Trading ambiguity: a tale of two heterogeneities *

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Abstract

We consider financial markets with heterogeneously ambiguous assets and heterogeneously ambiguity averse investors. Investors' preferences, a version of the smooth ambiguity model, are a parsimonious extension of the standard mean-variance framework. We consider, in turn, portfolio choice, equilibrium prices, and trade upon arrival of public information, and show, in each case, there are departures from the outcome in standard theory. These departures are of significance as they occur in the direction of empirical regularities that belie the standard theory.

Keywords: ambiguity, ambiguity aversion, asset pricing, cross-sectional returns, earnings announcements, parameter uncertainty, portfolio choice, trading volume

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1 Introduction

Modern decision theory uses the term *ambiguity* to describe uncertainty about a data generating process. The decision-maker believes that the data comes from an unknown member of a set of possible models. Knight (1921) and Ellsberg (1961) intuitively argue that concern about this uncertainty induces a decision-maker to want decision rules that work robustly across the set of models believed to be possible. The argument is formalized in pioneering contributions by Schmeidler (1989) and Gilboa and Schmeidler (1989) followed by a body of subsequent work including robust control theory (Hansen and Sargent (2008)) and the theory of smooth ambiguity aversion (Klibanoff et al. (2005)).

The financial literature largely proceeds from the assumption that investors behave as if they know the distributions of returns, ruling out ambiguity. However, this assumption is hard to justify. Finer sampling would, arguably, virtually eliminate estimation errors for second moments of return distributions, but it is well established that first moments (i.e., means) are extremely difficult to estimate (Merton (1980), Cochrane (1997), Blanchard (1993), and Anderson et al. (2003)). This paper considers investors who are concerned about the ambiguity of return distributions, more specifically, the ambiguity due to the uncertainty about the means of returns. We conceive of this *parameter uncertainty* in a Bayesian fashion: unknown means are treated as random variables. More concretely, combining a prior over the means for the set of assets considered with observations from the data, we take the resulting posterior joint distribution of the means to describe the parameter uncertainty, i.e., the ambiguity about the return distributions.[1] The ambiguity averse investor is inclined to choose a portfolio position whose value is less affected by, and hence robust to, the parameter uncertainty.[2]

We use *smooth ambiguity* to model investors' ambiguity aversion.[3] If the data generating process were known, then a smooth ambiguity averse investor would evaluate a portfolio by its expected utility given that process. However, given the parameter uncertainty, the posterior joint distribution of the asset mean returns together with the portfolio induces a distribution over possible expected utility values. If one takes two portfolio positions a and a' such that the distribution induced by a' is a mean-preserving spread of the distribution induced by a, then the smooth ambiguity-averse investor prefers a to a'. Increasing the ambiguity aversion makes the preference for a over a' stronger; passing to the limit, the maximally ambiguity averse investor evaluates a portfolio by only considering its minimum expected utility value, as e.g., in Garlappi et al. (2007). The (smooth) ambiguity-neutral investor, on the other hand, is indifferent between two such portfolios and hence maximizes his expected utility by investing in a Bayesian optimal portfolio. The pioneering works of Klein and Bawa (1976) and Brown (1979)

[1] An alternative conception of parameter uncertainty, in the classical rather than Bayesian approach, is offered by Garlappi et al. (2007) in the form of *confidence intervals* around the point estimates of means.

[2] See,e.g., Dow and Werlang (1992), Epstein and Wang (1994), Mukerji and Tallon (2001), Uppal and Wang (2003), Garlappi et al. (2007), Hansen and Sargent (2010), Gollier (2011), Boyle et al. (2012), Maccheroni et al. (2013), and *inter alia*.

[3] Trautmann and van de Kuilen (2015) surveys experimental evidence on ambiguity aversion. See Cubitt et al. (2018), and references to the literature therein, for experimental evidence comparing the smooth ambiguity model to other alternatives. Dimmock et al. (2016) and Bianchi and Tallon (2018) use survey data to relate evidence on individuals' ambiguity aversion to their behavior in financial markets, for instance, portfolio holdings. Chew et al. (2018) document that ambiguity aversion is more prevalent among analytically sophisticated subjects.

study portfolio choice and pricing implications of parameter uncertainty by modeling investors as choosing Bayesian optimal portfolios.

Recent contributions have demonstrated ambiguity aversion can significantly help to explain empirical regularities involving intertemporal price movements of aggregate uncertainty.[4] In this paper, we consider market implications when there are multiple uncertain assets with heterogeneous ambiguity, i.e., assets that can be ranked in terms of uncertainty about mean returns. The paper considers, in turn, portfolio choice, equilibrium prices and returns, and trade upon arrival of public information, given such heterogeneity in the cross-section of assets. We find that there can be departures from the predictions of standard theory, given the presence of another key ingredient – a second heterogeneity: multiple agents who are heterogeneously ambiguity averse. Significantly, these departures occur in the direction of empirical regularities that belie the standard theory. In this way, novel to the literature, ambiguity is shown to have the potential for explaining cross-sectional empirical regularities including aspects of nature and volume of trade. That ambiguity has this potential is important since many of the other new theories at the vanguard of consumption-based asset pricing, which are good at explaining aggregate puzzles such as equity premium and excess volatility, do not have similar explanatory power when it comes to cross-sectional puzzles (see, e.g., Lettau and Ludvigson (2009)).[5]

Our specification of the smooth ambiguity model simplifies to a parsimonious extension of the standard mean-variance framework. In this extension the investor faces a three-way trade-off between expected return, risk and ambiguity. As intuition suggests, we find that more ambiguity averse investors resolve this trade-off by putting more weight on ambiguity and thus are more partial to the less ambiguous assets. This key, single driving force connects our results on portfolio choice, equilibrium asset prices, and trade upon arrival of public information. By doing so, our theory speaks to several puzzling phenomena in a unified fashion: the asset allocation puzzle (Canner et al. (1997)), the size and value premia (Fama and French (1992), (1993)), the empirical security market line being flatter than the one predicted by the CAPM (Black et al. (1972) and Treynor and Black (1973)), and the observation that earnings announcements are often followed by significant trading volume with small price change (Kandel and Pearson (1995)).

As we noted, investors' response to heterogeneity in the ambiguity of assets is key to our results. Intuitively, this heterogeneity may arise for at least a couple of reasons. One reason may be that one asset is structurally more exposed to uncertainty quite generally, whether it be risk or ambiguity. For instance, a firm's stock return is structurally more exposed to uncertainty than its bond return as stock is a residual claim. Moreover, bond returns are exposed to only downside uncertainty whereas stock returns are exposed to both downside and upside uncertainty. A second reason is about the fundamentals of the underlying asset. For instance,

[4]For instance, see Maenhout (2004), Epstein and Schneider (2008), Illeditsch (2011), Condie et al. (2015), and in a more macro-finance tradition, Ju and Miao (2012), Hansen and Sargent (2010), and Collard et al. (2018).

[5]The new theories considered in Lettau and Ludvigson (2009) are (i) the representative agent external habit-persistence paradigm of Campbell and Cochrane (1999) that has been modified to accommodate a cross-section of tradable risky assets as in Menzly et al. (2004), (ii) the representative agent long-run risk model based on recursive preferences of Bansal and Yaron (2004), and (iii) the limited participation model of Guvenen (2009). The cross sectional puzzles considered in the study include size and book-market sorted portfolio returns emphasized by Fama and French (1992), (1993). At the time of Lettau and Ludvigson's study, the empirical implications of ambiguity aversion for intertemporal price movements of aggregate uncertainty were not fully appreciated.

new-technology companies or companies exploring new markets would have fundamentals whose risks have not been fully learned. Also, firms which are more exposed to aggregate uncertainty shocks, for instance, because of financial distress or reliance on external financing, would be in this category. The success of the macro-finance literature incorporating ambiguity in explaining price dynamics lends support to the idea of treating aggregate uncertainty as ambiguous. In that literature, the assumed source of the ambiguity in the agent's beliefs is the occurrence of periodic, temporary changes in the probability distribution governing next period's growth outcome due to the effect of the business cycle. In the literature on financial intermediation, Caballero and Krishnamurthy (2008), Uhlig (2010) and Dicks and Fulghieri (2019) give theoretical models that explain the role of ambiguity and ambiguity aversion in exacerbating financial crises through bank runs. In these models uncertainty about the extent of liquidity and profitability shocks affecting banks, can trigger episodes of "flight-to-quality", credit crunch and contagion, which in turn amplify uncertainty shocks and their implications for the macroeconomy.

We adopt the framework of Hara and Honda (2016) to describe investors' common beliefs about asset returns. The asset returns are jointly normally distributed with a known covariance matrix and an unknown mean vector. The means are also taken to be jointly normally distributed, and this latter distribution describes the ambiguity. Given these assumptions on beliefs and appropriate parametric restrictions on the smooth ambiguity model, the investors' evaluation of final payoff is given by a generalization of the mean-variance preference model which Maccheroni et al. (2013) refer to as *robust mean-variance*.[6] This evaluation is a linear function of three terms, namely, the mean of the payoff according to the predictive distribution,[7] the variance of the payoff according to the predictive distribution, and the variance of the unknown mean. The first two terms are as in the mean-variance framework. The third term is novel and encapsulates *ambiguity*. In the evaluation, the second term is weighted by the coefficient of risk aversion whereas the third term is weighted by the coefficient of ambiguity aversion.

We first analyze portfolio choice in a static setting with two uncertain assets and a risk-free asset. Hara and Honda (2016) show that the mutual fund theorem of Tobin (1958) holds if agents are homogeneously ambiguity averse but may fail otherwise.[8] We add to this finding by demonstrating that the key direction of departure from the mutual fund theorem is the tendency of the more ambiguity averse agent to hold less of the more ambiguous asset. This sets the foundation for our subsequent results on equilibrium pricing and trade following public signals. The direction of departure is also empirically compelling to the extent that it is consistent with the so-called *asset allocation puzzle* of Canner et al. (1997) if we were to identify more ambiguity averse agents with more conservative attitudes towards financial investments. Dimmock et al. (2016) and Bianchi and Tallon (2018) provide direct evidence that ambiguity aversion is an important explanatory variable of individual investors' portfolios, in particular portfolio allocation

[6]Our use of the smooth ambiguity model is motivated, in part, by the distinctive parametric separation of ambiguity from ambiguity attitudes it facilitates. This makes it possible to incorporate the two heterogeneities, crucial to our analysis.

[7]The predictive (unconditional) distribution of returns is the weighted mixture of return distributions conditional on means where the weights are dictated by the distribution of means.

[8]This result implies that the mutual fund theorem holds when investors are ambiguity-neutral. This was earlier shown by Klein and Bawa (1976) who model investors facing parameter uncertainty as choosing Bayesian optimal portfolios.

to equities broadly as well as between categories of stocks.

Next we close the model, endogenizing prices, while maintaining multiplicity of agents with heterogeneous ambiguity attitudes. If investors were homogeneously ambiguity averse, then the mutual fund theorem holds and a CAPM-like single-factor pricing formula, where the factor is the excess return of the market portfolio, naturally follows. This echoes the findings of Chen and Epstein (2002) for the case of a single-agent economy in a continuous-time framework and Ruffino (2014) for a static economy where investors have robust mean-variance preferences with homogeneous ambiguity aversion. Our key contribution here is to show that the single-factor pricing formula continues to hold despite the failure of the mutual fund theorem due to the heterogeneity in ambiguity aversion.[9] The pricing formula depends on what we refer to as the market ambiguity aversion and characterize as a function of the distribution of ambiguity aversion in the economy. In particular, we find that agents with intermediate levels of ambiguity aversion have the highest impact on the market ambiguity aversion and therefore pricing.

In the CAPM-like formula we obtain, the standard CAPM beta is adjusted depending on the extent to which the ambiguity of the asset return is correlated with the ambiguity of the market portfolio return. The expected excess return of an asset is shown to be composed of two uncertainty premia: (CAPM-predicted) risk premium, and ambiguity premium. The latter provides a candidate explanation for the so-called *size premium* and *value premium* (or *book-to-market premium*) famously documented by Fama and French (1992), (1993). High book-to-market firms, which tend to be in financial distress, and small-cap firms, due to their over-reliance on external financing, likely carry a high ambiguity premium. This is because, as discussed earlier, such firms have relatively high exposure to aggregate uncertainty. The empirical literature also established that the security market line (i.e., the relation between CAPM betas and excess returns of assets) is flatter than what is predicted by CAPM (Black et al. (1972) and Treynor and Black (1973)). We argue our theory gives a plausible explanation for this observation.

Finally, we propose two dynamic extensions of the static model in order to study how prices and trade respond to the arrival of public information. In the first dynamic extension, in the interim period the agents receive a public signal drawn from the same process which governs the realization of the final-period return. We interpret this signal as an earnings announcement. In the second dynamic extension, we consider uncertainty shocks: the public signal is an event which directly increases or decreases the parameter uncertainty (i.e., ambiguity). To fix ideas, think of the Brexit vote outcome announcement: a binary signal with two possible outcomes, the "leave" signal outcome making a larger parameter uncertainty imminent than a "remain" outcome. In both extensions, we show arrival of public signals leads to trading if and only if agents are heterogeneously ambiguity averse, and trade may occur with no or very small price movements. Most earnings announcements do not convey surprising news, in which case our theory predicts that associated price changes are typically small but there is still significant trading volume. On the other hand, we show, large and surprising uncertainty shocks are associated with both large trading volumes and large price changes.

In the theory developed here, the nature of trading is dictated by uncertainty sharing con-

[9] Our result implies that the standard CAPM holds when all investors are ambiguity neutral, a result that goes back to Brown (1979) who, like Klein and Bawa (1976), model investors facing parameter uncertainty as choosing Bayesian optimal portfolios.

siderations. Following the public signal, the return-risk-ambiguity trade-off changes, making investors seek a different allocation of more and less ambiguous assets depending on their different tolerances for ambiguity. This requires portfolio rebalancing, i.e., changes in the composition of the portfolio of uncertain assets, thus mutually beneficial exchange of such assets. In particular, larger uncertainty shocks cause individual portfolios to move further away from the market portfolio, leading to larger trading volumes. There is evidence of trading caused by ambiguity averse investors' portfolio rebalancing in response to uncertainty shocks. Empirical findings in Dimmock et al. (2016) show that "ambiguity aversion interacts with time-varying levels of economic uncertainty: [investors] with higher ambiguity aversion were more likely to actively reduce their equity holdings during the financial crisis".

Kandel and Pearson (1995) document significant trading volume around earnings announcements often without large price movements, a fact hard to reconcile with standard theory. They show that differential interpretation of public signals can lead to trade without an accompanying price change. Banerjee and Kremer (2010) develop a model where agents agree to disagree, to explain volume of trade. In both models, trade stems from the individuals' willingness to (speculatively) bet against one another based on their differences in beliefs. Thinking solely of the trade as a consequence of differences in beliefs, it is difficult to draw clear welfare conclusions (e.g., see Gilboa et al. (2014)).

As we will show, the robust mean-variance formula of an ambiguity averse investor's evaluation of the portfolio can be re-written *as if* it were the evaluation of a *standard* mean-variance utility investor with a subjective belief where the subjectivity is only evident via an adjustment to the variance term (but not the mean term). The adjustment depends, in part, on the investor's ambiguity aversion. This as if interpretation connects the findings of this paper to those in the differences-of-opinion literature, showing how it builds on some of its insights while departing from others, as we elaborate in Remark 1. That the mechanism of ambiguity aversion may act through the channel of as if subjective beliefs has been observed by, for instance, Hansen and Sargent (2008)(p.9, para.3), Strzalecki and Werner (2011), Gollier (2011) and Collard et al. (2018) in relation to robust control theory, uncertainty sharing, portfolio choice, and asset pricing, respectively.

Finally, in the concluding section, we discuss strategies for empirical testing of the theoretical results and predictions developed in the paper. In particular, we discuss how ambiguity of returns of individual assets may be estimated and how such estimates may then be used to test the predictions for portfolio choice, the effect of ambiguity on returns across the cross-section, and the mechanism of portfolio rebalancing through which changes in ambiguity following public announcements cause episodes of stock trading.

2 The Base Setup

In this section we describe the domain of choice, agents, their beliefs and preferences. We consider a model with two uncertain assets $i = 1, 2$ and a risk-free asset f. The price of the risk-free asset, p_f, is normalized to 1. The price of uncertain asset i is denoted by p_i. There is a finite set of agents, $\{1, \ldots, n, \ldots, N\}$. Agent n's holdings of the assets are denoted by $q_{i,n}$; for convenience, we will write $q_n = (q_{1,n}, q_{2,n})$.[10] We denote by $a_{i,n} \equiv p_i q_{i,n}$ the monetary amount invested in asset i by agent n, and write $a_n = (a_{1,n}, a_{2,n})$ to denote the monetary investment (or equivalently, monetary holdings) in uncertain assets. Given the normalization $p_f = 1$, $a_{f,n} = q_{f,n}$ denotes the monetary holding of the risk-free asset by agent n. Agent n's endowment of asset i is $e_{i,n}$, and the aggregate endowment of the asset is e_i. All agents have zero endowment of the risk-free asset, and therefore there is zero aggregate supply of the risk-free asset so that $e_f = 0$. Both risk-free and uncertain returns are exogenous. The gross (monetary) returns of the risk-free asset and uncertain assets are R_f and R_i, $i = 1, 2$, respectively.[11] That is, if agent n invests $a_{j,n}$ in asset j, where $j = f, 1, 2$, then his payoff is $R_j a_{j,n}$.

The uncertain returns are ambiguous in the sense that agents are uncertain about the probability distribution governing each return: they believe that the returns data are generated by an unknown member of a set of possible models. Formally, we have a random vector $M \equiv (M_1, M_2)$, the model, whose realization fixes the vector of conditional distribution of returns $R|M \equiv (R_1|M_1, R_2|M_2)$. The uncertainty about returns conditional on a model M (i.e., $R|M$) is referred to as the *first-order uncertainty* while uncertainty about the model M itself is referred to as the *second-order uncertainty.* We adapt the setup of Hara and Honda (2016) to describe the agents' common beliefs about the uncertainty governing returns.[12] In this setup, both first- and second-order uncertainties are Gaussian. We further impose the following assumptions.[13]

Assumption 1. *The mean return of asset i conditional on model M is $M_i, i = 1, 2$. That is,*

$$E\left[R \mid M\right] = M.$$

Assumption 2. *Models and asset returns are jointly normally distributed with*

$$cov\left(R, M\right) = var\left(M\right) \equiv \Sigma_M.$$

That is,

$$\begin{pmatrix} M \\ R \end{pmatrix} \sim N\left(\begin{pmatrix} E[M] \\ E[R] \end{pmatrix}, \begin{pmatrix} \Sigma_M & \Sigma_M \\ \Sigma_M & \Sigma_R \end{pmatrix}\right)$$

where

$$\Sigma_M = \begin{pmatrix} \left(\sigma_1^M\right)^2 & \sigma_{12}^M \\ \sigma_{12}^M & \left(\sigma_2^M\right)^2 \end{pmatrix} \textit{ and } \Sigma_R = \begin{pmatrix} {\sigma_1}^2 & \sigma_{12} \\ \sigma_{12} & {\sigma_2}^2 \end{pmatrix}.$$

Assumption 2 is actually all about the joint normality of R and M; the restriction that $cov\left(R, M\right) = var\left(M\right)$ does not result in loss of generality as pointed out by Hara and Honda (2016). The matrix Σ_R is the variance-covariance matrix of the unconditional distribution of

[10] All vectors are taken to be column vectors. Transposes are row vectors.

[11] We use R to denote the column vector (R_1, R_2).

[12] Assuming common (second-order) beliefs has the consequence, as shown by Rigotti et al. (2008), that *uncertainty averse* agents (which includes the class of agents we consider in this paper) will not want to enter into speculative trades, in the sense that, absent aggregate uncertainty, the full insurance allocations are Pareto optimal.

[13] We slightly abuse notation by denoting both the model random variable and a particular realization of the variable by M.

returns R. Henceforth, we will refer to $(\sigma_i^M)^2$ and σ_i^M as the *model variance of asset* i and *model standard deviation of asset* i, respectively, and Σ_M as the model variance-covariance matrix. The Projection Theorem, together with Assumptions 1 and 2, yields

$$M = E[R \mid M] = E[R] + [cov(R, M)] [var(M)]^{-1} (M - E[M]) = E[R] + M - E[M],$$

which in turn implies that $E[R] = E[M] \equiv \mu \equiv (\mu_1, \mu_2)$. We assume that $\mu_i > R_f$, $i = 1, 2$, so that risk- and ambiguity-averse agents do not rule out investment into uncertain assets. Also, following Assumption 2 and the Projection Theorem, we have

$$\Sigma \equiv var(R \mid M) = var(R) - [cov(R, M)] [var(M)]^{-1} [cov(R, M)] = \Sigma_R - \Sigma_M. \quad (1)$$

Hence, we obtain that the conditional asset return is distributed as:

$$R \mid M \sim N(M, \Sigma_R - \Sigma_M).$$

In particular, the variance of returns conditioned on the realization of M is independent of the realized value. In this setup, the uncertainty about the model *only* affects the (conditional) mean of the return, not the (conditional) variance, and thus reduces to *parameter uncertainty* about the mean.

We apply the framework of the smooth model of Klibanoff et al. (2005) to describe how the agents incorporate uncertainty in the evaluation of portfolios. In this model, if the agent were to know the realization of M, that is he faces no parameter uncertainty, the evaluation is the usual expected utility evaluation using the random variable $R|M$. Uncertainty about M makes this expected utility evaluation (based on $R|M$) uncertain; the agent is ambiguity averse if he dislikes mean preserving spreads in this uncertainty. More specifically, consider a portfolio $(a_{f,n}, a_{1,n}, a_{2,n})$ which yields a final contingent wealth equal to $W(a_{f,n}, a_{1,n}, a_{2,n}) = a_{f,n}R_f + a_{1,n}R_1 + a_{2,n}R_2$. Agent n evaluates such a portfolio according to:[14]

$$E_M \left[\phi_n \left(E_{R|M}[u_n (W(a_{f,n}, a_{1,n}, a_{2,n}))]\right)\right], \quad (2)$$

where u_n, a utility function, incorporates the agent's attitude to risk, and ϕ_n, an increasing concave function, reflects the agent's ambiguity aversion. Thus, the ambiguity and the ambiguity aversion are represented distinctly through the random variable M and the function ϕ_n, respectively. This parametric separation is useful in that it is possible to hold an agent's beliefs (perceived ambiguity) fixed while varying their ambiguity attitude, say from aversion to neutrality (i.e., replacing a concave ϕ_n with an affine one reduces the preference to expected utility while retaining the same beliefs).[15] As in Hara and Honda (2016), we further specify $u_n(x) = -exp(-\theta_n x)$ and $\phi_n(y) = -(-y)^{\gamma_n/\theta_n}$. Note, if $\gamma_n = \theta_n$, then agent n is ambiguity neutral and a CARA (expected) utility maximizer. If $\gamma_n > \theta_n$, the agent is ambiguity averse.

[14] E_X denotes an expectation operator which integrates over the realization of the random variable X.

[15] An ambiguity neutral, standard expected utility agent cares only about the unconditional uncertainty, represented by the random variable R.

Denote by

$$\eta_n \equiv -\ y\ \frac{\phi_n''(y)}{\phi_n'(y)} = \frac{\gamma_n}{\theta_n} - 1 = \frac{\gamma_n - \theta_n}{\theta_n},$$

the coefficient of (relative) ambiguity aversion of the agent. In the rest of the paper, we will always assume that $\eta_n \geq 0$ for all n, i.e., we never consider ambiguity seeking.

We now set up the maximization problem the agents solve. Given Assumptions 1 and 2, and the specifications of u_n and ϕ_n as above, Lemma 1 of Hara and Honda (2016) shows that maximizing (2) is equivalent to choosing a portfolio $(a_{f,n}, a_{1,n}, a_{2,n})$ that maximizes

$$V_n(a_{f,n}, a_{1,n}, a_{2,n}) \ \equiv\ \underbrace{a_{f,n} R_f + \mu^\top a_n}_{\text{mean return}} - \frac{\theta_n}{2}\ \underbrace{a_n^\top \Sigma_R a_n}_{\text{risk}} - \frac{\gamma_n - \theta_n}{2}\ \underbrace{a_n^\top \Sigma_M a_n}_{\text{ambiguity}}, \tag{3}$$

where $a_n = (a_{1,n}, a_{2,n})$. This formulation therefore generalizes the standard and commonly used mean-variance model of Markowitz (1952): $\frac{\theta_n}{2}\ a_n^\top \Sigma_R a_n$ is the standard risk adjustment to the evaluation while $\frac{\gamma_n - \theta_n}{2}\ a_n^\top \Sigma_M a_n$ introduces an ambiguity adjustment. Maccheroni et al. (2013) obtain the formulation (3), which they refer to as the *robust mean-variance model*, as a second-order (Arrow-Pratt) approximation of the certainty equivalent of a smooth ambiguity evaluation where u_n, ϕ_n and beliefs are arbitrarily specified. This is analogous to the fact that the standard mean-variance formulation may be seen as a quadratic approximation of the certainty equivalent of an expected utility evaluation where utility and beliefs are arbitrarily specified. Thus, while the route here to an exact robust mean-variance formulation relied on particular specification of utilities and beliefs following Hara and Honda (2016), we may nevertheless interpret the formulation as an approximation to a more general specification.

Observe, we may rewrite the formulation in (3) as

$$V_n(a_{f,n}, a_{1,n}, a_{2,n}) \ =\ a_{f,n} R_f + \mu^\top a_n - \frac{\theta_n}{2}\ a_n^\top \left(\Sigma_R + \eta_n \Sigma_M\right) a_n. \tag{4}$$

Hence, our ambiguity averse agent's evaluation of the portfolio can be read as if it were the evaluation of a *standard* (as opposed to *robust*) mean-variance utility agent with absolute risk aversion parameter θ_n and an *as if* subjective belief that uncertain assets' return distribution is given by $N(\mu, \Sigma_R + \eta_n \Sigma_M)$. Thus, our population of robust mean-variance agents with identical beliefs but heterogeneous ambiguity aversion may be equivalently seen as a population of standard mean-variance agents with heterogeneous as if beliefs which differ in the variance *but not in the mean* of returns. The disagreement in as if beliefs is in the term $\eta_n \Sigma_M$, hence it stems from differences in η_n and the differences are magnified by Σ_M. Given that the ambiguity is about the mean returns, one might have expected that more ambiguity averse decision makers would behave as if they believe the mean returns were lower, but they instead behave as if the return variances are higher. This as if beliefs interpretation is helpful in gaining intuition for some of the results we obtain in the subsequent analysis. The following remark elaborates on how we build on and add to the insights from the differences-of-opinion literature.

Remark 1. *The differences-of-opinion literature focuses mostly on subjective beliefs where the subjectivity about the expected value of returns drives agents to engage in speculative trades – speculative in the sense that they would pursue trade even if they were risk neutral (as in,* *e.g., Harrison and Kreps (1978), Morris (1996), Kandel and Pearson (1995) and Banerjee and Kremer (2010)). On the other hand, as we note from (4), agents with heterogeneous ambiguity aversion act as if they have subjective beliefs where the subjectivity is not about the expected value of returns but instead about the return variances.*[16] *In this sense, the trade in our model is not for speculative reasons but for uncertainty sharing purposes, as agents disagree on how to optimally diversify. There is a second way in which the as if beliefs formulation given in (4) adds to the differences-of-opinion literature. The subjectivity in as if beliefs is characterized by $\eta_n \Sigma_M$ and thus connected to objective, and in principle measurable, properties of return distributions encapsulated in Σ_M. Ambiguity aversion, which forges the link in this connection, is also measurable (see, e.g., Dimmock et al. (2016)). Hence, compared to the differences-of-opinion literature, the as if beliefs here are imparted with more discipline which in turn yields sharper, testable predictions, as we discuss in Section 6.*[17]

3 Portfolio choice and the asset allocation puzzle

3.1 Portfolio choice

We study here how the composition of the optimal portfolio is determined by ambiguity aversion given exogenous asset prices. Consider agent n with initial wealth $W_n \equiv p_1 e_{1,n} + p_2 e_{2,n}$. The maximization problem he faces is

$$\begin{aligned} \max_{a_{f,n},\, a_{1,n},\, a_{2,n}} \quad & a_{f,n} R_f + \mu^\top a_n - \frac{\theta_n}{2} a_n^\top \Sigma_R a_n - \frac{\gamma_n - \theta_n}{2} a_n^\top \Sigma_M a_n \\ & \text{s. to} \quad a_{f,n} + a_{1,n} + a_{2,n} \le W_n. \end{aligned} \tag{5}$$

Solving for the optimal monetary holdings of uncertain assets, $a_n = (a_{1,n}, a_{2,n})$, yields:

$$a_n = \frac{1}{\theta_n} (\Sigma_R + \eta_n \Sigma_M)^{-1} (\mu - R_f 1), \tag{6}$$

or more explicitly,

$$a_{i,n} = \frac{(\mu_i - R_f) A_{j,n} - (\mu_j - R_f) B_{12,n}}{A_{1,n} A_{2,n} + (B_{12,n})^2}, \quad i = 1, 2, \tag{7}$$

where

$$\begin{aligned} A_{i,n} &= \theta_n \sigma_i^2 + (\gamma_n - \theta_n)(\sigma_i^M)^2 = \theta_n \left[\sigma_i^2 + \eta_n (\sigma_i^M)^2 \right], \\ B_{12,n} &= \theta_n \sigma_{12} + (\gamma_n - \theta_n) \sigma_{12}^M = \theta_n \left[\sigma_{12} + \eta_n \sigma_{12}^M \right]. \end{aligned}$$

Therefore, we have

$$\frac{a_{1,n}}{a_{2,n}} = \frac{(\mu_1 - R_f) \left[\sigma_2^2 + \eta_n (\sigma_2^M)^2 \right] - (\mu_2 - R_f) \left[\sigma_{12} + \eta_n \sigma_{12}^M \right]}{(\mu_2 - R_f) \left[\sigma_1^2 + \eta_n (\sigma_1^M)^2 \right] - (\mu_1 - R_f) \left[\sigma_{12} + \eta_n \sigma_{12}^M \right]}. \tag{8}$$

[16] That the agents share the same as if beliefs about the expected value of returns follows from our assumption of common (second-order) beliefs – see footnote 12.

[17] As if beliefs in the macro-finance literature with ambiguity aversion have an analogous interpretation – see, for instance, the related discussion in Remark 1 of Collard et al. (2018).

So, the ratio of monetary investments in uncertain assets is independent of the agent's risk aversion parameter θ_n. If everyone is ambiguity neutral (i.e., $\eta_n = 0$ for all n), this implies nothing but the classical mutual fund theorem (Tobin (1958)): ambiguity neutral agents, regardless of their risk aversion, invest in the same proportion across uncertain assets, and therefore hold the same portfolio of uncertain assets. The ratio in (8) does depend, however, on the agent's ambiguity aversion parameter η_n.[18] The mutual fund theorem continues to hold if agents are homogeneously ambiguity averse. However, generically, two agents n and n' with different ambiguity aversion parameters will have different ratios of monetary investments in uncertain assets. Noting $\frac{a_{1,n}}{a_{2,n}} = \frac{p_1 q_{1,n}}{p_2 q_{2,n}}$, this also means that they will hold uncertain assets in different proportions, i.e., $\frac{q_{1,n}}{q_{2,n}} \neq \frac{q_{1,n'}}{q_{2,n'}}$. The following remark summarizes these observations:

Remark 2. *If agents are homogeneous in ambiguity aversion, i.e., $\eta_n = \eta_{n'}$ for all n, n', then the mutual fund theorem holds, that is, for optimal portfolio choices it holds that $\frac{a_{1,n}}{a_{2,n}} = \frac{a_{1,n'}}{a_{2,n'}}$ and $\frac{q_{1,n}}{q_{2,n}} = \frac{q_{1,n'}}{q_{2,n'}}$ for all n, n'. If, on the other hand, agents are heterogeneous in ambiguity aversion, then the mutual fund theorem generically fails.*

To see the intuition, recall the observation in (4) and the discussion following that observation. When agents are homogeneously ambiguity averse, they act as if they have standard mean-variance preferences with the same beliefs, and hence the mutual fund theorem follows. On the other hand, agents with heterogeneous ambiguity aversion act as if they have different beliefs about return variances, which leads them to hold different portfolios as they disagree on how to optimally diversify. The latter point is made explicit by (6): optimal monetary holdings are a function of $\eta_n \Sigma_M$, the distinguishing aspect of the as if belief.

That the mutual fund theorem holds with homogeneous ambiguity aversion and fails with heterogeneous ambiguity aversion was already noted in Hara and Honda (2016). In the next subsection, we go beyond this observation by characterizing the departure from the mutual fund theorem in terms of agents' ambiguity aversion and ambiguity of the assets.

3.2 Comparative statics

Let $S_i \equiv \frac{\mu_i - R_f}{\sigma_i}$ and $S_i^{Amb} \equiv \frac{\mu_i - R_f}{\sigma_i^M}$, $i = 1, 2$. S_i is, of course, the standard Sharpe ratio of asset i, and we will refer to S_i^{Amb} as the *ambiguity Sharpe ratio* of asset i. We also let $\rho \equiv \frac{\sigma_{12}}{\sigma_1 \sigma_2}$ and $\rho^M \equiv \frac{\sigma_{12}^M}{\sigma_1^M \sigma_2^M}$. Lemma A.1 in the Appendix gives a full characterization of the comparative statics of portfolio choice with respect to ambiguity aversion. The characterizing condition in Lemma A.1 is a mouthful, but at its heart lies a three-way trade-off between excess return, risk and ambiguity – glimpsed through the interplay between the (standard and ambiguity) Sharpe ratios and the two correlation terms ρ and ρ^M. We explore the content of the characterization through a proposition and a couple of corollaries, to follow.

Proposition 1. *Let agent n be more ambiguity averse than agent n', i.e., $\eta_n > \eta_{n'}$. Then, for*

[18] Unless very specific stringent restrictions on the asset return structure hold, such as, obviously, if the two assets are identical with $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\sigma_1^M = \sigma_2^M$, or if $\sigma_i = \sigma_i^M$ and $\sigma_{12} = \sigma_{12}^M$.

optimal portfolio choices it holds that

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \text{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}.$$

if either of the following conditions holds:

(a) $S_1 = S_2$, $S_1^{Amb} < S_2^{Amb}$ *and* $\rho \neq 1$, *or*

(b) $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$ *and* $\rho\left(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}\right) \leq \rho^M \leq \rho\left(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2}\right)$.

In (a), we make the agent indifferent between the two assets in terms of return compensation per unit risk, but partial to asset 2 in terms of return compensation per unit ambiguity. If there is no room for risk diversification (i.e., if $\rho = 1$), then absent ambiguity aversion, agent's choice between the uncertain assets would be indeterminate, hence we could not say how more ambiguity aversion would affect the agent's choice. However, when $\rho \neq 1$, risk diversification uniquely determines the optimal portfolio choice absent ambiguity aversion, and introducing ambiguity aversion tilts the optimal choice away from asset 1. Therefore, the agent would hold proportionately less of asset 1 if he were more ambiguity averse.

In (b), the restriction $\rho\left(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}\right) \leq \rho^M \leq \rho\left(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2}\right)$ implies that the correlations ρ and ρ^M have the same sign, and thus, that diversification opportunities in risk are aligned, as would seem plausible, with those in ambiguity. Assuming such an alignment ensures that the trade-off between risk diversification and ambiguity diversification does not play a significant role in heterogeneously ambiguity averse agents' portfolio choices. The first-order effect that leads to differences in differently ambiguity averse agents' portfolios, instead, comes from these agents' different evaluations of each asset's ambiguity in comparison to its risk-return profile. Specifically, if asset 1 offers lower return per unit ambiguity in proportion to return per unit risk compared to asset 2, that is, if $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$, then the more ambiguity averse agent holds proportionately less of asset 1.

Intuition suggests that more ambiguity averse agents should put more weight on ambiguity in the three-way trade-off between return, risk and ambiguity and therefore be more partial to the less ambiguous assets. We articulate this intuition more precisely in the following corollaries, obtained by applying two formal notions of one asset being more affected by ambiguity than another, developed by Jewitt and Mukerji (2017). Given the class of preferences in our setup, asset 1 is *more ambiguous (I)* than asset 2 if and only if $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\sigma_1^M > \sigma_2^M$, and asset 1 is *more ambiguous (II)* than asset 2 if and only if $\sigma_1^M > \sigma_2^M$.[19] Using this new vernacular, we have the following result as a direct consequence of Proposition 1:

Corollary 1. *Let agent* n *be more ambiguity averse than agent* n', *i.e.,* $\eta_n > \eta_{n'}$. *If*

(i) asset 1 is more ambiguous (I) than asset 2 and $\rho \neq 1$, *or*

(ii) asset 1 is sufficiently more ambiguous (II) than asset 2 so that $\sigma_1^M > \sigma_2^M\left(\frac{\sigma_1}{\sigma_2}\right)$ *and* $\rho\left(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}\right) \leq \rho^M \leq \rho\left(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2}\right)$,

[19] For characterizations of both "more ambiguous (I)" and "more ambiguous (II)" for the class of preferences considered here, see Example 4.2 in Jewitt and Mukerji (2017).

then for optimal portfolio choices it holds that

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \textit{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}.$$

Next, we take one uncertain asset to be more ambiguous (II) and to have a lower ambiguity Sharpe ratio compared to the other one, and ask how an agent would optimally allocate her wealth between the uncertain assets if she were *sufficiently* ambiguity averse.

Corollary 2. *Let $S_1^{Amb} < S_2^{Amb}$. If agent n is sufficiently ambiguity averse, and asset 1 is more ambiguous (II) than asset 2, then the agent optimally allocates a smaller portion of his wealth to asset* 1 *compared to asset* 2*, that is,* $a_{1,n} < a_{2,n}$.

3.3 Asset allocation puzzle

Following Corollaries 1 and 2, the departure from the mutual fund theorem is in a particular direction in that more ambiguity aversion leads to a portfolio that has proportionately more of the less ambiguous asset. This seems to accord well with the widely recognized deviation from the mutual fund theorem, the *asset allocation puzzle*, first noted in Canner et al. (1997): it is very common to observe in financial planning advice that more conservative investors are encouraged to hold more bonds, relative to stocks. Table I reproduced from Canner et al. (1997) illustrates the puzzle.

TABLE I - ASSET ALLOCATIONS RECOMMENDED BY FINANCIAL ADVISORS

	Percent of portfolio			
Advisor and investor type	Cash	Bonds	Stocks	Ratio of bonds to stocks
A. *Fidelity*				
Conservative	50	30	20	1.50
Moderate	20	40	40	1.00
Aggressive	5	30	65	0.46
B. *Merrill Lynch*				
Conservative	20	35	45	0.78
Moderate	5	40	55	0.73
Aggressive	5	20	75	0.27
C. *Jane Bryant Quinn*				
Conservative	50	30	20	1.50
Moderate	10	40	50	0.80
Aggressive	0	0	100	0.00
D. *New York Times*				
Conservative	20	40	40	1.00
Moderate	10	30	60	0.50
Aggressive	0	20	80	0.25

As documented by Canner et al. (1997) and as expected, stocks are riskier than long-term bonds which are in turn riskier than Treasury bills, where risk is proxied by standard deviation. It follows that the standard error of the estimate of mean returns for each asset class is also ranked the same way. Hence, the confidence interval around the estimated mean returns gets wider as we move from Treasury bills to bonds to stocks, suggesting a greater degree of uncertainty or poorer knowledge about what the true mean is. This in turn suggests stock returns may be perceived as more ambiguous than bond returns and bond returns may be perceived as more ambiguous than Treasury bill returns. Staying within the standard mean-variance framework, the mutual fund theorem holds and therefore interpreting conservatism as risk aversion does not explain the asset allocation puzzle. On the other hand, the popular financial advice is accommodated in the robust mean-variance framework when we interpret conservatism as ambiguity aversion.

Canner et al. (1997) discuss various possible explanations of the puzzle (though not ambiguity aversion) and find them unsatisfactory. In particular, they point out that subjective beliefs cannot be an explanation as the financial advisor's subjective belief about asset returns presumably does not change depending on whom they advise.[20] In their concluding remarks, interestingly, the authors conjecture that non-standard preferences may help explain the puzzle. More recently, Dimmock et al. (2016) test the relation between ambiguity aversion and the fraction of financial assets allocated to stocks, and find it to be statistically and economically significantly negative, lending support to our explanation. Bajeux-Besnainou et al. (2001) and Campbell and Viceira (2002) suggest alternative, complementary explanations based on intertemporal hedging incentives. A distinctive feature of our explanation based on heterogeneity in asset ambiguity and agent ambiguity aversion is that it also provides a basis for candidate explanations for some well-known departures from CAPM and of the nature of trade following earnings announcements, as we show in subsequent analysis.

4 Static equilibrium asset pricing and cross-sectional implications

We now study the equilibrium of a static multi-agent economy and derive properties of asset prices as a function of asset ambiguity and agent ambiguity aversion.

4.1 A generalized CAPM with heterogeneous ambiguity aversion

In this subsection, we establish a single-factor CAPM-like asset pricing formula with heterogeneously ambiguous assets and heterogeneously ambiguity averse agents. As in the CAPM, the single pricing factor is the excess return of the market portfolio, however the factor loading now accounts for the existence of ambiguity as well as risk. In the formula we obtain, the standard CAPM beta is adjusted depending on the extent to which the ambiguity of the asset return is correlated with the ambiguity of the market portfolio return. As long as there is an ambiguity averse agent in the market, the expected excess return of an asset is shown to be composed of two uncertainty premia: the (CAPM-predicted) risk premium and the ambiguity premium.

That we obtain a CAPM-like pricing formula with heterogeneously ambiguity averse agents is surprising since we know from the results in the previous section that with such heterogeneity the classical mutual fund theorem fails, and therefore agents hold portfolios distinct from the market portfolio.[21] We are able to obtain the single-factor pricing formula by constructing what is

[20] Of course, the as if belief interpretation in (4) is not subject to the same critique.

[21] Ruffino (2014) has a similar CAPM-like formula but with homogeneously ambiguity averse agents. In the author's notation, the assumption that the ratio of θ to λ is equal for all investors is equivalent to assuming homogeneous ambiguity aversion among investors, i.e., a common η in the notation here (see p. 5, Ruffino

effectively a representative agent for pricing purposes. By obtaining an explicit characterization of the representative agent, we show how heterogeneity in ambiguity aversion affects equilibrium asset prices. In particular, our finding suggests that the agents with intermediate levels of ambiguity aversion matter most for the size of ambiguity premium.

An *equilibrium* of the (static) economy is given by asset holdings $\{(q^*_{f,n}, q^*_{1,n}, q^*_{2,n})\}_{n=1}^N$ and asset prices (p^*_1, p^*_2) such that monetary holdings $\{(q^*_{f,n}, p^*_1 q^*_{1,n}, p^*_2 q^*_{2,n})\}_{n=1}^N$ are a solution to the agents' maximization problem (5), and market clears, i.e.,

$$\sum_n q^*_{i,n} = \sum_n e_{i,n} = e_i, \ i = 1, 2, \text{ and } \ \sum_n q^*_{f,n} = 0.$$

To state our asset pricing result, we need to introduce some further notation. The return of a generic portfolio of uncertain assets held by agent n is denoted by $R(a_n)$ where,

$$R(a_n) \equiv \frac{a_{1,n}R_1 + a_{2,n}R_2}{a_{1,n} + a_{2,n}} = \frac{p_1 q_{1,n} R_1 + p_2 q_{2,n} R_2}{p_1 q_{1,n} + p_2 q_{2,n}}.$$

We also let

$$cov(R(a_n), R_i) \equiv \frac{a_{1,n}\sigma_{1i} + a_{2,n}\sigma_{2i}}{a_{1,n} + a_{2,n}}, \qquad var(R(a_n)) \equiv \frac{(a_{1,n})^2(\sigma_1)^2 + 2a_{1,n}a_{2,n}\sigma_{12} + (a_{2,n})^2(\sigma_2)^2}{(a_{1,n} + a_{2,n})^2},$$
$$cov^M(R(a_n), R_i) \equiv \frac{a_{1,n}\sigma^M_{1i} + a_{2,n}\sigma^M_{2i}}{a_{1,n} + a_{2,n}}, \qquad var^M(R(a_n)) \equiv \frac{(a_{1,n})^2(\sigma^M_1)^2 + 2a_{1,n}a_{2,n}\sigma^M_{12} + (a_{2,n})^2(\sigma^M_2)^2}{(a_{1,n} + a_{2,n})^2}.$$

As is standard, agent n is said to *hold the market portfolio* if his holdings of uncertain assets is proportional to the aggregate endowment of uncertain assets, i.e., $(q_{1,n}, q_{2,n}) = (k_n e_1, k_n e_2)$ for some real scalar k_n. This implies that agent n holds the market portfolio if and only if $\frac{q_{1,n}}{q_{2,n}} = \frac{e_1}{e_2}$. We let R_{market} denote *the market portfolio return*, that is,

$$R_{market} \equiv \frac{p^*_1 e_1 R_1 + p^*_2 e_2 R_2}{p^*_1 e_1 + p^*_2 e_2}. \tag{9}$$

Note that the market portfolio return, R_{market}, is determined in equilibrium as it depends on equilibrium prices (p^*_1, p^*_2), even though asset returns R_i are exogenous. Let us define asset i's *beta*, β_i, in the standard way, and *ambiguity beta*, β^{Amb}_i, analogously:

$$\beta_i \equiv \frac{cov(R_{market}, R_i)}{var(R_{market})}, \quad \beta^{Amb}_i \equiv \frac{cov^M(R_{market}, R_i)}{var^M(R_{market})}.$$

We are now ready to state the anticipated result:

Proposition 2. *Let $\underline{\eta} \equiv \min_n \{\eta_n\}$ and $\bar{\eta} \equiv \max_n \{\eta_n\}$. Then, there exists a unique $\eta_{market} \in [\underline{\eta}, \bar{\eta}]$ such that the following equality holds:*

$$E[R_i] - R_f = \frac{cov(R_{market}, R_i) + \eta_{market}\ cov^M(R_{market}, R_i)}{var(R_{market}) + \eta_{market}\ var^M(R_{market})} \left(E[R_{market}] - R_f\right) \tag{10}$$
$$= \alpha_i + \beta_i \left(E[R_{market}] - R_f\right), \tag{11}$$

(2014)). As shown in the previous section and by Hara and Honda (2016), the mutual fund theorem holds with homogeneity in ambiguity aversion.

where

$$\alpha_i \equiv \frac{\eta_{market}\ var^M(R_{market})}{var(R_{market}) + \eta_{market}\ var^M(R_{market})} \left(\beta_i^{Amb} - \beta_i\right) \left(E[R_{market}] - R_f\right). \tag{12}$$

Furthermore, $\eta_{market} > 0$ *if and only if* $\bar{\eta} > 0$*.*

We show in the Appendix (Lemma B.1) that, given an agent's optimal portfolio choice, expected excess asset returns can be explained by a single-factor pricing formula, where the single factor is the excess return of the agent's optimal portfolio (of the uncertain assets). With homogeneous ambiguity aversion, all agents' optimal portfolios coincide with the market portfolio since the mutual fund theorem holds. Therefore, setting η_{market} equal to the agents' common ambiguity aversion coefficient, the CAPM-like formulation in Proposition 2 obtains (as Ruffino (2014) had shown). The novel contribution here is that we establish the same formula continues to hold under heterogeneous ambiguity aversion, a case where the mutual fund theorem fails. That we can follows from the fact that we effectively establish a representative agent for pricing purposes. Specifically, under heterogeneous ambiguity aversion, we show that one may construct an ambiguity aversion coefficient, namely η_{market}, for which the corresponding optimal portfolio is the market portfolio, thereby giving us the proposition. In Section 4.3, we present a full characterization of η_{market} and show that it is determined (in part) by the distribution of ambiguity aversion among the agents.

The excess return of an asset is the compensation for *systematic* (non-diversifiable) uncertainty in that asset. Like in the standard CAPM, the single pricing factor is the expected excess return on the market portfolio. The difference ambiguity and ambiguity aversion bring is the adjustment to the factor loading, as can be seen from (10). The overall adjustment to the standard CAPM formula is encapsulated in α_i, which we refer to as asset i's *ambiguity premium.* Asset i's ambiguity premium is zero if (i) all agents are ambiguity neutral so that $\eta_{market} = 0$, or (ii) all uncertain assets carry no ambiguity so that $var^M(R_{market}) = 0$, or (iii) $\beta_i^{Amb} = \beta_i$. Given some ambiguity and ambiguity aversion, $\alpha_i > 0$ if and only if $\beta_i^{Amb} > \beta_i$, that is, if the asset's exposure to systematic ambiguity (or, exposure to systematic second-order uncertainty) is greater than its exposure to systematic unconditional uncertainty. Note, however, even if an asset's returns are ambiguous, the asset will carry an ambiguity discount if its systematic ambiguity exposure is relatively low. The main take-away from Proposition 2 is that the cross-sectional heterogeneity in exposure to systematic ambiguity is a determinant for the cross-sectional expected returns. This has empirical implications, as we discuss next.

4.2 Discussion of cross-sectional empirical implications

If we were to regress the excess returns of asset i on the excess returns of the market portfolio over time, we would get $\beta_i = \frac{cov(R_{market}, R_i)}{var(R_{market})}$ as the OLS slope coefficient. Note, this OLS coefficient is the CAPM factor loading and therefore the sole determinant for the cross-sectional expected returns in CAPM. Absent ambiguity aversion, the asset pricing prediction of our theory is identical to that of CAPM. However, even with a single ambiguity averse agent in the economy, there is an additional term in the excess return predicted by our theory, the ambiguity premium

α_i, driven by asset i's exposure to systematic ambiguity as can be seen from (10). Thus, by our theory, the cross-sectional expected returns are not only determined by the standard CAPM predictor, β_i, based on the OLS coefficient, but also by i's ambiguity exposure. The natural question then, is whether some of the well-known, systematic departures from CAPM can be accounted for by such exposure.

Consider the *size* and *value premia* documented in the literature critiquing CAPM (e.g., see Fama and French (1992) and (1993)). These premia refer to the observation that, controlling for CAPM betas, firms with small market capitalizations and high book-to-market equity ratios generate, on average, higher stock returns. There is no theoretical consensus on why these premia exist. It has been suggested that size premium may be partly an outcome of the fact that small firms are more dependent on outside financing, leaving them more exposed to business cycles and broader financial conditions.[22] When it comes to value premium, many associate it with financial distress risk.[23] Financially distressed firms tend to have high book-to-market equity ratios: distress causes a collapse in the market value of equity whereas it has little effect on the book value of equity since the latter reflects historical cost of acquisition of assets making up the firm less the (promised) liabilities. We argue, next, that cross-sectional heterogeneity in exposure to systematic ambiguity is a common thread that runs through these informal, uncertainty-based explanations. The success of the macro-finance literature incorporating ambiguity in explaining price dynamics of aggregate uncertainty is a justification for treating systematic uncertainty as ambiguous.[24] This common thread provides an articulation of these explanations that has the advantage of being formal as well as accommodated within a CAPM-style framework with a single factor.[25]

Absent financing considerations, stock returns are simply driven by operating returns on capital investments. If availability of sufficient financing is uncertain for a firm, then it is harder for investors to forecast the firm's stock returns as they need to consider potential shortfall in capital investments which has *sustained* implications for operating returns and thus the potential to affect *expected* returns. Also, since stock returns are operating returns net of outside financing costs, exposure to uncertain financing costs driven by broad financial conditions makes them less predictable. Therefore, if the financing conditions worsen due to business or financial cycles, firms more reliant on outside financing, such as small and distressed ones, generate stock returns that are more ambiguous from the perspective of investors. The ambiguity is also systematic: dependence on outside financing implies higher exposure to economy-wide conditions, and therefore all small and distressed firms are commonly and strongly affected by cycles.

The empirical literature also established that even though there is a positive relation between

[22] See, for instance, Perez-Quiros and Timmermann (2000) for empirical evidence supporting this line of argument.

[23] Fama and French (1995) and Vassalou and Xing (2004) provide empirical evidence for the association between value premium and financial distress risk.

[24] The macro-finance literature has established connections between uncertainty shocks, recessionary episodes and ambiguity. See, for instance, the discussion in Section 3.3.3 of Collard et al. (2018) relating evolution of perceived ambiguity to the macroeconomic uncertainty indices, such as the ones developed by Jurado et al. (2015) and Carriero et al. (2018).

[25] There is a strand of literature (Lakonishok et al. (1994), La Porta (1996) and La Porta et al. (1997)) which offers an explanation for the value premium, behavioral in nature, that is an alternative to the risk-based explanation. These authors argue that the returns associated with value investing are due to naive investor expectations of future growth that result in mispricing.

an asset's CAPM beta and its average return this relation (i.e., the security market line) is too flat in the data (Black et al. (1972) and Treynor and Black (1973)). That is, compared to what is predicted by the CAPM, the returns on the low beta assets are too high and the returns on the high beta assets are too low. Our theory is consistent with this observation: a flatter security market line is obtained when low beta assets carry a positive ambiguity premium and high beta assets carry a negative ambiguity premium. This would happen if the cross-sectional variation in ambiguity beta were lower than the cross-sectional variation in beta so that ambiguity beta would be higher than beta for low beta assets and lower for high beta assets. In our framework, ambiguity beta is driven by exposure to aggregate shocks that affect models governing the return generation processes (more specifically, means of returns) whereas beta captures both such exposure and also exposure to aggregate shocks that affect return realizations given the models. Consider a recession: it implies a lower *mean return* for most firms due to its economy-wide and persistent effect, however *realized returns* depend on firm characteristics such as brand loyalty, disposable income of the client base and price elasticity of the product or service offered. Therefore, arguably, there is less cross-sectional variation in ambiguity betas compared to betas.

The relationship between expected returns and dispersion in earnings forecasts has received considerable attention and is empirically contested. Some papers claim to document a negative relationship (e.g., Diether et al. (2002) and Hong and Stein (2003)), but others claim to find a positive relationship between the two (e.g., Qu et al. (2004) and Banerjee (2011)). Dispersion of opinions or lack of consensus need not reflect high levels of ambiguity to the extent that it arises from private information. However, dispersion of opinions among experts may instil ambiguity in the mind of non-expert decision makers who rely on experts for guidance. Arguably, then, there is a positive association between scenarios where there is dispersion of expert opinion about returns and scenarios where there is ambiguity about returns. The theory presented here characterizes conditions under which the relationship between expected returns and dispersion in earnings forecasts would be positive or negative to the extent that systematic or idiosyncratic ambiguity is responsible for the dispersion of opinion. In particular, our theory predicts, for firms which carry positive (negative) ambiguity premium, there is a positive (negative) relationship between their expected returns and dispersion in earnings forecasts when there is also dispersion in macroeconomic forecasts – the latter indicating that dispersion of opinion is due to high levels of systematic ambiguity rather than differential information.

4.3 Heterogeneity and the market ambiguity aversion

We now return to the question of how distribution of ambiguity aversion among agents affects the market ambiguity aversion, η_{market}, which in turn affects the ambiguity premia/discounts of assets. As a key step towards establishing this, we first give an explicit characterization of the market ambiguity aversion coefficient.

Proposition 3. *The market ambiguity aversion coefficient introduced in Proposition 2, η_{market}, is given by*

$$\eta_{market} = \left(\sum_{k=1}^{N} \Pi_k\right)^{-1} \sum_{n=1}^{N} \Pi_n \eta_n,$$

where, for $n = 1, \cdots, N$,

$$\Pi_n = \left(\frac{1}{\theta_n}\right)\left(\frac{1}{\left(\sigma_1^2 + \eta_n(\sigma_1^M)^2\right)\left(\sigma_2^2 + \eta_n(\sigma_2^M)^2\right) + \left(\sigma_{12} + \eta_n\sigma_{12}^M\right)^2}\right).$$

The corollary below gives us more insight into how an agent's level of ambiguity aversion impacts the level of market ambiguity aversion. Even though the weights Π_n decrease with η_n, as the overall contribution of an agent's ambiguity aversion coefficient to the market ambiguity aversion is determined by the product of the coefficient itself and its corresponding weight, it is ex-ante unclear whether highly ambiguity averse agents have more or less impact on the market ambiguity aversion. Intuitively, there are two forces working in opposite directions: higher ambiguity aversion requires a greater compensation for ambiguity; on the other hand, it also causes the agent to take a less aggressive position in the market and thus to affect the prices less. Corollary 3 shows there is a threshold level of ambiguity aversion, $\hat{\eta}$, below which the first force dominates, and above which the second dominates.

Corollary 3. *Without loss of generality, re-index the agents so that* $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_{N-1} \leq \eta_N$. *Let*

$$n^* = \min\left\{n \in \{1, \cdots, N\} : \eta_n > \hat{\eta} \equiv \sqrt{\frac{\sigma_1^2\sigma_2^2 + \sigma_{12}^2}{(\sigma_1^M)^2(\sigma_2^M)^2 + (\sigma_{12}^M)^2}}\right\}.$$

For Π_n, $n = 1, \cdots, N$, *as given in Proposition 3, the following conditions hold:*

(a) if $n^* = \emptyset$ *or* $n^* = N$, *then* $\theta_1\Pi_1\eta_1 \leq \theta_2\Pi_2\eta_2 \leq \cdots \leq \theta_N\Pi_N\eta_N$;

(b) if $n^* = 1$, *then* $\theta_1\Pi_1\eta_1 \geq \theta_2\Pi_2\eta_2 \geq \cdots \geq \theta_N\Pi_N\eta_N$;

(c) otherwise, $\theta_1\Pi_1\eta_1 \leq \theta_2\Pi_2\eta_2 \leq \cdots \leq \theta_{n^*}\Pi_{n^*}\eta_{n^*}$ *and* $\theta_{n^*}\Pi_{n^*}\eta_{n^*} \geq \theta_{n^*+1}\Pi_{n^*+1}\eta_{n^*+1} \geq \cdots \geq \theta_N\Pi_N\eta_N$.

If $\eta_n < \eta_{n+1}$ *for some* n, *then the corresponding inequalities for* $\theta_n\Pi_n\eta_n$ *and* $\theta_{n+1}\Pi_{n+1}\eta_{n+1}$ *in conditions (a)-(c) are also strict.*

Note that $\hat{\eta} > 0$ and that the smaller the contribution of ambiguity to overall uncertainty (i.e., the smaller $(\sigma_1^M)^2(\sigma_2^M)^2 + (\sigma_{12}^M)^2$ is in relation to $\sigma_1^2\sigma_2^2 + \sigma_{12}^2$) the greater is the $\hat{\eta}$. This has two implications. First, the presence of agents whose ambiguity aversion coefficients are less than $\hat{\eta}$ (which naturally includes ambiguity neutral agents) would render condition (b) void. Secondly, condition (a) is likely to hold only when ambiguity is a relatively insignificant part of the overall uncertainty. Hence, if ambiguity contributes significantly to overall uncertainty and there is also enough heterogeneity in ambiguity aversion across agents, then the more salient case in point is that described in condition (c).

To isolate the pure effect of heterogeneity in levels of ambiguity aversion, hold the level of risk aversion to be constant across the population, that is, $\theta_1 = \cdots = \theta_N$. Then, in the case described in condition (c), among agents whose ambiguity aversion coefficients are lower than the threshold $\hat{\eta}$, the higher the agent's level of ambiguity aversion, the higher is his contribution to the market ambiguity aversion. On the other hand, among agents whose ambiguity aversion coefficients are greater than $\hat{\eta}$, the relationship is reversed. As a result, the highest impact agents in terms of contribution to the market ambiguity aversion are the ones with intermediate levels of ambiguity aversion.

5 Dynamic equilibrium analysis with public signals

We now turn to two dynamic extensions of our static model where we study how prices and trade respond to the arrival of a *public* signal. The two dynamic extensions differ in the content of this signal. In the first extension, the agents receive a public signal drawn from the same process which governs the realization of the liquidating dividends of uncertain assets. Upon receiving the signal, the agents update their beliefs about the distribution of the model M. We interpret these signals as earnings announcements. In the second dynamic extension, the signal informs only about the model variance; we see the realization of such a signal as a realization of a (model) uncertainty shock. The next subsection presents the dynamic structure common to both extensions.

5.1 The common dynamic structure and notion of equilibrium

We model a three-period economy whose structure and timeline are as follows:

- In the initial period, $t = 0$, agents trade and choose a portfolio of three assets, two uncertain (indexed by $1, 2$) and one risk-free (indexed by f).
- In the interim period, $t = 1$, agents receive a public signal $S = (S_1, S_2)$ about the liquidating dividends of the uncertain assets, update their beliefs and have an opportunity to trade again in all the assets. The risk-free asset pays off R_f for each dollar invested in $t = 0$. No dividend from the uncertain assets is realized, however their prices change endogenously following the signal. No consumption takes place in this period.
- In the final period, $t = 2$, no decisions are taken – the risk-free asset pays off R_f for each dollar invested in $t = 1$, liquidating dividends of the uncertain assets realize (more precisely, asset i pays off R_i for each dollar invested in $t = 1$), and agents consume.

In $t = 0$, asset i's ($i = 1, 2, f$) price is denoted by p_i^0, its quantity held by agent n is $q_{i,n}^0$, and the corresponding monetary holding is $a_{i,n}^0 \equiv p_i^0 q_{i,n}^0$. In $t = 1$, these variables depend on the realization of the signal. When talking about price and holdings conditional on the realization of a signal S, we write p_i^S, $q_{i,n}^S$ and $a_{i,n}^S \equiv p_i^S q_{i,n}^S$. The price of the risk-free asset is normalized to 1 in $t = 0, 1$. Uncertain asset i's ($i = 1, 2$) gross return from $t = 0$ to $t = 1$ is equal to $\frac{p_i^S}{p_i^0}$. We refer to this as asset i's *interim return.* Abusing notation, we let $\frac{p^S}{p^0} \equiv \left[\frac{p_1^S}{p_1^0}, \frac{p_2^S}{p_2^0}\right]$. The uncertain asset i pays off R_i in $t = 2$ for each dollar invested in $t = 1$. We refer to this as asset i's *return.* Note, the asset i pays off $\frac{p_i^S}{p_i^0} R_i$ in $t = 2$ for each dollar invested in $t = 0$. As in the static model, the aggregate endowment of the uncertain asset i is denoted by e_i – to simplify, we take the endowment to be the same for both uncertain assets so that $e_i = e$ for $i = 1, 2$. There is zero aggregate supply of the risk-free asset so that $e_f = 0$.

An *equilibrium* is given by prices and holdings of uncertain and risk-free assets in periods 0 and 1 such that the holdings are optimal, given the prices and information, and clear the markets in both periods. We refer to equilibrium prices and holdings in period 0 and in period 1 as *ex ante equilibrium* and *interim equilibrium*, respectively. We say that an *equilibrium entails trivial trading* if the composition of the uncertain portfolio stays the same across periods 0 and 1, but the quantity of the risk-free asset held changes. An *equilibrium entails non-trivial trading* if the composition of the uncertain portfolio changes from period 0 to period 1. Formally,

· an *equilibrium entails trivial trading at signal realization S* if, at the equilibrium,

$$\frac{q^S_{1,n}}{q^S_{2,n}} = \frac{q^0_{1,n}}{q^0_{2,n}} \quad \text{for all } n \quad \text{and} \quad q^S_{f,n'} \neq q^0_{f,n'} \quad \text{for some } n',$$

· an *equilibrium entails non-trivial trading at signal realization S* if, at the equilibrium,

$$\frac{q^S_{1,n}}{q^S_{2,n}} \neq \frac{q^0_{1,n}}{q^0_{2,n}} \quad \text{for some } n,$$

· an *equilibrium entails no trade at signal realization S* if it entails neither trivial nor non-trivial trading.

We now define interim and ex ante preferences according to the recursive smooth ambiguity formulation of Klibanoff et al. (2009) which, given the recursive construction, guarantees dynamically consistent behavior. After realization of the signal S, the agent updates his beliefs through Bayes rule. Let $M' \equiv M|S$ and $R' \equiv R|S$ denote the updated beliefs over M and R. Then, the interim utility from an interim portfolio $(a^S_{f,n}, a^S_n)$ is given by[26]

$$U^S_n(a^S_{f,n}, a^S_n) \equiv \phi_n^{-1}\left(E_{M'}\left[\phi_n(E_{R'|M'}\left[u_n\left(W^2_n(a^S_{f,n}, a^S_n)\right)\right])\right]\right), \tag{13}$$

where $W^2_n(a^S_{f,n}, a^S_n) = (a^S_n)^\top R + a^S_{f,n} R_f$ is the final wealth obtained after the liquidation of the dividends. Ex ante, prior to the realization of the signal, the utility from an initial portfolio $(a^0_{f,n}, a^0_n)$ is, via recursion as stipulated by Klibanoff et al. (2009),

$$U^0_n(a^0_{f,n}, a^0_n) \equiv \phi_n^{-1}\left(E_M\left[\phi_n\left(E_{S|M}\left(U^S_n(a^{*,S}_{f,n}, a^{*,S}_n)\right)\right)\right]\right), \tag{14}$$

where $(a^{*,S}_{f,n}, a^{*,S}_n)$ is a solution to

$$\max_{a^S_n, a^S_{f,n}} U^S_n(a^S_{f,n}, a^S_n) \tag{15}$$

subject to the budget constraint

$$(a^S_n)^\top 1 + a^S_{f,n} \leq (a^0_n)^\top \frac{p^S}{p^0} + a^0_{f,n} R_f \equiv W^S_n(a^0_{f,n}, a^0_n). \tag{16}$$

Observe that $(a^{*,S}_{f,n}, a^{*,S}_n)$ depends on $(a^0_{f,n}, a^0_n)$. In (13) and in (14), we assume $u_n(x) =$

[26]Recall, ϕ_n is an increasing function and thus maximizing $U^S_n(a^S_{f,n}, a^S_n)$ is equivalent to maximizing $E_{M'}\left[\phi_n(E_{R'|M'}\left[u_n\left(W^2_n(a^S_{f,n}, a^S_n)\right)\right])\right]$.

$-exp(-\theta_n x)$ and $\phi_n(y) = -(-y)^{\gamma_n/\theta_n}$ as in the static analysis of Section 2. Note, (13) yields a robust mean-variance formulation while (14) does not.

5.2 Earnings announcements

5.2.1 Modeling earnings announcements

We formalize an earnings announcement as a publicly observed signal drawn from the same stochastic process governing uncertain asset returns. This signal allows the agents to update their common prior on models believed to generate returns – thus leaving them better informed about returns. We let $S = (S_1, S_2)$ to be the public signal about uncertain assets 1 and 2. Conditional on a model M, return R and signal S are i.i.d. Consistent with the notation in (1) of the static setup, we let $\Sigma = \Sigma_R - \Sigma_M$ where Σ_R and Σ_M are as given in Assumption 2, and assume that the beliefs conditional on the model are

$$\begin{pmatrix} R|M \\ S|M \end{pmatrix} \sim N\left(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}\right). \tag{17}$$

As is the case in the static setup, let $M \sim N(\mu, \Sigma_M)$. Then, it follows from Bayes' Rule that $M' \equiv M|S \sim N(\mu_S, \Sigma_S)$, where

$$\mu_S = \left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\left(\Sigma_M^{-1}\mu + \Sigma^{-1}S\right), \tag{18}$$

$$\Sigma_S^{-1} = \Sigma_M^{-1} + \Sigma^{-1}. \tag{19}$$

Notice that μ_S is a linear function of S while Σ_S does not depend on the realized value of S. However, the precision of model uncertainty increases following the signal as evident from (19), where the left hand side shows the precision of M' while the precision of M is given by Σ_M^{-1}. Analogous to Assumption 2 of the static setup, we assume that $Cov(R', M') = Var(M')$. Hence, $Var(R') \equiv \Sigma_{R'} = \Sigma + \Sigma_S$ and the updated beliefs are given by

$$\begin{pmatrix} M' \\ R' \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_S \\ \mu_S \end{pmatrix}, \begin{pmatrix} \Sigma_S & \Sigma_S \\ \Sigma_S & \Sigma + \Sigma_S \end{pmatrix}\right). \tag{20}$$

Also, following from (20), $R'|M' \sim N(M', \Sigma)$, which is analogous to (1) in the static model.

5.2.2 Equilibrium analysis

We solve the equilibrium backwards, first deriving the interim equilibrium and then the ex ante. For the interim analysis, we place ourselves at period 1 once S is realized and observed by all agents. Note, the analysis is essentially the same as the one performed for the static case, adapted to the updated beliefs. Recalling the interim maximand (15), the equivalent robust mean-variance form (3), the updated beliefs (20), and the budget constraint (16), the maximization problem of agent n reduces to

$$\max_{a_n^S}\left\{\left(W_n^S(a_{f,n}^0, a_n^0) - (a_n^S)1\right)R_f + (a_n^S)^\top \mu_S - \frac{\theta_n}{2}(a_n^S)^\top(\Sigma + \Sigma_S)a_n^S - \frac{\gamma_n - \theta_n}{2}(a_n^S)^\top \Sigma_S a_n^S\right\} \tag{21}$$

where $W_n^S(a_{f,n}^0, a_n^0) = (a_n^0)^\top \frac{p^S}{p^0} + a_{f,n}^0 R_f$ is the wealth agent n derives from his portfolio $(a_{f,n}^0, a_n^0)$ when S is realized.

The interim equilibrium is characterized in Lemmas C.1 and C.2 in the Appendix. The main take-aways from these results are as follows: As in the static case, agents hold the market portfolio in the interim period if they are homogeneous in ambiguity aversion. Otherwise, they generically hold different uncertain portfolios which vary with the realized signal S and their ambiguity aversions. Also, interim prices are linear functions of the signal.

Ex ante, agent n seeks to maximize $U_n^0\left(a_{f,n}^0, a_n^0\right)$ as defined in (14) –and where $(a_{f,n}^{*,S}, a_n^{*,S})$ is a solution to (21)– subject to the budget constraint $(q_n^0)^\top p^0 + q_{f,n}^0 \leq W_n^0$, in which W_n^0 is agent n's wealth at time 0. Lemmas C.3 and C.4 in the Appendix characterize the ex ante equilibrium. We find that, under homogeneous ambiguity aversion, agents hold the market portfolio in the initial period and their risk-free holdings remain the same across both initial and interim periods. On the other hand, if agents are heterogeneous in ambiguity aversion, ex ante they hold uncertain portfolios which vary only with their ambiguity aversions. Recall that interim portfolios not only depend on ambiguity aversion but also vary with the signal realization. Therefore, there is no trade with homogeneity in ambiguity aversion whereas equilibrium generically entails non-trivial trading with heterogeneity – as we formally state in the following proposition:

Proposition 4. *If agents are homogeneous in ambiguity aversion, i.e., $\eta_n = \eta_{n'}$ for all n, n', then the equilibrium entails no trade at any signal realization. The equilibrium entails non-trivial trading at almost all signal realizations if agents are heterogeneous in ambiguity aversion, i.e., if there exist n, n' such that $\eta_n \neq \eta_{n'}$.*

Recall from (4), heterogeneously ambiguity averse agents can be interpreted as mean-variance agents with different as-if beliefs, different only with respect to the variance term. On the other hand, the economy with homogeneously ambiguity averse agents can be formally interpreted as a standard mean-variance economy with common beliefs, and is therefore *effectively complete* with everyone holding the market portfolio before and after the public signal.[27] The effective completeness of the economy implies that the ex ante equilibrium allocation is Pareto efficient, hence following a public signal there is no trade (not even trivial trading) under homogeneous ambiguity aversion.

For an intuition of why non-trivial trading arises with heterogeneous ambiguity aversion, recall that interim portfolio choice problem is the same as the static one, adapted to updated beliefs. As formalized in (20), upon the realization of a signal S, the beliefs get updated and therefore the ratio of interim period monetary holdings, $\frac{a_{1,n}^S}{a_{2,n}^S}$, varies with S.[28] And, a key take-away from our static analysis is that this ratio would also vary with ambiguity aversion given any signal realization S. The ratio of initial-period monetary holdings, $\frac{a_{1,n}^0}{a_{2,n}^0}$, on the other hand, does not depend on the signal. Hence, the interim and initial ratios differ for almost all signal realizations. Can the inter-temporal change in the asset prices fully account for the inter-temporal change in the ratio of monetary holdings? The answer is negative if agents

[27] See the discussion of Rubinstein (1974) in pp. 54-58 of Back (2017).

[28] While adapting the static portfolio choice characterization in (8) to the interim signal, the vector of parameters $(\mu_1, \mu_2; \sigma_1, \sigma_2, \sigma_{12}; \sigma_1^M, \sigma_2^M, \sigma_{12}^M)$ gets updated and the update varies with S.

are heterogeneously ambiguity averse so that $\eta_n \neq \eta_{n'}$ for some n, n': then, for a given signal realization, the inter-temporal change in the ratio of asset prices which would completely account for the inter-temporal change in the ratio of monetary holding of agent n cannot be the same as the change in the ratio of prices that would fully account for the ratio change in the monetary holding of agent n'.

Differences in as if beliefs, under heterogeneous ambiguity aversion, explains why there is trade, but the idea is less helpful in understanding the *non-trivial* trade posited in Proposition 4. For that, an uncertainty sharing perspective takes us further. Following the public signal, the return-risk-ambiguity trade-off changes, making agents seek a different allocation of more and less ambiguous assets depending on their different tolerances for ambiguity. This prompts portfolio rebalancing and thus trade in uncertain assets. Moreover, in the uncertainty sharing perspective the trade is mutually beneficial, that is, welfare increasing. Thinking solely of the trade as a consequence of differences in beliefs, it is difficult to draw clear welfare conclusions (e.g., see Gilboa et al. (2014)).

5.2.3 Trade with and without price movements

Kandel and Pearson (1995) document that earnings announcements are usually followed by a significant rise in trading volume – not necessarily associated with large price changes:

> *Using the announcement dates of quarterly (interim) earnings from the Compustat quarterly files and daily data on the returns and volumes of common stocks, we find that there are economically and statistically significant positive abnormal volumes associated with quarterly earnings announcements even when prices do not change in response to the announcements. It is notable that there appear to be abnormal volumes that are unrelated to the magnitudes of the price changes. This is inconsistent with most existing models of volume around public announcements in which agents have identical interpretations of public signals.*

By *abnormal volume*, the authors refer to the additional volume due to the public information release beyond the usual volume observed in no-announcement days which results from trading due to life-cycle considerations or trading to exploit private information.[29] Since there is no life-cycle consideration or private information in our model, the trading volume attained in our analysis is "abnormal" in this sense.

We saw our theory gives an explanation for non-trivial trade following public signals (earnings announcements). But what does it have to say about price changes associated with such trade? To that end, first we have the following proposition:

Proposition 5. *With heterogeneous ambiguity aversion, the equilibrium generically entails non-trivial trading at the signal realization which yields no price change across periods.*

Since equilibrium prices and holdings are continuous in the signal, it follows from the proposition that, given heterogeneous ambiguity aversion, in the neighbourhood of the signal which

[29] Empirically, they define abnormal volume to be the difference between volume in a 3-day period around the announcement and the mean 3-day volume in pre-announcement periods with the same stock return.

yields no price change, there are equilibria entailing non-trivial trading with small price changes. To get an idea of the quantitative significance of the trading volume and associated price changes, later on in this subsection we report a numerical exercise. Before we do so, we discuss how our theory of trade following public announcements stands in relation to the existing literature.

A number of papers explain trading volume in dynamic settings with heterogeneous information (e.g., Kim and Verrachia (1994), and He and Wang (1995)). In these models, there is no trading volume due to public announcements unless agents also have private information. Also, importantly there is no trade without an associated change in price contrary to the key observation of Kandel and Pearson (1995) noted above. Another strand of literature explains trading volume in dynamic settings with heterogeneous prior beliefs (e.g., Harris and Raviv (1993), Kandel and Pearson (1995), and Banerjee and Kremer (2010)). In Harris and Raviv (1993), trade cannot occur in the absence of a price change. However, in the models of Kandel and Pearson (1995) and Banerjee and Kremer (2010) such trade can occur. In the latter two models, agents interpret the public signal differently due to their different prior beliefs. Trade occurs in the absence of a price change if and only if agents have different priors about the mean of the public signal hence, as noted in Remark 1, trade is driven by agents' speculatively betting against each other. Also, note that in the papers cited above the economies consist of one uncertain asset and one risk-free asset, and hence any trade generated in these papers is what we refer to as trivial trade in our context.

We now provide the numerical exercise that reports data on interim returns and trade generated by simulated signals in a calibrated dynamic economic equilibrium. The returns of the two uncertain assets are assumed to have the same mean and variance so that $M_1 = M_2$ and $\sigma_1^2 = \sigma_2^2$. The uncertainty about the mean (i.e., ambiguity) is assumed to be different across the assets so that $(\sigma_1^M)^2 \neq (\sigma_2^M)^2$. The parameter values for the risk-free asset and the uncertain assets are chosen based on the 1974-2015 nominal annual returns of the 3-month US T-Bills and the S&P500 index, respectively: R_f is rounded up as 1.05 from the sample mean of the 3-month US T-Bill (gross) returns, and $\sigma_1^2 = \sigma_2^2$ are rounded up as 0.04 from the sample variance of the S&P500 returns. We generate a return history for the uncertain assets by taking them to be the same as the 1974-2015 S&P500 index returns. Combining the non-informative prior over the mean with observations from this history, we find the variance of the resulting posterior distribution to be 0.0010. The parameters $(\sigma_1^M)^2 = 0.0006$ and $(\sigma_2^M)^2 = 0.0016$ are chosen around the latter figure so as to generate an economy with heterogeneously ambiguous assets. By rounding up the expected value of the posterior, we obtain $E[M_1] = E[M_2] = \mu_1 = \mu_2 = 1.12$ (which is a gross figure). We set $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$ and $\rho^M = \frac{\sigma_{12}^M}{\sigma_1^M \sigma_2^M}$ to be 0.5.

Following (1) and (17), conditional on the true means of the uncertain assets, the public signals (i.e., earnings announcements) are distributed such that $S|M \sim N(M, \Sigma_R - \Sigma_M)$. For our numerical exercise, we take the true means to be the same as the unconditional means, that is, we assume $M_1 = M_2 = 1.12$, and draw 1000 signal realizations from the distribution

$$N\left(\begin{pmatrix} 1.12 \\ 1.12 \end{pmatrix}, \begin{pmatrix} 0.04 - 0.0016 & (0.5)\sqrt{0.04}\sqrt{0.04} - (0.5)\sqrt{0.0006}\sqrt{0.0016} \\ (0.5)\sqrt{0.04}\sqrt{0.04} - (0.5)\sqrt{0.0006}\sqrt{0.0016} & 0.04 - 0.0006 \end{pmatrix}\right)$$

so as to simulate interim period returns and trading volumes generated by the signals. The economy is assumed to have four agents, who are heterogeneous in ambiguity aversion but homogeneous in risk aversion, with $\eta_1 = 0$, $\eta_2 = 3$, $\eta_3 = 9$, $\eta_4 = 12$ and $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$. The aggregate endowments of uncertain assets are normalized to 1. As is standard, asset i's trading volume is given by $\frac{\sum_n |q_{i,n}^S - q_{i,n}^0|}{2}$, where $q_{i,n}^S$ and $q_{i,n}^0$ are the quantities of asset i held by agent n after realization of signal S and prior to it, respectively. As mentioned, we may interpret this as abnormal trading volume in the sense of Kandel and Pearson (1995).

Figure 1: The graph in the left panel plots abnormal trading volume in relation to (interim period) absolute returns for asset 1 for a thousand draws of public signals. The graph in the right panel shows the frequency distribution of (interim period) returns for asset 1.

The graph in the left panel of Figure 1 shows substantial abnormal trading volume which is relatively unvarying – staying within a narrow band of the 2%-3% of the aggregate endowment and accompanied by both low and high (interim period) absolute returns. Prices may spike or dive following public signals, but more often than not the price changes are modest (see the graph in the right panel of Figure 1). Hence, echoing the observation of Kandel and Pearson (1995), the more frequent occurrence is relatively small interim period returns accompanied by 2%-3% abnormal trading volume.

What accounts for these findings? Recall from the equilibrium analysis (Section 5.2.2) that interim period prices are linear functions of public signal realizations, hence the distribution of interim returns follows that of signals. Extreme interim returns are low probability events, occurring only following signal realizations which are surprises. Small interim returns are more frequent and follow signal realizations which are closer to agents' expectations. The trading volume, on the other hand, follows the portfolio adjustments prompted by the change in return-risk-ambiguity trade-off following the signals. The change in ambiguity does not depend on the realized value of the signal, unlike interim returns. Thus, even when interim returns are small, the more common occurrence, the trading volume can still be significant.[30] In summary, the price change is smaller the more the signal confirms expectations, while the trading volume is mostly dictated by the change in ambiguity. This insight is explored further in the second dynamic extension where the ambiguity may be made to vary more completely.

[30]Furthermore, in results not reported here, the numerical exercise also shows that increasing heterogeneity in ambiguity aversion increases the trading volume: when the economy is assumed to have only two agents with ambiguity aversion coefficients equal to 3 and 9, the trading volume shifts slightly to the left.

5.3 Uncertainty shocks

We next explore the issue of reaction to arrival of information in a somewhat more stylized model, where the signal realizations directly inform the ambient ambiguity. More precisely, the signal directly determines the variance of second order beliefs (i.e., model uncertainty) but not the means of the returns. This modeling strategy allows us to look into the relation between changes in the level of ambiguity and trading volume unencumbered by any change in mean returns. Therefore, the analytical staging of this extension is different from that in the previous one where mean returns and the level of ambiguity were affected by the signal but only the mean returns varied with the realized value of the signal.

To fix ideas, one may think of such signals as uncertainty shocks, shocks which determine the level of uncertainty in the environment. As a concrete example, think of the Brexit vote outcome as one of two possible signal realizations: Brexit or the status quo. Each realization could determine a distinct level of uncertainty; in particular, a larger parameter uncertainty would follow the Brexit outcome.

5.3.1 Modeling uncertainty shocks

We consider a version of the model introduced in Section 5.1 with a special signal structure to model uncertainty shocks – there are three possible realizations of the signal, which we call interim states, $S = H(\text{igh}), I(\text{ntermediate}), L(\text{ow})$, that directly inform on the level of ambiguity: the defining feature of an interim state is the associated model variance denoted by $\bar{\Sigma}_S$. We assume, for simplicity, that the signal realizations are unambiguous events, i.e., the probability of S, denoted by $\pi(S)$, is the same under any model M. Like in the previous extension, we let $R|M \sim N(M, \Sigma)$ where $\Sigma = \Sigma_R - \Sigma_M$.

As before, we denote by $M' = M|S$ and $R' = R|S$ the updated beliefs over M and R. Analogous to the setup introduced in section 5.2.1, we assume that $Cov(R', M') = Var(M')$. Hence, as was the case in (20), we have

$$\begin{pmatrix} M' \\ R' \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \bar{\Sigma}_S & \bar{\Sigma}_S \\ \bar{\Sigma}_S & \Sigma + \bar{\Sigma}_S \end{pmatrix} \right)$$

and $R'|M' \sim N(\mu, \Sigma)$. Note, however, unlike in (20), $E[M']$ and $E[R']$ do not depend on S.

5.3.2 Equilibrium analysis and implications

We assume only two agents, $n = 1, 2$, in our equilibrium analysis. As in the analysis of section 5.2.1, we start with the interim period, once the signal is realized. Analogously, the interim period maximization program of agent n is given by

$$\max_{a_n^S} \left\{ \left(W_n^S(a_{f,n}^0, a_n^0) - (a_n^S)1\right) R_f + (a_n^S)^\top \mu - \frac{\theta_n}{2}(a_n^S)^\top(\Sigma + \bar{\Sigma}_S)a_n^S - \frac{\gamma_n - \theta_n}{2}(a_n^S)^\top \bar{\Sigma}_S a_n^S \right\}$$

where $W_n^S(a_{f,n}^0, a_n^0) = (a_n^0)^\top \frac{p^S}{p^0} + a_{f,n}^0 R_f$ is the wealth agent n derives from his portfolio $(a_{f,n}^0, a_n^0)$ when S is realized. The solution of this maximization problem for uncertain assets is:

$$a_n^{*,S} = \frac{1}{\theta_n}(\Sigma + \bar{\Sigma}_S + \eta_n \bar{\Sigma}_S)^{-1}(\mu - R_f 1) \tag{22}$$

The interim period market clearing condition is $\sum_n a_{i,n}^S = p^S \sum_n e_{i,n}$. Given our assumption that asset endowments are equal, $\sum_n e_{1,n} = \sum_n e_{2,n} \equiv e$, we have that at an interim equilibrium

$$p^S = \frac{1}{e}\sum_n \frac{1}{\theta_n}(\Sigma + \bar{\Sigma}_S + \eta_n \bar{\Sigma}_S)^{-1}(\mu - R_f 1) \tag{23}$$

and asset holding in interim period state S is

$$q_{i,n}^{*,S} = \frac{a_{i,n}^{*,S}}{p_i^S}. \tag{24}$$

Ex ante, an agent seeks to maximize $U_n^0\left(a_{f,n}^0, a_n^0\right)$ as defined in (14) –and where $(a_{f,n}^{*,S}, a_n^{*,S})$ is given by (22)– subject to the budget constraint $(q_n^0)^\top p^0 + q_{f,n}^0 \leq W_n^0$. Given the assumption that the probability of each state S is the same under each model M, (14) simplifies to:

$$U_n^0(a_{f,n}^0, a_n^0) = E_S\left(U_n^S(a_{f,n}^{*,S}, a_n^{*,S})\right).$$

Lemma C.5 in the Appendix shows that agents hold (a proportion of) the *market portfolio* as their uncertain portfolio in period 0 and agent n's market portfolio holding is proportional to his risk tolerance $\frac{1}{\theta_n}$. We know from our earlier static analysis that interim period holdings will be exactly of the same form if agents are homogeneously ambiguity averse. However, with heterogeneity in ambiguity aversion we have trading over time, just as we obtained in Proposition 4.

Proposition 6. *The equilibrium entails non-trivial trading if agents are heterogeneous in ambiguity aversion, i.e., $\eta_1 \neq \eta_2$. If agents are homogeneous in ambiguity aversion, i.e., $\eta_1 = \eta_2$, then the equilibrium entails no trade at any signal realization.*

Since agents hold the *market portfolio* in period 0, the ex ante portfolio holdings do not depend on the probabilities of the three interim states. The interim portfolio holdings are chosen after the state realizes, hence they do not depend on the state probabilities either. Therefore, trading volume does not depend on interim state probabilities. All the action that comes from these probabilities is subsumed by prices: as can be seen from (57) in the Appendix, the ex-ante equilibrium price is a weighted sum of the interim equilibrium prices, where the weights are proportional to the probabilities of the states. For instance, if the probability of an interim state tends to 1, then the ex ante price converges to that state's price discounted by the risk-free rate. In this case, we will observe a small price change if the likely state were to arise. Furthermore, we know from (23) that the interim equilibrium prices vary with the signal realization S and are inversely related to $\bar{\Sigma}_S$, the level of ambiguity embodied by the realized signal. Therefore, if the level of ambiguity varies significantly across states and a state with small probability but relatively high ambiguity level were to arise, then there would be a significant drop in the asset prices. The price changes in this model are significant if and only if the state realizations are surprises and the ambiguity levels vary significantly across the states.

Next we relate the trading volume generated in our model to the level of ambiguity in the interim period. We focus on a scenario where the uncertainty shock does not affect all assets, in particular, it affects just asset 1, assumed to be more ambiguous (I) than asset 2. For a clearer statement of the result, we relabel the two agents in the economy as n and n'.

Proposition 7. *Assume $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, $\bar{\sigma}_1^H > \bar{\sigma}_1^I > \bar{\sigma}_1^L > \bar{\sigma}_2$, and*

$$\bar{\Sigma}_S = \begin{pmatrix} \left(\bar{\sigma}_1^S\right)^2 & \bar{\sigma}_{12} \\ \bar{\sigma}_{12} & \left(\bar{\sigma}_2\right)^2 \end{pmatrix},$$

where $S = H, I, L$. Let $\eta_{n'} > \eta_n$ and Λ be a constant whose expression is given by (58) in the Appendix. If

$$\left(\bar{\sigma}_1^S\right)^2 < \Lambda \quad \textit{for all } S, \tag{25}$$

then

$$\frac{a_{1,n}^{*,H}}{a_{2,n}^{*,H}} - \frac{a_{1,n'}^{*,H}}{a_{2,n'}^{*,H}} \; > \; \frac{a_{1,n}^{*,I}}{a_{2,n}^{*,I}} - \frac{a_{1,n'}^{*,I}}{a_{2,n'}^{*,I}} \; > \; \frac{a_{1,n}^{*,L}}{a_{2,n}^{*,L}} - \frac{a_{1,n'}^{*,L}}{a_{2,n'}^{*,L}} \; > \; 0.$$

This proposition posits that the higher the level of ambiguity following the uncertainty shock, *modulo the upper bound (25)*, the bigger is the difference of the interim asset allocation ratios (i.e., the ratio of monies allocated to asset 1 versus asset 2) between the less and the more ambiguity averse agents. Given that all agents, ex ante, hold the market portfolio, for each agent the change in the asset allocation ratio across time is bigger following a shock resulting in a higher level of ambiguity. This implies that the dollar volume of trading is monotonically increasing in the level of ambiguity (uncertainty shock).

The conclusions we draw from Proposition 7 are tempered by the restriction on the ambiguity level given by (25). Why does this restriction arise? Recall from Corollary 1 that, if agent n' is more ambiguity averse than agent n and asset 1 is more ambiguous (I) than asset 2, then the difference in asset allocation ratios, $\frac{a_{1,n}^{*,S}}{a_{2,n}^{*,S}} - \frac{a_{1,n'}^{*,S}}{a_{2,n'}^{*,S}}$, is strictly positive for any given ambiguity level S. What Proposition 7 adds to that corollary is the implication about how these differences compare across the ambiguity levels. Notice though, the asset allocation ratios here are the ratios of monetary holdings, i.e., price *times* quantity, while the corollary was true for quantity allocations, too. Asset prices differ across ambiguity levels (revealed through signal realizations); in particular, typically, the higher the ambiguity level of an asset the lower is that asset's price. Hence it is possible that even if a higher ambiguity level leads to a bigger difference in the asset allocation ratio in quantities, the lower price attained in that ambiguity level would make the difference in monies smaller. The role of the restriction (25) is essentially to limit the ambiguity increase to a level that does not depress prices enough to reverse the monotonicity.

Dimmock et al. (2016) find empirical support for the notion that ambiguity aversion interacts with time-varying levels of economic uncertainty: in a representative US household survey, conditional on holding stocks prior to the 2008-09 financial crisis, more ambiguity averse households were more likely to actively reduce their equity holdings following the onset of the crisis. This is in line with the main take-away from Proposition 7: the higher the level of ambiguity following an uncertainty shock, the higher is the dollar trading volume as more ambiguity averse agents fly to safety, i.e., less ambiguous assets.

To summarize the insights from the discussions in this section, trading volume is positively associated with the variation in ambiguity across periods while price changes are inversely related to the probability of the realized state (the extent to which the state is anticipated). Hence, even if the realized state is not a surprise, the level of trading volume can be significant because the resolution of ambiguity always changes the return-risk-ambiguity trade-off. If we assume that most announcements are not surprises, then, according to our model, non-negligible trading volume with small price movements would be a common occurrence following announcements. However, on occasion a big surprise transpires, often accompanied by a relatively big change in ambiguity. In such a case, the big surprise would be associated with both a big price change and significant trading volume.

6 Concluding discussion: empirical testability

We have demonstrated the potential of our theory to provide explanations for several interesting phenomena regarding portfolio choice, cross-sectional asset pricing and trading volume. The explanations are based on intuitive, connected narratives of agents' perception of and reaction to uncertainty. The narratives do not stand up if the uncertainty were formalized as risk but do when formalized as ambiguity, mirroring the findings of the recent macro-finance literature that treats systematic uncertainty as ambiguous. The next natural step is to empirically test this potential, especially its quantitative significance. To that end, the critical question is the measurement of ambiguity of returns. While some work in this regard has taken place in the macro-finance context, there has been scarcely any work in incorporating ambiguity into cross-sectional empirical analysis, with the notable exception of Garlappi et al. (2007). To conclude, we next discuss some empirical strategies to test our results, though carrying out formal tests or even providing the details of empirical implementation are beyond the scope of this paper.

In the narrower perspective of our theory, ambiguity about the return distribution is taken to be the uncertainty about the mean parameter. This may be measured by obtaining a Bayesian estimate of the parameter. That is, combining a prior over the means with observations from the returns data, the covariance matrix of the resulting posterior joint distribution of the means can be taken as the measure of ambiguity for the assets under consideration. For instance, Gallant et al. (2018) do this for aggregate asset returns in order to estimate the impact of ambiguity on the dynamics of equity premium. An alternative approach for measurement of ambiguity, in the classical rather than Bayesian fashion, is to use the width of confidence intervals around the point estimates of means for the quantification of the measure, as in Garlappi et al. (2007) whose objective is to construct optimal portfolios for agents concerned with parameter uncertainty.

Having a measure of asset ambiguity allows us to test whether our theory addresses the asset allocation puzzle. A first hypothesis to test is whether financial advisors identify portfolios tilted towards less ambiguous assets as those suitable for more conservative investors. A stronger test would involve measurement of ambiguity aversion of individual investors and compare their portfolio allocations. This could be done following the approaches taken by Bianchi and Tallon (2018) and Dimmock et al. (2016). Using household survey data, both papers measure ambiguity aversion by asking subjects to choose between lotteries with known versus unknown probability distributions over the final payoffs and match this measurement with the subjects' portfolio allocations. Thus, to directly link our theory to the asset allocation puzzle, we need to test two hypotheses: first, that households who choose portfolios recommended to conservative investors by financial advisors tend to be more ambiguity averse, and if so, that their recommended portfolios, on average, carry less ambiguous assets.

Turning to testable cross-sectional pricing implications, the first issue to be tackled is to measure β^{Amb}. It can be estimated making use of the covariance matrix of the posterior joint distribution of the means of assets and the market portfolio. On the other hand, β can be estimated following standard procedures, i.e., regressing individual asset excess returns against market portfolio excess returns and taking the resulting slope coefficient as the estimate. These estimates would be enough to get us to a first basic test of the result presented in Proposition 2. The proposition suggests that the proportion of ambiguity premia across any two assets, i and j, is given by

$$\frac{\alpha_i}{\alpha_j} = \frac{E[R_i] - R_f - \beta_i(E[R_{market}] - R_f)}{E[R_j] - R_f - \beta_j(E[R_{market}] - R_f)} = \frac{\beta_i^{Amb} - \beta_i}{\beta_j^{Amb} - \beta_j}.$$

Therefore, a hypothesis to be tested is whether stocks with higher $\beta^{Amb} - \beta$ generate higher abnormal returns, on average, where abnormal returns are taken to be returns in excess of the ones predicted by CAPM. If this hypothesis were not rejected, then abnormal returns are positively correlated with ambiguity premium, giving qualitative support for (11). For the next step of obtaining a more powerful quantitative test of our theory, we need an estimate for market ambiguity aversion, i.e., η_{market}. To that end, one can use (11) again and assume that, in each period t, realized return of each asset i satisfies the following relationship:

$$R_{i,t} - R_{f,t} = \left[\frac{\eta_{market}\ var^M(R_{market})}{var(R_{market}) + \eta_{market}\ var^M(R_{market})}\left(\beta_i^{Amb} - \beta_i\right) + \beta_i\right](R_{market,t} - R_{f,t}) + \epsilon_{i,t},$$

where $\epsilon_{i,t}$ is taken to be an unbiased error term i.i.d. across assets and periods. Based on this relationship, η_{market} can be estimated by first plugging in β_i's, β_i^{Amb}'s and $var^M(R_{market})$ whose measurements are discussed above and then by matching moments using a panel dataset of realized returns. Then, a statistical goodness-of-fit test can be applied to measure the difference between the empirical distribution of realized returns and the predicted distribution of returns obtained using the estimated η_{market}, and thus to evaluate how well our theory explains returns compared to (standard) CAPM.

We may test whether ambiguity has explanatory power for the value and size premia along the following lines. We start by constructing quantile portfolios sorted according to $\beta^{Amb} - \beta$. Then, within each such quantile, we further sort stocks into portfolios ranked according to their book-to-market ratios or market capitalizations. Sorting according to $\beta^{Amb} - \beta$ allows us to control for cross-sectional differences in exposure to systematic ambiguity. If the difference between the abnormal returns of high and low book-to-market quantile stocks is significantly lower when exposure to systematic ambiguity is controlled for (i.e., within each $\beta^{Amb} - \beta$ quantile) compared to the case when this exposure is not controlled for, then one would have supporting evidence for ambiguity (at least, partially) explaining value premium. A similar test can be run for size premium along the same lines. Alternatively, as a short cut approach to proxy cross-sectional variation in exposure to systematic ambiguity, we can make use of macroeconomic uncertainty indices developed by Jurado et al. (2015) and Carriero et al. (2018). In particular, stock returns can be regressed against these indices and stocks with higher slope coefficients can be identified as those with higher exposure to macroeconomic uncertainty and therefore likely with higher exposure to systematic ambiguity. Using a similar empirical strategy, one can also test whether ambiguity has explanatory power for the flatter empirical security market line phenomenon.

In closing, we turn to a testing strategy for the theory of ambiguity driven trading discussed in Section 5. Proposition 7 suggests that an increase (decrease) in ambiguity following a public announcement would lead to more ambiguity averse investors' portfolios to diverge away from (converge towards) those of less ambiguity averse. One may test this with a panel dataset that tracks differently ambiguity averse investors' portfolio allocations over time, by checking how investors' portfolio positions vary with changes in macroeconomic uncertainty indices that follow public announcements.

Appendix:
Proofs of results in the main text and additional related results

A Section 3: Portfolio choice

Lemma A.1. *Let agent n be more ambiguity averse than agent n', i.e., $\eta_n > \eta_{n'}$. Then, for optimal portfolio choices it holds that*

$$\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}} \quad \text{and} \quad \frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}.$$

if and only if

$$\left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb}\ S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 \ + \ \left(S_1^2 - S_2^2\right)\ \sigma_{12}^M\ \sigma_1^2\ \sigma_2^2 \\ - \left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right)\ \sigma_{12} \left(\sigma_1^M\right)^2 \left(\sigma_2^M\right)^2 \quad < \quad 0. \tag{26}$$

Proof of Lemma A.1. It follows from (8) that

$$\frac{\partial \left(\frac{a_{1,n}}{a_{2,n}}\right)}{\partial \eta_n} = \frac{C}{\left[(\mu_2 - R_f)\left(\sigma_1^2 + \eta_n (\sigma_1^M)^2\right) - (\mu_1 - R_f)\left(\sigma_{12} + \eta_n \sigma_{12}^M\right)\right]^2}, \tag{27}$$

where

$$C = \left[(\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M\right]\left[(\mu_2 - R_f)(\sigma_1)^2 - (\mu_1 - R_f)\sigma_{12}\right] \\ - \left[(\mu_2 - R_f)(\sigma_1^M)^2 - (\mu_1 - R_f)\sigma_{12}^M\right]\left[(\mu_1 - R_f)(\sigma_2)^2 - (\mu_2 - R_f)\sigma_{12}\right].$$

As $S_i = \frac{\mu_i - R_f}{\sigma_i}$ and $S_i^{Amb} = \frac{\mu_i - R_f}{\sigma_i^M}$, $i = 1, 2$, we can re-write C as follows after some tedious but straightforward calculations:

$$C = \left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb} S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 + \left(S_1^2 - S_2^2\right) \sigma_{12}^M \sigma_1^2 \sigma_2^2 \\ - \left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right) \sigma_{12} \left(\sigma_1^M\right)^2 \left(\sigma_2^M\right)^2. \tag{28}$$

Therefore, following from (27) and (28), $\frac{\partial \left(\frac{a_{1,n}}{a_{2,n}}\right)}{\partial \eta_n} < 0$ if and only if

$$\left(\left(\frac{S_1^{Amb}}{S_1}\right)^2 - \left(\frac{S_2^{Amb}}{S_2}\right)^2\right) S_1^{Amb} S_2^{Amb} \left(\sigma_1^M\right)^3 \left(\sigma_2^M\right)^3 + \left(S_1^2 - S_2^2\right) \sigma_{12}^M \sigma_1^2 \sigma_2^2 \\ - \left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right) \sigma_{12} \left(\sigma_1^M\right)^2 \left(\sigma_2^M\right)^2 < 0. \tag{29}$$

Hence, $\frac{a_{1,n}}{a_{2,n}} < \frac{a_{1,n'}}{a_{2,n'}}$ for $\eta_n > \eta_{n'}$ iff (29) holds. Since $\frac{a_{1,n}}{a_{2,n}} = \frac{p_1 q_{1,n}}{p_2 q_{2,n}}$ for all n, it also holds that

$\frac{q_{1,n}}{q_{2,n}} < \frac{q_{1,n'}}{q_{2,n'}}$ for $\eta_n > \eta_{n'}$ iff (29) holds. $\square$

Proof of Proposition 1.

(a) Let $S_1 = S_2$. Then the left-hand side of (26) reduces to

$$\left(\left(S_1^{Amb}\right)^2 - \left(S_2^{Amb}\right)^2\right)\left(\sigma_1^M\right)^2\left(\sigma_2^M\right)^2\sigma_1\sigma_2\left(1-\rho\right),$$

which is strictly less than 0 if $\rho \neq 1$ and $S_1^{Amb} < S_2^{Amb}$. Hence, the desired result follows from Lemma A.1. $\square$

(b) After somewhat tedious but routine calculations the left-hand side of (26) can be re-written as

$$\begin{aligned}&\left(\mu_1 - R_f\right)\left(\mu_2 - R_f\right)\times\\&\left[\left(\sigma_1^2\left(\sigma_2^M\right)^2 - \sigma_2^2\left(\sigma_1^M\right)^2\right) + \frac{\mu_1 - R_f}{\mu_2 - R_f}\left(\sigma_2^2\sigma_{12}^M - \left(\sigma_2^M\right)^2\sigma_{12}\right) + \frac{\mu_2 - R_f}{\mu_1 - R_f}\left(\left(\sigma_1^M\right)^2\sigma_{12} - \sigma_1^2\sigma_{12}^M\right)\right].\end{aligned} \tag{30}$$

Assume that $\rho\left(\frac{\sigma_1^M}{\sigma_2^M}\frac{\sigma_2}{\sigma_1}\right) \leq \rho^M \leq \rho\left(\frac{\sigma_2^M}{\sigma_1^M}\frac{\sigma_1}{\sigma_2}\right)$. Then the expression in (30), and therefore the left-hand side of (26), is less than or equal to

$$\begin{aligned}&\left(\mu_1 - R_f\right)\left(\mu_2 - R_f\right)\left(\sigma_1^2\left(\sigma_2^M\right)^2 - \sigma_2^2\left(\sigma_1^M\right)^2\right)\\=\;&\left(\mu_1 - R_f\right)\left(\mu_2 - R_f\right)\sigma_2^2\left(\sigma_2^M\right)^2\left(\left(\frac{\sigma_1}{\sigma_2}\right)^2 - \left(\frac{\sigma_1^M}{\sigma_2^M}\right)^2\right).\end{aligned}$$

The above expression is strictly less than 0 if $S_1^{Amb}/S_1 < S_2^{Amb}/S_2$ (which is equivalent to $\sigma_1^M/\sigma_2^M > \sigma_1/\sigma_2$). Therefore, the desired result follows from Lemma A.1. $\square$

Proof of Corollary 1. This is a direct corollary of Proposition 1. $\square$

Proof of Corollary 2. Observe from (8) that

$$\lim_{\eta_n\to\infty}\frac{a_{1,n}}{a_{2,n}} = \frac{(\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M}{(\mu_2 - R_f)(\sigma_1^M)^2 - (\mu_1 - R_f)\sigma_{12}^M}.$$

If $S_1^{Amb} < S_2^{Amb}$ and $\sigma_1^M > \sigma_2^M$, then

$$\frac{\mu_1 - R_f}{\mu_2 - R_f} < \frac{\sigma_1^M}{\sigma_2^M} \leq \left(\frac{\sigma_1^M}{\sigma_2^M}\right)\left(\frac{\sigma_1^M + \rho^M\sigma_2^M}{\sigma_2^M + \rho^M\sigma_1^M}\right) = \frac{(\sigma_1^M)^2 + \sigma_{12}^M}{(\sigma_2^M)^2 + \sigma_{12}^M}.$$

This further implies that $\lim_{\eta_n\to\infty}\frac{a_{1,n}}{a_{2,n}} < 1$. $\square$

B Section 4: Static equilibrium asset pricing

Lemma B.1. *For any given $n = 1, \dots, N$, let a_n be agent n's optimal portfolio so that a_n solves the maximization problem (5). Then the following holds:*

$$E[R_i] - R_f = \frac{cov(R(a_n), R_i) + \eta_n cov^M(R(a_n), R_i)}{var(R(a_n)) + \eta_n var^M(R(a_n))}(E[R(a_n)] - R_f), \quad i = 1, 2. \tag{31}$$

Proof of Lemma B.1. Recall that the optimal portfolio is given by (7):

$$a_{i,n} = \frac{(\mu_i - R_f)A_{j,n} - (\mu_j - R_f)B_{12,n}}{A_{1,n}A_{2,n} + B_{12,n}^2}, \quad i, j = 1, 2, i \neq j$$

where $A_{i,n} = \theta_n \left[\sigma_i^2 + \eta_n(\sigma_i^M)^2\right]$ and $B_{12} = \theta_n \left[\sigma_{12} + \eta_n \sigma_{12}^M\right]$. Hence, we have the following:

$$\mu_i - R_f = (a_{i,n}A_{i,n} + a_{j,n}B_{12,n}) \frac{A_{1,n}A_{2,n} + B_{12,n}^2}{A_{1,n}A_{2,n} - B_{12,n}^2} \tag{32}$$

This further implies

$$\begin{aligned} E[R(a_n)] - R_f &\equiv \frac{a_{1,n}(\mu_1 - R_f) + a_{2,n}(\mu_2 - R_f)}{a_{1,n} + a_{2,n}} \\ &= \frac{(a_{1,n})^2 A_{1,n} + 2a_{1,n}a_{2,n}B_{12,n} + (a_{2,n})^2 A_{2,n}}{a_{1,n} + a_{2,n}} \times \frac{A_{1,n}A_{2,n} + B_{12,n}^2}{A_{1,n}A_{2,n} - B_{12,n}^2}. \end{aligned}$$

Together with (32), this implies that:

$$\frac{\mu_i - R_f}{a_{i,n}A_{i,n} + a_{j,n}B_{12,n}} = \frac{E[R(a_n)] - R_f}{\frac{(a_{1,n})^2 A_{1,n} + 2a_{1,n}a_{2,n}B_{12,n} + (a_{2,n})^2 A_{2,n}}{a_{1,n} + a_{2,n}}}. \tag{33}$$

Next, observe the following:

$$\begin{aligned} a_{i,n}A_{i,n} + a_{j,n}A_{j,n} &= \theta_n(a_{1,n} + a_{2,n}) \left(cov(R(a_n), R_i) + \eta_n cov^M(R(a_n), R_i)\right), \\ (a_{1,n})^2 A_{1,n} + 2a_{1,n}a_{2,n}B_{12,n} + (a_{2,n})^2 A_{2,n} &= \theta_n(a_{1,n} + a_{2,n})^2 \left(var(R(a_n)) + \eta_n var^M(R(a_n))\right). \end{aligned}$$

Therefore, (33) can be re-written as

$$E[R_i] - R_f = \frac{cov(R(a_n), R_i) + \eta_n cov^M(R(a_n), R_i)}{var(R(a_n)) + \eta_n var^M(R(a_n))}(E[R(a_n)] - R_f),$$

which is the desired result. $\square$

Proof of Proposition 2. Proving there exists $\eta_{market} \in \left[\underline{\eta}, \bar{\eta}\right]$ such that an agent with ambiguity aversion parameter η_{market} would optimally hold the market portfolio is sufficient to establish existence of $\eta_{market} \in \left[\underline{\eta}, \bar{\eta}\right]$ such that (10)-(12) hold. This is so because the antecedent implies via Lemma B.1 that (10) holds, which can then be re-written as (11)-(12). From

(7), we know that agent n's optimal monetary holding of asset i is $a_{i,n} = \frac{K_i(\eta_n)}{\theta_n}$ where

$$K_i(\eta_n) \equiv \frac{(\mu_i - R_f)(\sigma_j^2 + \eta_n(\sigma_j^M)^2) - (\mu_j - R_f)(\sigma_{12} + \eta_n \sigma_{12}^M)}{(\sigma_1^2 + \eta_n(\sigma_1^M)^2)(\sigma_2^2 + \eta_n(\sigma_2^M)^2) + (\sigma_{12} + \eta_n \sigma_{12}^M)^2}, \quad j \neq i.$$

Therefore agent n's optimal portfolio return is given by

$$\frac{a_{1,n}R_1 + a_{2,n}R_2}{a_{1,n} + a_{2,n}} = \frac{K_1(\eta_n)R_1 + K_2(\eta_n)R_2}{K_1(\eta_n) + K_2(\eta_n)}.$$

Following the equation above and (9), the optimal portfolio return of an agent with ambiguity aversion parameter η_{market} is equal to the market portfolio return if and only if

$$\frac{K_1(\eta_{market})R_1 + K_2(\eta_{market})R_2}{K_1(\eta_{market}) + K_2(\eta_{market})} = \frac{p_1^* e_1 R_1 + p_2^* e_2 R_2}{p_1^* e_1 + p_2^* e_2}. \tag{34}$$

Moreover, if the above equality holds for all realizations of exogenous asset returns R_1 and R_2, then the agent with ambiguity aversion η_{market} optimally holds the market portfolio. Now note that market clearing implies

$$\sum_n \frac{K_i(\eta_n)}{\theta_n} = p_i^* e_i \quad i = 1, 2. \tag{35}$$

Hence, following (34)-(35), an agent with ambiguity aversion parameter η_{market} optimally holds the market portfolio if and only if[31]

$$\frac{K_1(\eta_{market})}{K_1(\eta_{market}) + K_2(\eta_{market})} = \frac{\sum_n \frac{K_1(\eta_n)}{\theta_n}}{\sum_n \frac{K_1(\eta_n)}{\theta_n} + \sum_n \frac{K_2(\eta_n)}{\theta_n}}. \tag{36}$$

Re-writing the above equation, we obtain

$$\begin{aligned} & \frac{(\mu_1 - R_f)\sigma_2^2 - (\mu_2 - R_f)\sigma_{12} + \eta_{market}((\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M)}{\sum_{i=1,2,\ j\neq i}[(\mu_i - R_f)\sigma_j^2 - (\mu_j - R_f)\sigma_{12} + \eta_{market}((\mu_i - R_f)(\sigma_j^M)^2 - (\mu_j - R_f)\sigma_{12}^M)]} \\ = & \frac{\sum_n \frac{1}{\theta_n} \frac{(\mu_1 - R_f)\sigma_2^2 - (\mu_2 - R_f)\sigma_{12} + \eta_n((\mu_1 - R_f)(\sigma_2^M)^2 - (\mu_2 - R_f)\sigma_{12}^M)}{(\sigma_1^2 + \eta_n(\sigma_1^M)^2)(\sigma_2^2 + \eta_n(\sigma_2^M)^2) + (\sigma_{12} + \eta_n\sigma_{12}^M)^2}}{\sum_{i=1,2,\ j\neq i}\sum_n \frac{1}{\theta_n} \frac{(\mu_i - R_f)\sigma_j^2 - (\mu_j - R_f)\sigma_{12} + \eta_n((\mu_i - R_f)(\sigma_j^M)^2 - (\mu_j - R_f)\sigma_{12}^M)}{(\sigma_1^2 + \eta_n(\sigma_1^M)^2)(\sigma_2^2 + \eta_n(\sigma_2^M)^2) + (\sigma_{12} + \eta_n\sigma_{12}^M)^2}}. \end{aligned}$$

This uniquely yields

$$\eta_{market} = \left(\sum_{k=1}^{N} \Pi_k \right)^{-1} \sum_{n=1}^{N} \Pi_n \eta_n, \tag{37}$$

where, for $n = 1, \cdots, N$,

$$\Pi_n = \left(\frac{1}{\theta_n} \right) \left(\frac{1}{\left(\sigma_1^2 + \eta_n(\sigma_1^M)^2\right)\left(\sigma_2^2 + \eta_n(\sigma_2^M)^2\right) + \left(\sigma_{12} + \eta_n \sigma_{12}^M\right)^2} \right). \tag{38}$$

[31]Note that, if (36) holds, then it necessarily holds that

$$\frac{K_2(\eta_{market})}{K_1(\eta_{market}) + K_2(\eta_{market})} = \frac{\sum_n \frac{K_2(\eta_n)}{\theta_n}}{\sum_n \frac{K_1(\eta_n)}{\theta_n} + \sum_n \frac{K_2(\eta_n)}{\theta_n}}.$$

Since $\Pi_n > 0$ for all n, it follows that $\min_n\{\eta_n\} = \underline{\eta} < \eta_{market} < \bar{\eta} = \max_n\{\eta_n\}$ provided that $\underline{\eta} \neq \bar{\eta}$. Hence, we have the desired result. $\square$

Proof of Proposition 3. This follows from (37)-(38) in the proof of Proposition 2. $\square$

Proof of Corollary 3. It follows from Proposition 3 that $\frac{\partial \theta_n \Pi_n \eta_n}{\partial \eta_n}$ is strictly negative if and only if $\eta_n > \sqrt{\frac{\sigma_1^2\sigma_2^2+\sigma_{12}^2}{(\sigma_1^M)^2(\sigma_2^M)^2+(\sigma_{12}^M)^2}}$. Therefore, $\theta_n \Pi_n \eta_n$ is an increasing function of η_n if $\eta_n \leq \sqrt{\frac{\sigma_1^2\sigma_2^2+\sigma_{12}^2}{(\sigma_1^M)^2(\sigma_2^M)^2+(\sigma_{12}^M)^2}}$ and it is a decreasing function of η_n otherwise. This gives us the desired result. $\square$

C Section 5: Dynamic equilibrium analysis with public signals

Lemma C.1. *Optimal interim monetary holdings of agent n are given by*

$$a_n^S = \frac{1}{\theta_n}\left(A_n + B_n S\right), \tag{39}$$

where

$$\begin{aligned} A_n &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left[\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma_M^{-1}\mu - R_f 1\right], \\ B_n &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma^{-1}. \end{aligned}$$

Furthermore, B_n is a symmetric matrix.

Proof of Lemma C.1. Recall from (21), the interim maximization problem of agent n reduces to

$$\max_{a_n^S}\left\{\left(W_n^S(a_{n,f}^0, a_n^0) - (a_n^S)1\right)R_f + (a_n^S)^\top \mu_S - \frac{\theta_n}{2}(a_n^S)^\top(\Sigma+\Sigma_S)a_n^S - \frac{\gamma_n - \theta_n}{2}(a_n^S)^\top \Sigma_S a_n^S\right\}.$$

First order condition for this problem yields

$$-R_f 1 + \mu_S - \theta_n\left(\Sigma + \Sigma_S\right)a_n^S - (\gamma_n - \theta_n)\Sigma_S a_n^S = 0.$$

Since $\eta_n = \frac{\gamma_n - \theta_n}{\theta_n}$,

$$-R_f 1 + \mu_S - \theta_n\left(\Sigma + \Sigma_S\right)a_n^S - \theta_n\eta_n\Sigma_S a_n^S = 0$$

and thus

$$a_n^S = \frac{1}{\theta_n}\left(\Sigma + \Sigma_S + \eta_n\Sigma_S\right)^{-1}\left(\mu_S - R_f 1\right) = \frac{1}{\theta_n}\left(\Sigma + (\eta_n+1)\Sigma_S\right)^{-1}\left(\mu_S - R_f 1\right)$$

or, recalling that $\Sigma_{R'} = \Sigma + \Sigma_S$,

$$a_n^S = \frac{1}{\theta_n}\left(\Sigma_{R'} + \eta_n\Sigma_S\right)^{-1}\left(\mu_S - R_f 1\right).$$

Note that μ_S is a function of S, whereas Σ_S and Σ (and thus $\Sigma_{R'}$) are not. Plugging the expression for $\mu_S = \left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\left(\Sigma_M^{-1}\mu + \Sigma^{-1}S\right)$, we obtain:

$$a_n^S = \frac{1}{\theta_n}\left(A_n + B_n S\right),$$

where

$$\begin{aligned} A_n &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left[\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma_M^{-1}\mu - R_f 1\right], \\ B_n &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma^{-1}. \end{aligned}$$

Also, observe that

$$\begin{aligned} B_n &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\Sigma^{-1} \\ &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left(\Sigma\left(\Sigma_M^{-1} + \Sigma^{-1}\right)\right)^{-1} \\ &= (\Sigma + (\eta_n+1)\Sigma_S)^{-1}\left(\Sigma\Sigma_M^{-1} + I\right)^{-1} \\ &= \left[\left(\Sigma\Sigma_M^{-1} + I\right)(\Sigma + (\eta_n+1)\Sigma_S)\right]^{-1} \\ &= \left[\Sigma\Sigma_M^{-1}\Sigma + \Sigma + (\eta_n+1)\Sigma\Sigma_M^{-1}\Sigma_S + (\eta_n+1)\Sigma_S\right]^{-1} \\ &= \left[\Sigma\Sigma_M^{-1}\Sigma + \Sigma + (\eta_n+1)\Sigma\underbrace{\left(\Sigma_S^{-1} - \Sigma^{-1}\right)}_{\text{follows from Eqn. (19)}}\Sigma_S + (\eta_n+1)\Sigma_S\right]^{-1} \\ &= \left[\Sigma\Sigma_M^{-1}\Sigma + \Sigma + (\eta_n+1)\left(\Sigma - \Sigma_S\right) + (\eta_n+1)\Sigma_S\right]^{-1} \\ &= \left[\Sigma\Sigma_M^{-1}\Sigma + \Sigma + (\eta_n+1)\Sigma\right]^{-1}, \end{aligned}$$

which is symmetric since $\Sigma\Sigma_M^{-1}\Sigma$ and Σ are both symmetric. □

Lemma C.2. *At interim equilibrium, for any $S \in \mathbb{R}$, $i \in \{1,2\}$ and $n \in \{1,\cdots,N\}$, agent n's holding of asset i is given by:*

$$q_{i,n}^S = \begin{cases} \frac{\frac{1}{\theta_n}}{\sum_k \frac{1}{\theta_k}} \times e & : \ \eta_n = \eta_k \quad \forall k \in \{1,\cdots,N\} \\ \frac{\frac{1}{\theta_n}(A_n + B_n S)_i}{\sum_k \frac{1}{\theta_k}(A_k + B_k S)_i} \times e & : \ otherwise, \end{cases} \tag{40}$$

where A_n and B_n are as defined in Lemma C.1, and

$$p^S = A + BS \tag{41}$$

where $A \equiv \frac{1}{e}\sum_n \frac{1}{\theta_n}A_n$ and $B \equiv \frac{1}{e}\sum_n \frac{1}{\theta_n}B_n$.[32]

Proof of Lemma C.2. This lemma follows directly from Lemma C.1 above and market clearing. □

[32]For a vector V, we to denote the i^{th} component of the vector by $(V)_i$.

Remark 3. *As noted earlier, ex ante preferences defined in (14) do not take the basic robust mean-variance form. Rather, we apply the recursive smooth model framework of Klibanoff et al. (2009) to embed the robust mean-variance form in the dynamic setting, which complicates the ex ante equilibrium analysis compared to the static analysis.*

Lemma C.3. *At ex ante equilibrium, for any $i \in \{1,2\}$ and $k, n \in \{1, \cdots, N\}$, the ratio of agents k and n's asset i holdings is given by:*

$$\frac{q_{i,k}^0}{q_{i,n}^0} = \begin{cases} \frac{\theta_n}{\theta_k} & : \ \eta_n = \eta_k \\ \frac{\theta_n}{\theta_k} \frac{(Q(\eta_k, p^0))_i}{(Q(\eta_n, p^0))_i} & : \ otherwise, \end{cases}$$

where $Q(\eta_n, p^0)$ is a vector whose expression is as given by (43) in the proof below.

Proof of Lemma C.3. First note that, for given monetary holdings $(a_{f,n}^0, a_n^0)$:

$$\begin{aligned}
&E_{R'|M'}\left[u_n\left(W_n^2(a_{f,s}^S, a_n^S)\right)\right] = \\
&= E_{R'|M'}\left[u_n\left((a_n^S)^\top R + \left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f\right)\right] \\
&= -exp\left(-\theta_n\left[\left(W_n^0 - (q_{f,n}^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f\right]\right) \times \\
&\quad exp\left(-\theta_n (a_n^S)^\top E_{R'|M'}[R] + \frac{1}{2}\theta_n^2 (a_n^S)^\top var_{R'|M'}(R) a_S^n\right) \\
&= -exp\left(-\theta_n\left[\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f + (a_n^S)^\top M' - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_S^n\right]\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&E_{M'}\left[\phi_n\left(E_{R'|M'}\left[u_n\left(W_n^2(a_{f,s}^S, a_n^S)\right)\right]\right)\right] = \\
&= E_{M'}\Big[-exp\Big(-\gamma_n\Big[\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f + (a_n^S)^\top M' - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_n^S\Big]\Big)\Big] \\
&= -exp\Big(-\gamma_n\Big[\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_n^S\Big]\Big) \times \\
&\quad exp\left(-\gamma_n (a_n^S)^\top \mu_S + \frac{\gamma_n^2}{2}(a_n^S)^\top \Sigma_S a_n^S\right).
\end{aligned}$$

Since $\phi_n^{-1}(z) = -(-z)^{\frac{\theta_n}{\gamma_n}}$, we have

$$\begin{aligned}
U_n^S(a_{f,s}^S, a_n^S) &= \phi_n^{-1}\left(E_{M'}\left[\phi_n\left(E_{R'|M'}\left[u_n\left(W_n^2(a_{f,s}^S, a_n^S)\right)\right]\right)\right]\right) \\
&= -exp\Big(-\theta_n\Big[\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f + (a_n^S)^\top \mu_S \\
&\qquad - \frac{\theta_n}{2}(a_n^S)^\top \Sigma a_n^S - \frac{\gamma_n}{2}(a_n^S)^\top \Sigma_S a_n^S\Big]\Big).
\end{aligned}$$

Plugging in a_n^S and p^S from (39) and (41), and using the fact that $\mu_S = \left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1}\left(\Sigma_M^{-1}\mu + \Sigma^{-1}S\right)$, we obtain:

$$U_n^S(a_{f,s}^{*,S}, a_n^{*,S}) = -exp\Big(-\theta_n\Big[\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + (q_n^0)^\top (A + BS) R_f + \frac{1}{\theta_n}(A_n + B_n S)^\top (\mu_S - 1 R_f)$$

$$
\begin{aligned}
& - \frac{1}{2\theta_n}(A_n + B_n S)^\top \Sigma (A_n + B_n S) - \frac{\gamma_n}{2\theta_n^2}(A_n + B_n S)^\top \Sigma_S (A_n + B_n S)\Big]\Big) \\
= \quad & -\Gamma_n \; exp\left(S^\top C_n S + d_n^\top S + e_n \right),
\end{aligned}
$$

where $\Gamma_n > 0$ does not depend on q_n^0, S or M, and

$$
\begin{aligned}
C_n &= \frac{1}{2}B_n^\top \Sigma B_n + \frac{1}{2}(\eta_n + 1)B_n^\top \Sigma_S B_n - \left(\left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1} \Sigma^{-1} \right)^\top \left(\Sigma + (\eta_n + 1)\Sigma_S\right)^{-1} \left(\Sigma_M^{-1} + \Sigma^{-1}\right)^{-1} \Sigma^{-1}, \\
d_n^\top &= -\theta_n (q_n^0)^\top B R_f - A_n^\top \Sigma_S \Sigma^{-1} - \left(\Sigma_S \Sigma_M^{-1} \mu - 1 R_f\right)^\top B_n + A_n^\top \Sigma B_n + (\eta_n + 1) A_n^\top \Sigma_S B_n, \\
e_n &= -\theta_n (q_n^0)^\top A R_f + \theta_n (q_n^0)^\top p^0 R_f^2.
\end{aligned}
$$

Note, we rely on the fact that B_n is symmetric and on its expression given in Lemma C.1 in the derivation of C_n. Next, letting $Y \equiv S - M$ yields

$$
U_n^S(a_{f,s}^{*,S}, a_n^{*,S}) = -\Gamma_n \; exp\left(Y^\top C_n Y + (d_n^\top + 2M^\top C_n)Y + e_n + M^\top C_n M + {d_n}^\top M \right).
$$

We now introduce a well-known result about multivariate normal distributions (e.g., see Anderson (1984), Ch. 2 or Brunnermeier (2001), p. 64):

Mathematical Preliminary. *Let* $\omega \sim N\left(0, \bar{\Sigma}\right)$. *Then,*

$$
E\left[\exp\left(\omega^\top \bar{A} \omega + \bar{b}^\top \omega + \bar{c}\right)\right] = |I - 2\bar{\Sigma}\bar{A}|^{-1/2} \exp\left(\frac{1}{2}\bar{b}^\top \left(I - 2\bar{\Sigma}\bar{A}\right)^{-1} \bar{\Sigma}\bar{b} + \bar{c} \right)
$$

where $\bar{A}$ *is a symmetric matrix,* $\bar{b}$ *a vector and* $\bar{c}$ *a scalar.*

Note that $Y|M \sim N\left(0, \Sigma\right)$. Also, observe that C_n is a symmetric matrix. Therefore, using the Mathematical Preliminary, we get the following:

$$
\begin{aligned}
& E_{S|M}\left[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\right] = \\
& = -\Gamma_n \; E_{Y|M}\left[exp\left(Y_n^\top C_n Y + ({d_n}^\top + 2M^\top C_n)Y + e_n + M^\top C_n M + {d_n}^\top M \right) \right] \\
& = -\Gamma_n \; |I - 2\Sigma C_n|^{-1/2} \times \\
& \qquad exp\Big(\frac{1}{2}({d_n}^\top + 2M^\top C_n)\left(I - 2\Sigma C_n\right)^{-1} \Sigma \left({d_n}^\top + 2M^\top C_n\right)^\top + e_n + M^\top C_n M + {d_n}^\top M \Big) \\
& = -\Gamma_n \; |I - 2\Sigma C_n|^{-1/2} \; exp\Big(M^\top D_n M + f_n^\top M + g_n \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
D_n &= 2\left(C_n^{-1}\Sigma^{-1}C_n^{-1} - 2C_n^{-1}\right)^{-1} + C_n, \\
f_n^\top &= 2d_n^\top \left(I - 2\Sigma C_n\right)^{-1} \Sigma C_n + d_n^\top, \\
g_n &= \frac{1}{2}{d_n}^\top \left(I - 2\Sigma C_n\right)^{-1} \Sigma d_n + e_n.
\end{aligned}
$$

Note, D_n is symmetric.

Making use of the Mathematical Preliminary once again, we get:

$$
\begin{aligned}
&\phi_n\left(U_n^0\left(a_{f,n}^0, a_n^0\right)\right) = \\
&= E_M\left[\phi_n\left(E_{S|M}\left[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\right]\right)\right] \\
&= -\Gamma_n^{\gamma_n/\theta_n}\ |I-2\Sigma C_n|^{-\gamma_n/2\theta_n}\ E_M\left[exp\left(\frac{\gamma_n}{\theta_n}\left[M^\top D_n M + f_n^\top M + g_n\right]\right)\right] \\
&= -\Gamma_n^{\gamma_n/\theta_n}\ |I-2\Sigma C_n|^{-\gamma_n/2\theta_n} \times \\
&\quad E_M\left[exp\left(\frac{\gamma_n}{\theta_n}\left[(M-\mu)^\top D_n(M-\mu) + (f_n^\top + 2\mu^\top D_n)(M-\mu) + g_n + f_n^\top\mu + \mu^\top D_n\mu\right]\right)\right] \\
&= -\Gamma_n^{\gamma_n/\theta_n}\ |I-2\Sigma C_n|^{-\gamma_n/2\theta_n}\ |I-2\Sigma_M\frac{\gamma_n}{\theta_n}D_n|^{-1/2} \times \\
&\quad exp\left(\frac{1}{2}\frac{\gamma_n^2}{\theta_n^2}(f_n^\top + 2\mu^\top D_n)(I-2\Sigma_M\frac{\gamma_n}{\theta_n}D_n)^{-1}\Sigma_M(f_n^\top + 2\mu^\top D_n)^\top + \frac{\gamma_n}{\theta_n}(g_n + f_n^\top\mu + \mu^\top D_n\mu)\right).
\end{aligned}
$$

Since agent n maximizes over q_n^0 (which is equivalent to maximizing over a_n^0 given prices), we can focus only on elements containing q_n^0 in the objective function. Therefore,

$$
\begin{aligned}
&argmax_{q_n^0}\ E_M\left[\phi_n\left(E_S\left[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\right]\right)\right] = \\
&argmax_{q_n^0}\ \left\{-\frac{1}{2}\frac{\gamma_n}{\theta_n}(f_n^\top + 2\mu^\top D_n)E_n(f_n^\top + 2\mu^\top D_n)^\top - g_n - f_n^\top\mu\right\}
\end{aligned}
$$

where $E_n = \left(I - 2\Sigma_M\frac{\gamma_n}{\theta_n}D_n\right)^{-1}\Sigma_M = \left(\Sigma_M^{-1} - 2(\eta_n+1)D_n\right)^{-1}$, which depends only on η_n and not on θ_n. Next, let

$$
\begin{aligned}
F_n &= (I-2\Sigma C_n)^{-1}\Sigma, \\
G_n &= -A_n^\top\Sigma_S\Sigma^{-1} - \left(\Sigma_S\Sigma_M^{-1}\mu - 1R_f\right)^\top B_n + A_n^\top\Sigma B_n + (\eta_n+1)A_n^\top\Sigma_S B_n.
\end{aligned}
$$

Note, both F_n and G_n depend only on η_n and not on θ_n. Observe that

$$
\begin{aligned}
&argmax_{q_n^0}\ E_M\left[\phi_n E_S\left[U_n^S(a_{f,s}^{*,S}, a_n^{*,S})\right]\right] = \\
&argmax_{q_n^0}\ \Bigg\{ -\frac{1}{2}\gamma_n\theta_n(R_f)^2(q_n^0)^\top B(2F_nC_n+I)E_n(2F_nC_n+I)^\top B^\top q_n^0 \\
&\qquad + 2\gamma_n R_f(q_n^0)^\top B(2F_nC_n+I)E_n\mu_n^\top D_n + \theta_n R_f(q_n^0)^\top BF_nG_n \\
&\qquad - \frac{1}{2}\theta_n^2R_f^2(q_n^0)^\top BF_nB^\top q_n^0 + \theta_n(q_n^0)^\top(p^0R_f^2 - AR_f) + \theta_nR_f(q_n^0)^\top B(2F_nC_n+I)\mu\Bigg\}.
\end{aligned}
$$

The first order condition for the above maximization problem yields:

$$
\begin{aligned}
0 \;=\; & -\gamma_n\theta_nR_fB(2F_nC_n+I)E_n(2F_nC_n+I)^\top B^Tq_n^0 + 2\gamma_nB(2F_nC_n+I)E_n\mu^\top D_n + \theta_nBF_nG_n \\
& -\theta_n^2R_fBF_nB^\top q_n^0 + \theta_n(p^0R_f - A) + \theta_nB(2F_nC_n+I)\mu.
\end{aligned}
$$

This implies that

$$
q_n^0 = \frac{1}{\theta_nR_f}\ Q(\eta_n, p^0), \tag{42}
$$

where

$$\begin{aligned} Q(\eta_n, p^0) &= \Big((\eta_n+1)B(2F_nC_n+I)E_n(2F_nC_n+I)^\top B^\top + BF_nB^\top\Big)^{-1} \times \\ &\quad \Big(2(\eta_n+1)B(2F_nC_n+I)E_n\mu^\top D_n + BF_nG_n - A + B(2F_nC_n+I)\mu + p^0R_f\Big). \end{aligned} \tag{43}$$

Observe that the vector $Q(\eta_n, p^0)$ depends neither on θ_n nor on S. Taking the ratio of ex ante equilibrium asset i holdings of agents k and n given by (42), we get the desired result. $\square$

Lemma C.4. *At ex ante equilibrium, for any $i \in \{1, 2\}$ and $n \in \{1, \cdots, N\}$, agent n's holding of asset i is*

$$q^0_{i,n} = \begin{cases} \frac{\frac{1}{\theta_n}}{\sum_n \frac{1}{\theta_n}} \times e & : \ \eta_n = \eta_k \quad \forall k \in \{1, \cdots, N\} \\ \frac{1}{\theta_n R_f} \times (Q^\star(\eta_n))_i & : \ \textit{otherwise}, \end{cases}$$

where $Q^\star(\eta_n)$ is a vector whose expression is as given by (45) in the proof below.

Proof of Lemma C.4. Re-write the vector $Q(\eta_n, p^0)$ given by 43) as

$$Q(\eta_n, p^0) = (Y_n)^{-1}(Z_n + p^0R_f), \tag{44}$$

where

$$\begin{aligned} Y_n &= (\eta_n+1)B(2F_nC_n+I)E_n(2F_nC_n+I)^\top B^\top + BF_nB^\top, \\ Z_n &= 2(\eta_n+1)B(2F_nC_n+I)E_n\mu^\top D_n + BF_nG_n - A + B(2F_nC_n+I)\mu. \end{aligned}$$

The market clearing condition implies that $\sum_k \frac{1}{\theta_k}(Y_k)^{-1}(Z_k + p^0R_f) \ = \ R_f\ e$. Hence, at an ex ante equilibrium, it holds that

$$p^0\ R_f \ = \ \left(\sum_k \frac{1}{\theta_k}(Y_k)^{-1}\right)^{-1}\left(R_f\ e - \sum_k \frac{1}{\theta_k}(Y_k)^{-1}Z_k\right).$$

Plugging this back into (44) yields

$$Q^\star(\eta_n) = (Y_n)^{-1}\left[Z_n + \left(\sum_k \frac{1}{\theta_k}(Y_k)^{-1}\right)^{-1}\left(R_f\ e - \sum_k \frac{1}{\theta_k}(Y_k)^{-1}Z_k\right)\right]. \tag{45}$$

Hence, it follows from (42) that

$$q^0_{i,n} \ = \ \frac{1}{\theta_n R_f} \times (Q^\star(\eta_n))_i\,. \tag{46}$$

Under homogeneity of ambiguity aversion, we know from Lemma C.3 that $q^0_{i,k} = q^0_{i,1}\ \frac{\theta_1}{\theta_k}$ for all $k \in \{1, \cdots, N\}$. Therefore, for any $i = 1, 2$, it follows from the market clearing condition that $q^0_{i,1}\ \theta_1 \sum_k \frac{1}{\theta_k} \ = e$. This implies that $q^0_{i,1} \ = \ \frac{e}{\theta_1}\ \frac{1}{\sum_k \frac{1}{\theta_k}}$, which in turn implies

$$q^0_{i,n} \ = \ \frac{\frac{1}{\theta_n}}{\sum_k \frac{1}{\theta_k}} \times e \tag{47}$$

under homogeneous ambiguity aversion. (46) and (47) together yield the desired result. □

Proof of Proposition 4. Under homogeneity of ambiguity aversion, Lemmas C.2 and C.4 establish that agents' asset holdings are the same ex ante and interim regardless of the signal realization. Therefore, the equilibrium entails no trade for any signal realization under homogeneous ambiguity aversion.

Under heterogeneous ambiguity aversion, observe from Lemmas C.2 and C.4 that the ratio of interim asset holdings, $\frac{q^S_{1,n}}{q^S_{2,n}}$, depends on the realization of S, whereas the ratio of ex ante asset holdings, $\frac{q^0_{1,n}}{q^0_{2,n}}$, does not. Therefore, for almost all signal realizations, the equilibrium entails non-trivial trading with heterogeneous ambiguity aversion. □

Proof of Proposition 5. We know that, upon observing signal S, the interim period equilibrium asset price vector is given by $p^S = \frac{\sum_n \frac{1}{\theta_n}(A_n+B_nS)}{e} \equiv A + BS$. Since B, a symmetric matrix, is generically invertible, there generically exists a signal realization, call it $\check{S}$, such that the interim period equilibrium asset price vector, p^S, is equal to the initial period equilibrium asset price vector, p^0: $\check{S} = B^{-1}\left(p^0 - A\right)$.

It follows from Lemma C.1 that

$$q^{\check{S}}_{i,n} = \frac{a^{\check{S}}_{i,n}}{p^{\check{S}}_i} = \frac{\frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_i}{\sum_n \frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_i} \times e, \qquad i = 1, 2,$$

and therefore

$$\frac{q^{\check{S}}_{1,n}}{q^{\check{S}}_{2,n}} \ = \ \frac{\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_1}{\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_2} \times \frac{\sum_n \frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_2}{\sum_n \frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_1}. \tag{48}$$

Also, from (42), we know that

$$\frac{q^0_{1,n}}{q^0_{2,n}} \ = \ \frac{(Q(\eta_n, p^0))_1}{(Q(\eta_n, p^0))_2}, \tag{49}$$

where $Q(\eta_n, p^0)$ is as defined in (43).

Let

$$\begin{aligned} f\left(\{\theta_k\}_{k=1}^N, \eta_n, p^0\right) \ &\equiv \ \frac{\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_1}{\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_2} \times \frac{\sum_n \frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_2}{\sum_n \frac{1}{\theta_n}\left(A_n + B_nB^{-1}\left(p^0 - A\right)\right)_1}, \\ g(\eta_n, p^0) \ &\equiv \ \frac{(Q(\eta_n, p^0))_1}{(Q(\eta_n, p^0))_2}. \end{aligned}$$

Following (48) and (49), the equilibrium does not entail non-trivial trading at $\check{S}$ if and only if

$$f\left(\{\theta_k\}_{k=1}^N, \eta_n, p^0\right) \;=\; g(\eta_n, p^0) \quad \text{for all } n. \tag{50}$$

Note that f generically depends on $\{\theta_k\}_{k=1}^N$ while g does not, therefore

$$F((\{\theta_k\}_{k=1}^N, \eta_n, \cdot)) \;\equiv\; f\left(\{\theta_k\}_{k=1}^N, \eta_n, \cdot\right) - g(\eta_n, \cdot)$$

is a non-degenerate function of $(\{\theta_k\})_{k=1}^N$ and η_n. Let $p\left(\{\theta_k\}_{k=1}^N, \eta_n\right)$ solve

$$F\left(\{\theta_k\}_{k=1}^N, \eta_n, \cdot\right) \;=\; 0.$$

Then $p((\{\theta_k\}_{k=1}^N, \eta_n)$ generically depends on η_n, implying that $p\left(\{\theta_k\}_{k=1}^N, \eta_n\right) \neq p\left(\{\theta_k\}_{k=1}^N, \eta_{n'}\right)$ if $\eta_n \neq \eta_{n'}$. This, in turn, implies that if $f\left(\{\theta_k\}_{k=1}^N, \eta_n, p^0\right) = g(\eta_n, p^0)$ then $f\left(\{\theta_k\}_{k=1}^N, \eta_{n'}, p^0\right) \neq g(\eta_{n'}, p^0)$ for $n \neq n'$. This violates the condition given in (50) and therefore we have the desired result. $\square$

Lemma C.5. *At ex ante equilibrium, for any $i \in \{1,2\}$ and $n \in \{1,2\}$, agent n's holding of asset i is*

$$q_{i,n}^0 \;=\; \frac{\frac{1}{\theta_n}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \times e.$$

Proof of Lemma C.5. Given monetary holdings $(a_{f,n}^S, a_n^S)$, we have

$$\begin{aligned}
&E_{R'|M'}\left[u_n(W_n^2(a_{f,n}^S, a_n^S))\right] = \\
&= E_{R'|M'}[-exp(-\theta_n W_n^2(a_{f,n}^S, a_n^S))] \\
&= -exp\left(-\theta_n\left[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\right]\right)\; E_{R'|M'}\left[exp\left(-\theta_n (a_n^S)^\top R\right)\right] \\
&= -exp\left(-\theta_n\left[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\right]\right) \times \\
&\qquad exp\left(-\theta_n (a_n^S)^\top E_{R'|M'}[R] + \frac{1}{2}\theta_n^2 (a_n^S)^\top var_{R'|M'}(R) a_n^S\right) \\
&= -exp\left(-\theta_n\left[(W_n^0 - (q_n^0)^\top p^0)R_f^2 + ((q_n^0)^\top p^S - (a_n^S)^\top 1)R_f\right]\right)\; exp\left(-\theta_n (a_n^S)^\top \mu + \frac{1}{2}\theta_n^2 (a_n^S)^\top \Sigma a_n^S\right).
\end{aligned}$$

Then,

$$\begin{aligned}
&E_{M'}\left(\phi_n\left[E_{R'|M'}\left[u_n(W_n^2(a_{f,n}^S, a_n^S))\right]\right]\right) = \\
&= -exp\Big(-\gamma_n\Big\{\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f\Big\}\Big) exp\left(-\gamma_n (a_n^S)^\top \mu + \frac{\gamma_n \theta_n}{2}(a_n^S)^\top \bar{\Sigma} a_n^S\right),
\end{aligned}$$

which in turn yields

$$\begin{aligned}
&U_n^S(a_{f,n}^S, a_n^S) \;=\; \phi_n^{-1}\left(E_{M'}\left(\phi_n\left[E_{R'|M'}\left[u_n(W_n^2(a_{f,n}^S, a_n^S))\right]\right]\right)\right) \;= \\
&= -exp\Big(-\theta_n\Big\{\left(W_n^0 - (q_n^0)^\top p^0\right) R_f^2 + \left((q_n^0)^\top p^S - (a_n^S)^\top 1\right) R_f + (a_n^S)^\top \mu - \frac{1}{2}\gamma_n (a_n^S)^\top \Sigma a_n^S\Big\}\Big)
\end{aligned}$$

$$= K^S\ exp\left(-\theta_n\left\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\right\}\right), \tag{51}$$

where

$$K^S = -exp\left(-\theta_n\left\{\left(W_n^0 R_f - (a_n^S)^\top 1\right) R_f + (a_n^S)^\top \mu - \frac{1}{2}\gamma_n (a_n^S)^\top \Sigma a_n^S\right\}\right).$$

Note, K^S does not depend on q_n^0. Following (51), agent n's maximization problem reduces to:

$$\max_{q_n^0} \sum_{S\in\{H,I,L\}} \pi(S)\ K^S\ exp\left(-\theta_n\left\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\right\}\right).$$

The first order condition of this problem is:

$$\begin{aligned}
0 \;=\;& \pi(H)\ K^H\ (p_i^H - p_i^0 R_f) R_f\ exp\left(-\theta_n\left\{(q_n^0)^\top p^H R_f - (q_n^0)^\top p^0 R_f^2\right\}\right) \\
&+ \pi(I)\ K^I\ (p_i^I - p_i^0 R_f) R_f\ exp\left(-\theta_n\left\{(q_n^0)^\top p^I R_f - (q_n^0)^\top p^0 R_f^2\right\}\right) \\
&+ \pi(L)\ K^L\ (p_i^L - p_i^0 R_f) R_f\ exp\left(-\theta_n\left\{(q_n^0)^\top p^L R_f - (q_n^0)^\top p^0 R_f^2\right\}\right), \quad i = 1, 2.
\end{aligned}$$

Let

$$A^S \;=\; exp\left(-\theta_n\left\{(q_n^0)^\top p^S R_f - (q_n^0)^\top p^0 R_f^2\right\}\right), \quad S = H, I, L. \tag{52}$$

Then, the first order condition above can be re-written as

$$0 \;=\; \pi(H) K^H (p_i^H - p_i^0 R_f) R_f A^H + \pi(I) K^I (p_i^I - p_i^0 R_f) R_f A^I + \pi(L) K^L (p_i^L - p_i^0 R_f) R_f A^L, \quad i = 1, 2,$$

which implies

$$\frac{A^H}{A^I} \;=\; \frac{K^I \pi(I)\left(R_f p_2^0 (p_1^L - p_1^I) + p_2^L p_1^I - p_1^L p_2^I + R_f p_1^0\left(p_2^I - p_2^L\right)\right)}{K^H \pi(H)\left(R_f p_2^0 (p_1^H - p_1^L) + p_2^H p_1^L - p_1^H p_2^L + R_f p_1^0\left(p_2^L - p_2^H\right)\right)}, \tag{53}$$

$$\frac{A^L}{A^I} \;=\; \frac{K^I \pi(I)\left(R_f p_2^0 (p_1^I - p_1^H) + p_1^H p_2^I - p_2^H p_1^I + R_f p_1^0\left(p_2^H - p_2^I\right)\right)}{K^L \pi(L)\left(R_f p_2^0 (p_1^H - p_1^L) + p_2^H p_1^L - p_1^H p_2^L + R_f p_1^0\left(p_2^L - p_2^H\right)\right)}. \tag{54}$$

Also, it follows from (52) that

$$\begin{aligned}
\frac{A^H}{A^I} \;&=\; exp\left(-\theta_n\left[q_{1,n}^0 (p_1^H - p_1^I) + q_{2,n}^0 (p_2^H - p_2^I) R_f\right]\right), \\
\frac{A^L}{A^I} \;&=\; exp\left(-\theta_n\left[q_{1,n}^0 (p_1^L - p_1^I) + q_{2,n}^0 (p_2^L - p_2^I) R_f\right]\right).
\end{aligned}$$

We have two equations with two unknowns, namely $q_{1,n}^0$ and $q_{2,n}^0$, above. Solving for the unknowns, we derive:

$$q_{1,n}^0 \;=\; \frac{\log\left(\frac{A^H}{A^I}\right)(p_2^L - p_2^I) + \log\left(\frac{A^L}{A^I}\right)(p_2^I - p_2^H)}{\theta_n R_f\left[p_2^H (p_1^L - p_1^I) + p_2^L (p_1^I - p_1^H) + p_2^I (p_1^H - p_1^L)\right]}, \tag{55}$$

$$q_{2,n}^0 \;=\; \frac{\log\left(\frac{A^H}{A^I}\right)(p_1^I - p_1^L) + \log\left(\frac{A^L}{A^I}\right)(p_1^H - p_1^L)}{\theta_n R_f\left[p_2^H (p_1^L - p_1^I) + p_2^L (p_1^I - p_1^H) + p_2^I (p_1^H - p_1^L)\right]}, \tag{56}$$

where $\frac{A^H}{A^I}$ and $\frac{A^L}{A^I}$ are as given in (53) and (54), i.e., in terms of prices (not quantities).

Note, (55) and (56) give us ex ante equilibrium quantities $q^0_{1,n}$ and $q^0_{2,n}$ in terms of ex ante equilibrium prices p^0_1 and p^0_2 and interim equilibrium prices p^S_1 and p^S_2. Next, we solve for ex ante equilibrium prices p^0_1 and p^0_2 using the market clearing condition

$$q^0_{i,1} + q^0_{i,2} = e, \quad i = 1, 2.$$

This yields:

$$p^0_i \;=\; \frac{D\ K^H\ \pi(H)\ p^H_i\ +\ E\ K^L\ \pi(L)\ p^L_i\ +\ K^I\ \pi(I)\ p^I_i}{R_f\,(D\ K^H\ \pi(H)\ +\ E\ K^L\ \pi(L)\ +\ K^I\ \pi(I))}, \quad i = 1, 2, \tag{57}$$

where

$$\begin{aligned} D &= exp\left(\frac{\operatorname{erf}\left(p^I_1 + p^I_2 - p^H_1 - p^H_2\right)\theta_1\theta_2}{\theta_1 + \theta_2}\right), \\ E &= exp\left(\frac{\operatorname{erf}\left(p^I_1 + p^I_2 - p^L_1 - p^L_2\right)\theta_1\theta_2}{\theta_1 + \theta_2}\right). \end{aligned}$$

In the above equations, $\operatorname{erf}(z)$ is the error function encountered in integrating the normal distribution, defined by

$$\operatorname{erf}(z) \;\equiv\; \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt.$$

Next, plugging (57) into (53)-(54) and then using the derived expressions for $\frac{A^H}{A^I}$ and $\frac{A^L}{A^I}$ in (55)-(56), we get the desired result:[33]

$$q^0_{i,n} \;=\; \frac{\theta_2}{\theta_1 + \theta_2} \times e \;=\; \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \times e, \quad i = 1, 2. \quad \square$$

Proof of Proposition 6. If there is homogeneous ambiguity aversion so that $\eta_1 = \eta_2$, then the interim equilibrium asset holdings given by (22) reduce to

$$q^S_{i,n} \;=\; \frac{\theta_2}{\theta_1 + \theta_2} \times e \;=\; \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \times e, \quad i = 1, 2,$$

for $S = H, I, L$, which are equal to the ex ante equilibrium asset holdings as derived in Lemma C.5. Hence, the equilibrium entails no trade for any signal realization under homogeneous ambiguity aversion.

Under heterogeneous ambiguity aversion, it follows from (22) that the ratio of interim asset holdings, $\frac{q^S_{1,n}}{q^S_{2,n}} = \frac{a^{*,S}_{1,n}}{a^{*,S}_{2,n}}$, depends on the signal realization S and is thus generically different from the ratio of ex ante asset holdings, $\frac{q^0_{1,n}}{q^0_{2,n}}$ (which does not depend on S as can be seen from Lemma C.5). $\square$

[33] We used Mathematica to carry out the algebraic simplifications and derive the result.

Proof of Proposition 7. To prove the result, it suffices to show that

$$\frac{\partial\left(\frac{a^{\star,S}_{1,n}}{a^{\star,S}_{2,n}}-\frac{a^{\star,S}_{1,n'}}{a^{\star,S}_{2,n'}}\right)}{\partial(\bar{\sigma}^S_1)^2}>0$$

since $\bar{\sigma}^S_1$ is the only model parameter that varies with S. Let $\sigma=\sigma_1=\sigma_2$. Under the assumption of the proposition,

$$\frac{a^{\star,S}_{1,n}}{a^{\star,S}_{2,n}} = \frac{(\sigma^2+(1+\eta_n)\bar{\sigma}^2_2)-(\sigma_{12}+(1+\eta_n)\bar{\sigma}_{12})}{(\sigma^2+(1+\eta_n)(\bar{\sigma}^S_1)^2)-(\sigma_{12}+(1+\eta_n)\bar{\sigma}_{12})}.$$

Thus,

$$\frac{a^{\star,S}_{1,n}}{a^{\star,S}_{2,n}}-\frac{a^{\star,S}_{1,n'}}{a^{\star,S}_{2,n'}} = \frac{(\eta_{n'}-\eta_n)(\sigma^2-\sigma_{12})((\bar{\sigma}^S_1)^2-\bar{\sigma}^2_2)}{(\sigma^2-\sigma_{12})^2+(2+\eta_n+\eta_{n'})(\sigma^2-\sigma_{12})\left((\bar{\sigma}^S_1)^2-\bar{\sigma}_{12}\right)+(1+\eta_n)(1+\eta_{n'})\left((\bar{\sigma}^S_1)^2-\bar{\sigma}_{12}\right)^2}.$$

Note that

$$\frac{a^{\star,S}_{1,n}}{a^{\star,S}_{2,n}}-\frac{a^{\star,S}_{1,n'}}{a^{\star,S}_{2,n'}}>0$$

for all S, because $\eta_{n'}>\eta_n$, $\sigma^2>\sigma_{12}$ and $\bar{\sigma}^S_1>\bar{\sigma}_2$ for all S. Taking the derivative of $\frac{a^{\star,S}_{1,n}}{a^{\star,S}_{2,n}}-\frac{a^{\star,S}_{1,n'}}{a^{\star,S}_{2,n'}}$ with respect to $(\bar{\sigma}^S_1)^2$ yields:

$$\frac{(\eta_{n'}-\eta_n)(\sigma^2-\sigma_{12})}{\left[(\sigma^2-\sigma_{12})^2+(2+\eta_n+\eta_{n'})(\sigma^2-\sigma_{12})\left((\bar{\sigma}^S_1)^2-\bar{\sigma}_{12}\right)+(1+\eta_n)(1+\eta_{n'})\left((\bar{\sigma}^S_1)^2-\bar{\sigma}_{12}\right)^2\right]^2}\times$$
$$\left\{(\sigma^2-\sigma_{12})^2+(2+\eta_n+\eta_{n'})(\sigma^2-\sigma_{12})((\bar{\sigma}_2)^2-\bar{\sigma}_{12})+(1+\eta_n)(1+\eta_{n'})((\bar{\sigma}^S_1)^2-\bar{\sigma}_{12})(2(\bar{\sigma}_2)^2-\bar{\sigma}_{12}-(\bar{\sigma}^S_1)^2)\right\}.$$

Given that $\eta_{n'}>\eta_n$ and $\sigma^2>\sigma_{12}$, the derivative is positive if the term in the curly brackets is positive. The latter is satisfied if $(\bar{\sigma}_1)^2<\Lambda$ where

$$\Lambda = (\bar{\sigma}_2)^2+2\sqrt{(\bar{\sigma}_2)^4+\frac{(\sigma^2-\sigma_{12})^2+(2+\eta_n+\eta_{n'})(\sigma^2-\sigma_{12})((\bar{\sigma}_2)^2-\bar{\sigma}_{12})+(1+\eta_n)(1+\eta_{n'})\bar{\sigma}_{12}(\bar{\sigma}_{12}-2(\bar{\sigma}_2)^2)}{(1+\eta_n)(1+\eta_{n'})}}. \tag{58}$$

Hence, we have the desired result. □

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