_i)$$

for all  $i$  since  $U_i$  is concave.

Since  $\pi$  belongs to  $\mathcal{P}_i$ , one gets that

$$\pi(\{s|C_i(s) < \bar{C}_i\}) + \pi(\{s|C_i(s) > \bar{C}_i\}) > 0.$$

Furthermore,  $\pi(\{s|C_i(s) < \bar{C}_i\}) = 0$  is impossible, for then  $\pi(\{s|C_i(s) > \bar{C}_i\}) > 0$ , implying by  $\sigma$ -additivity of  $\pi$  that  $E_\pi(C_i) > \bar{C}_i$ , a contradiction. Hence,  $\pi(\{s|C_i(s) < \bar{C}_i\}) > 0$  and, similarly,  $\pi(\{s|C_i(s) > \bar{C}_i\}) > 0$ .

It follows that  $V_i(C_i) < V_i(\bar{C}_i)$  since  $U_i$  is strictly concave. Therefore, the allocation  $\bar{C}$  Pareto dominates  $C$ , a contradiction.

To see that (ii) implies (iii), let  $C$  be a full insurance allocation. Assume, contrary to (iii), that it is not Pareto optimal, and is dominated by another allocation  $C'$ . By the same argument as above,  $\bar{C}'$  is at least as desirable as  $C'$  for every agent. By transitivity of Pareto domination,  $\bar{C}'$  Pareto dominates  $C$ . But this is a contradiction since both provide full insurance and there is only one good in the economy.

That (iii) implies (i) is obvious, and it remains to prove that (i) implies (iv). Suppose to the contrary that  $\cap_i \mathcal{P}_i = \emptyset$ , and let  $C$  be an interior Pareto optimal allocation that is a full-insurance allocation ( $C_i$  is constant for all  $i$  apart on a set of measure zero, the latter notion being defined unambiguously given our absolute mutual continuity assumption). By Theorem 2 (where  $X$  is  $B^*(S, \Sigma)$  endowed with the weak\*-topology), since  $\cap_i \mathcal{P}_i = \emptyset$ , there exists a nonempty set  $I$ , a point  $p$  and functionals  $h_i \in B^*(S, \Sigma)$ ,  $i \in I$ , such that:

- (a)  $\forall i \in I, h_i(q - p) > 0$  for all  $q \in \mathcal{P}_i$ ,
- (b)  $\sum_{i \in I} h_i = 0$ .

Recall that (see, e.g., Kelley and Namioka (1963, p. 155)) every weak\*-continuous linear functional on the conjugate space of a linear topological space  $E$  is the evaluation at some point of  $E$ . Hence, for all  $i \in I$ , there exists  $D_i \in \mathcal{B}(S, \Sigma)$  such that  $h_i(p) = p(D_i)$ , for all  $p \in B^*(S, \Sigma)$ .

Construct the allocation  $(\hat{C}_i)_{i=1, \dots, n}$  as follows:

$$\begin{aligned} \hat{C}_i &= C_i, \quad i \notin I, \\ \hat{C}_i &= C_i + \varepsilon [D_i - p(D_i)\mathbf{1}_S], \quad i \in I, \end{aligned}$$

with  $\varepsilon > 0$  small enough so that  $\hat{C}$  is an allocation.

We first check that this allocation is feasible:

$$\begin{aligned} \varepsilon \left[ \sum_{i \in I} D_i - \sum_{i \in I} p(D_i)\mathbf{1}_S \right] &= \varepsilon \left[ \sum_{i \in I} D_i - \sum_{i \in I} h_i(p) \right] \\ &= \varepsilon \sum_{i \in I} D_i \quad \text{since} \quad \sum_{i \in I} h_i = 0. \end{aligned}$$

Now,  $D_i$  is such that  $h_i(q) = q(D_i)$  for all  $q \in B^*(S, \Sigma)$  and hence  $q(\sum_{i \in I} D_i) = 0$  for all  $q \in B^*(S, \Sigma)$ .

To conclude that  $\sum_{i \in I} D_i = 0$ , suppose there exists  $s$  such that  $\sum_{i \in I} D_i(s) = a$ ,  $a \neq 0$ . The event  $\{s | \sum_{i \in I} D_i(s) = a\}$  is measurable because the  $D_i$  are measurable. Now, let  $q$  be the continuous linear functional in  $B^*(S, \Sigma)$  corresponding to the additive probability in  $\mathcal{B}(S, \Sigma)$  with the mass 1 on that event. Then  $q(\sum_{i \in I} D_i) = 0$  implies  $a = 0$ , a contradiction. Hence,  $\sum_{i \in I} D_i = 0$ .

Now, for  $i \in I$ , one has:

$$\begin{aligned} V_i(\hat{C}_i) &= E_{\hat{q}^\varepsilon} U_i(C_i + \varepsilon[D_i - p(D_i)\mathbf{1}_S]) \quad \text{for some } \hat{q}^\varepsilon \in \mathcal{P}_i \\ &= V_i(C_i) + \varepsilon U_i'(C_i)[\hat{q}^\varepsilon(D_i) - p(D_i)] + o(\varepsilon) \\ &= V_i(C_i) + \varepsilon U_i'(C_i)[h_i(\hat{q}^\varepsilon - p)] + o(\varepsilon) \\ &\geq V_i(C_i) + \varepsilon U_i'(C_i) \left[ \inf_{q \in \mathcal{P}_i} h_i(q - p) + \alpha(\varepsilon) \right] \end{aligned}$$

where  $\alpha(\varepsilon) = o(\varepsilon)/\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Since  $\inf_{q \in \mathcal{P}_i} h_i(q - p) > 0$  by continuity of  $h_i$  and compactness of  $\mathcal{P}_i$ , and  $\alpha(\varepsilon) \rightarrow 0$ , there exists  $\varepsilon$  small enough so that the term in bracket is strictly positive.

Hence,  $V_i(\hat{C}_i) > V_i(C_i)$  for  $i \in I$ , and we found a Pareto dominating allocation  $(\hat{C}_i)_{i=1, \dots, n}$ , a contradiction. Q.E.D.

PROOF OF THEOREM 2: We start with the following lemma:

LEMMA: Let  $X$  be a locally convex linear topological space and let  $\mathcal{P}_i \subseteq X$ ,  $1 \leq i \leq n$  be convex, nonempty, and compact. Assume that  $\bigcap_{i \leq n} \mathcal{P}_i = \emptyset$  but that for all  $l \leq n$ ,  $\bigcap_{i \neq l} \mathcal{P}_i \neq \emptyset$ . Then, there exist  $p \in \text{co}(\bigcup_{i=1}^n \mathcal{P}_i)$  and a continuous linear functional  $h_i: X \rightarrow \mathbb{R}$  for each  $i \leq n$  such that:

- (a)  $\forall i \leq n, h_i(q - p) > 0 \forall q \in \mathcal{P}_i$ ;
- (b)  $\sum_{i \leq n} h_i = 0$ .

The geometric interpretation of this lemma is as follows. Assume that  $n$  convex and compact sets have an empty intersection, but that every subset of them has a nonempty intersection. Then, we can find a point  $p$  that is not included in any set, but that is “in the middle” in the following sense: one can find, for each set  $\mathcal{P}_i$ , a hyperplane  $h_i$  that passes through  $p$  that is in the convex hull of the union of the  $\mathcal{P}_i$  and leaves the entire  $\mathcal{P}_i$  on one side, such that the normals of these hyperplanes, multiplied by appropriate positive constants, add up to zero. In the case  $n = 2$ , our lemma reduces to a standard separation theorem between two disjoint sets. For  $n > 2$ , the lemma may be considered as an  $n$ -way separation among  $n$  convex sets. See Figure 1 for an illustration of the case  $n = 3$ .

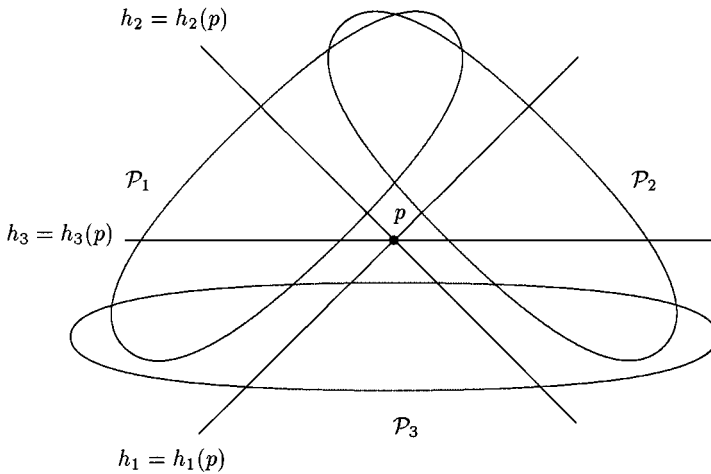


FIGURE 1.—Separation among three convex sets.

PROOF OF THE LEMMA: The proof is by induction on  $n$ . For  $n = 2$ , we have  $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$  and we use a standard separation theorem (cf. Kelley and Namioka (1963, p. 119, theorem on strong separation)) to conclude that there is a continuous linear functional  $h: X \rightarrow \mathbb{R}$  and a number  $\beta \in \mathbb{R}$  such that  $h(q) > \beta$  for  $q \in \mathcal{F}_1$  and  $h(q) < \beta$  for  $q \in \mathcal{F}_2$ . Choose  $p$  such that  $h(p) = \beta$ , and set  $h_1 = h$  and  $h_2 = -h$ . By linearity of  $h$  it is possible to choose  $p \in \text{co}(\mathcal{F}_1 \cup \mathcal{F}_2)$ .

Assume that the lemma holds for every  $n' < n$ . Let there be given  $(\mathcal{F}_i)_{i=1}^n$ . Set  $A = \bigcap_{i < n} \mathcal{F}_i$  and  $B = \mathcal{F}_n$ . Observe that  $A$  and  $B$  are convex, nonempty, and compact. Furthermore, they are disjoint since  $\bigcap_i \mathcal{F}_i = \emptyset$ . Apply the same separation theorem to conclude that there exist a continuous linear  $\tilde{h}_n: X \rightarrow \mathbb{R}$  and  $\beta \in \mathbb{R}$  such that

$$\tilde{h}_n(q) > \beta \quad \forall q \in B \quad \text{and} \quad \tilde{h}_n(q) < \beta \quad \forall q \in A.$$

Choose  $q_0 \in X$  such that  $\tilde{h}_n(q_0) = \beta$ . We shift the origin to  $q_0$ . Specifically, define for each  $i \leq n$ ,  $\hat{\mathcal{F}}_i = \{p - q_0 \mid p \in \mathcal{F}_i\} = \mathcal{F}_i - q_0$ . Naturally,  $(\hat{\mathcal{F}}_i)_{i=1}^n$  and their intersections inherit all relevant properties of  $(\mathcal{F}_i)$ . Denote  $\hat{B} = B - q_0 = \hat{\mathcal{F}}_n$  and  $\hat{A} = A - q_0 = \bigcap_{i < n} \hat{\mathcal{F}}_i$  and observe that  $\tilde{h}_n(q) > 0 \quad \forall q \in \hat{B}$  and  $\tilde{h}_n(q) < 0 \quad \forall q \in \hat{A}$ . Consider  $X' = \{q \in X \mid \tilde{h}_n(q) = 0\}$ .  $X'$  is a locally convex linear topological subspace of  $X$ . Focusing on this subspace, define  $\hat{\mathcal{F}}_i = \hat{\mathcal{F}}_i \cap X'$  for  $i < n$ . Obviously,  $\hat{\mathcal{F}}_i$  is convex and compact for every  $i < n$ . We argue that it is also nonempty. Indeed,  $\hat{\mathcal{F}}_i$  contains  $\hat{A}$ . On the other hand,  $\hat{\mathcal{F}}_i$  has a nonempty intersection with  $\hat{B} = \hat{\mathcal{F}}_n$ . By convexity of  $\hat{\mathcal{F}}_i$  and linearity of  $\tilde{h}_n$ ,  $\hat{\mathcal{F}}_i \neq \emptyset$ . Similarly, for  $l < n$ ,  $\bigcap_{i \neq l, n} \hat{\mathcal{F}}_i$  contains  $\hat{A}$  and intersects  $\hat{B}$  and we therefore get

$$\bigcap_{i \neq l, n} \hat{\mathcal{F}}_i \neq \emptyset \quad \forall l < n.$$

However,  $X'$  is a hyperplane separating  $\hat{B}$  from  $\hat{A}$ . Hence  $\bigcap_{i < n} \hat{\mathcal{F}}_i = \emptyset$ . It follows that  $(\hat{\mathcal{F}}_i)_{i < n}$  on  $X'$  satisfy the conditions of the lemma for  $n' = n - 1$ . Therefore, there exist a point  $\hat{p} \in \text{co}(\bigcup_{i=1}^{n-1} \hat{\mathcal{F}}_i)$  and continuous linear functionals  $H_i: X' \rightarrow \mathbb{R}$ ,  $i < n$ , such that  $H_i(q - \hat{p}) > 0 \quad \forall q \in \hat{\mathcal{F}}_i$ ,  $i < n$ , and  $\sum_{i < n} H_i = 0$  on  $X'$ . Using standard arguments (see Fact 1 below), we conclude that, for every  $i < n$ ,  $H_i$  on  $X'$  can be extended to  $h_i$  on all of  $X$  such that:

$$h_i(q - \hat{p}) > 0 \quad \forall q \in \hat{\mathcal{F}}_i.$$

Define  $h = \sum_{i < n} h_i$  on  $X$ . Observe that for every  $q \in X'$ ,

$$h(q) = \sum_{i < n} h_i(q) = \sum_{i < n} H_i(q) = 0.$$

Hence  $\tilde{h}_n$  and  $h$  are continuous linear functionals on  $X$  satisfying

$$\tilde{h}_n(q) = 0 \Rightarrow h(q) = 0 \quad \forall q \in X.$$

By standard arguments (see Fact 2 below), there exists  $\alpha \in \mathbb{R}$  such that  $h(q) = \alpha \tilde{h}_n(q) \quad \forall q \in X$ .

We wish to show that  $\alpha < 0$ . Consider  $q \in \hat{A} = \bigcap_{i < n} \hat{\mathcal{F}}_i$ . Since  $h_i(q - \hat{p}) > 0 \quad \forall i < n$  and  $h(\hat{p}) = 0$ , we obtain

$$h(q) = h(q - \hat{p}) = \sum_{i < n} h_i(q - \hat{p}) > 0.$$

On the other hand,  $\tilde{h}_n(q) < 0$  since  $q \in \hat{A}$ . It follows that  $\alpha < 0$ .

Define  $h_n = (-\alpha) \tilde{h}_n$ . Since  $(-\alpha) > 0$ ,  $h_n(q - \hat{p}) = h_n(q) > 0 \quad \forall q \in \hat{\mathcal{F}}_n$ .

To conclude, set  $p = \hat{p} + q_0$ . Observe that  $p \in \text{co}(\bigcup_{i=1}^{n-1} \mathcal{F}_i)$  and hence  $p \in \text{co}(\bigcup_{i=1}^n \mathcal{F}_i)$ . We claim that  $p$  and  $(h_i)_{i \leq n}$  satisfy (a) and (b). Indeed, for every  $i \leq n$ , and every  $q \in \mathcal{F}_i$ :

$$h_i(q - p) = h_i((q - q_0) - (p - q_0)) = h_i((q - q_0) - \hat{p}) > 0$$

since  $q - q_0 \in \hat{\mathcal{F}}_i$ . Finally,  $\sum_{i \leq n} h_i = 0$  by construction of  $h_n$ .

*Q.E.D.*



The following two facts, which are used in the proof above, are straightforward and/or well-known.

FACT 1: Let  $X$  be a locally convex linear topological space. Let  $\tilde{h}$  be a continuous linear functional and  $X' = \{p \in X \mid \tilde{h}(p) = 0\}$ . Assume that  $C \subseteq X$  is convex and compact, and that  $C \cap X' \neq \emptyset$ . Further assume that  $h': X' \rightarrow \mathbb{R}$  is a continuous linear functional such that  $h'(p) > 0 \forall p \in C \cap X'$ . Then,  $h'$  can be extended to a continuous linear functional  $h: X \rightarrow \mathbb{R}$  such that  $h(p) > 0 \forall p \in C$ .

PROOF OF FACT 1: Set  $D = \{p \in X' \mid h'(p) = 0\}$ . Observe that  $D \neq \emptyset$  since the origin is in  $D$ . Thus  $C$  and  $D$  are disjoint nonempty closed and convex sets in  $X$ , and  $C$  is compact. Let a continuous linear functional  $\tilde{h}: X \rightarrow \mathbb{R}$  and  $d \in \mathbb{R}$  be such that

$$\tilde{h}(p) < d \quad \forall p \in D \quad \text{and} \quad \tilde{h}(p) > d \quad \forall p \in C.$$

We claim that  $\tilde{h}$  has to be constant on  $D$ . Indeed, assume that for some  $p, q \in D$ ,  $\tilde{h}(p) \neq \tilde{h}(q)$ . Since  $p, q \in D$  implies  $\tilde{h}(p) = \tilde{h}(q) = 0$  and  $h'(p) = h'(q) = 0$ , we conclude that  $p + \alpha(q - p) \in D$  for all  $\alpha \in \mathbb{R}$ . Hence  $\{\tilde{h}(p + \alpha(q - p)) \mid \alpha \in \mathbb{R}\} = \mathbb{R}$ , a contradiction to the fact that  $\tilde{h}(p) < d \forall p \in D$ . Thus there is a  $c \in \mathbb{R}$  such that  $\tilde{h}(p) = c \forall p \in D$ . Since the origin is in  $D$ , we obtain  $c = 0$ . It follows that  $d > 0$  and therefore

$$\tilde{h}(p) > d > 0 \quad \forall p \in C.$$

We now wish to show that, up to multiplication by a positive constant,  $\tilde{h}$  extends  $h'$  on  $X$ . Restrict attention to  $X'$ . If  $p \in X'$  satisfies  $h'(p) = 0$ , then  $p \in D$  and we know that  $\tilde{h}(p) = 0$ . By Fact 2 below, there exists  $\alpha \in \mathbb{R}$  such that  $\tilde{h}(p) = \alpha h'(p) \forall p \in X'$ . However, on  $C \cap X'$ , both  $\tilde{h}$  and  $h'$  are positive. Therefore  $\alpha > 0$ . Hence  $h \equiv (1/\alpha)\tilde{h}$  extends  $h'$  on  $X$  and is positive on all of  $C$ . *Q.E.D.*

FACT 2: Let  $X$  be a linear space and let  $\tilde{h}, h: X \rightarrow \mathbb{R}$  be linear. Assume that

$$\tilde{h}(q) = 0 \Rightarrow h(q) = 0 \quad \forall q \in X.$$

Then there exists  $\alpha \in \mathbb{R}$  such that  $h(q) = \alpha \tilde{h}(q) \forall q \in X$ .

We skip the proof of this Fact and now turn to the proof of Theorem 2:

(i)  $\Rightarrow$  (ii). Assume that  $\bigcap_{i \leq n} \mathcal{P}_i = \emptyset$ . Let  $I$  be a minimal (with respect to set inclusion) subset of  $\{1, \dots, n\}$  with the property that  $\bigcap_{i \in I} \mathcal{P}_i = \emptyset$ . Since  $\bigcap_{i=1}^n \mathcal{P}_i = \emptyset$ , but  $\mathcal{P}_i \neq \emptyset$  for every  $i$ , such a set  $I$  exists and for every such set  $|I| \geq 2$ . Apply the Lemma to  $I$ .

(ii)  $\Rightarrow$  (i). Assume that a point  $p \in X$ , a set  $I \subseteq \{1, \dots, n\}$ , and functionals  $(h_i)_{i \in I}$  exist as required, and suppose, contrary to (i), that there exists  $q \in \bigcap_{i \leq n} \mathcal{P}_i$ . Then, by (a),  $\sum_{i \in I} h_i(q - p) > 0$ , contrary to (b). *Q.E.D.*

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