

Estimating Network Externalities in Undirected Link Formation Games

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Abstract

This paper explores the existence of network externalities in link formation games of incomplete information. It extends the structural estimation method in [Leung \(2015\)](#) to games where links are undirected and proposals are only partially observed. We provide an econometric characterization of the proposed two-step estimator and document its performance through a simulation exercise. When the estimation method is applied to data on risk-sharing arrangements in a Tanzanian village, results indicate that indirect connections matter.

Keywords: Undirected networks; Network externalities; Incomplete information; Risk-sharing

JEL codes: C45; D85; O12

1 Introduction

From its very first steps, network theory has claimed that the formation of links may depend strategically on the entire graph (Jackson and Wolinsky, 1996; Bala and Goyal, 2000). However, evidence based on experimental and observational data still lags behind, and empirical questions about the value of indirect connections remain largely unexplored.¹ Building on Leung (2015), we design an estimation protocol for network formation games where links are undirected and proposals are only partially observed. This procedure accommodates undirected links formed by bilateral or unilateral link formation rules. In our setting, agents play a simultaneous game of incomplete information where they form undirected links on the basis of their beliefs about the emerging network architecture. Assuming these beliefs satisfy a number of regularity conditions (discussed in Section 2), the estimation strategy boils down to a two-step procedure where the first stage estimates network statistics and the second stage estimates their role in agents' linking decisions.² We provide existence, consistency, and asymptotic normality results for the two-step estimator and conduct a set of simulation exercises to investigate its performance as sample size grows.

To illustrate the procedure, we then apply it to data on risk-sharing arrangements from the Tanzanian village of Nyakatoke. Lacking access to formal insurance, most households in developing countries rely on informal risk-sharing arrangements to cope with shocks such as health-related expenses, injuries, funerals and job losses. These arrangements have long captured the attention of economists, for several reasons. On the one hand, the prevalence of the phenomenon makes it of paramount importance for economic development.³ On the other hand, most arrangements do not take place at the level of the entire community but

¹Most of the available evidence relates to specific settings. For instance, the study of cross-firm collaborative networks suggests that information flows are insignificant for indirect neighbors (Breschi and Lissoni, 2005; Singh, 2005). On the other hand, experimental evidence with dictator games shows that further-away connections are relevant and decay with the inverse of distance (Goeree, McConnell, Mitchell, Tromp, and Yariv, 2010). Graham and Pelican (2019) provide a test for inter-dependencies in link-formation preferences and conclude for the presence of externalities in the same data we use here.

²A two-step approach is also taken by König, Liu, and Zenou (2019).

³Coate and Ravallion (1993), Townsend (1994), Udry (1994), Fafchamps and Lund (2003).

rather among pairs of households, which makes risk-sharing a compelling application of network theory for economists.⁴

We use the self-declared information in Nyakatoke data to draw the undirected village network and to investigate the role of network architecture. Specifically, we test whether agents choose between risk-sharing partners based solely on their individual characteristics, or whether the pattern of indirect connections also plays a role in these decisions. Results indicate that Nyakatoke villagers evaluate potential partners' connections in a positive manner. Assuming a bilateral link formation process, our estimates suggest that, for a given pair of potential partners ij , a one-percentage-point increase in the expected number of j 's friends raises the probability that i proposes a link to j by five percentage points.

From an econometric standpoint, testing whether network architecture predicts link formation has proved to be a complex task. Our paper deals with the case where the researcher observes a single network in a single period and wants to include network covariates in the agents' objective function. In this scenario, the structural equation can have multiple solutions (Bjorn and Vuong, 1984; Bresnahan and Reiss, 1991; Tamer, 2003), and the calculation may become intractable due to the combinatorial complexity of networks. One solution is provided by exponential random graph models in which a dynamic meeting protocol acts as an equilibrium selection mechanism (Hsieh and Lee, 2016; König, 2016; Mele, 2017; Badev, 2021). Another solution is to condition on models that replicate some observed topological patterns or to limit the degree to which other players can affect one's utility.⁵ Alternatively, one can simplify the estimation procedure by relying on incomplete information to induce symmetry and independence in agents strategies (Leung, 2015; De Paula and Tang, 2012),

⁴Risk-sharing networks have been studied from multiple angles, including the efficiency and sustainability of the resulting arrangements, the determinants of link formation and the structural properties of the network architecture (Genicot and Ray, 2003; Bramoullé and Kranton, 2007; Bloch, Genicot, and Ray, 2008; Jackson, Rodriguez-Barraquer, and Tan, 2012; Banerjee, Chandrasekhar, Duflo, and Jackson, 2013; Ambrus, Mobius, and Szeidl, 2014; Ambrus and Elliott, 2021).

⁵One can identify structural parameters by aggregating individuals into 'types' and assuming that agents have preferences only over the type of their partners (De Paula, Richards-Shubik, and Tamer, 2018), or by the rate at which various sub-graphs are observed in the overall network (Chandrasekhar and Jackson, 2025). Along similar lines, Boucher and Mourifie (2017) study a setting where individual preferences display weak homophily.

which is the approach we take here.

This paper’s main contribution is methodological: it develops a protocol to estimate network externalities in undirected link formation games of incomplete information. The suggested protocol relies heavily on [Leung \(2015\)](#), but differs from it in one substantive aspect: while [Leung \(2015\)](#)’s protocol requires data on directed links, which are interpreted as observed proposals in a game of unilateral link formation, our protocol is designed for undirected link data, which we interpret as the equilibrium outcome of a link formation process where proposals are only partially observed.

Extending [Leung \(2015\)](#)’s approach to undirected networks involves several challenges. First, while in directed networks “link proposals” are indistinguishable from “links”, in undirected networks one is required to specify how the first maps to the second – e.g. by a “bilateral rule”, where an undirected link forms if and only if *both* sides propose to one another, or by a “unilateral rule”, where an undirected link exists if and only if *at least one side* proposes to the other. We take the bilateral approach in the main text and discuss the unilateral approach in [Appendix B](#). Second, in undirected network games, observed links convey only partial information about the unobserved proposals of the two decision-makers. As a result, the corresponding log-likelihood takes a more complex form, reminiscent of the partial observability probit model proposed by [Poirier \(1980\)](#).⁶ Third, when undirected links are formed bilaterally, the solution concept used in [Leung \(2015\)](#) (Bayes Nash Equilibrium, henceforth BNE) gives rise to coordination failures where mutually beneficial links fail to form because neither side believes the other will extend a proposal. This is an issue because the bilateral formation process is meant to reflect the idea that pairs of agents are free to coordinate their actions. We propose to overcome it by restricting attention to the set of *admissible* BNE. Lastly, [Leung \(2015\)](#)’s “separability condition”, which we also impose, becomes more stringent under undirected links. In the directed case it requires that agent i ’s marginal utility from directing a link to j be independent of the links i directs to other agents.

⁶For a general discussion of the identification and efficiency challenges inherent in partial observability models, see [Poirier \(1980\)](#).

In the undirected case, in contrast, it requires independence from *all* other links involving i . As an example, this means that separability in undirected networks rules out statistics based on closed triads around i , while such statistics remain admissible in directed networks as long as the closed triads include a single outgoing edge from i (e.g., (ij, jk, ki)). In Subsection 3.2 we discuss the implications of separability in undirected networks and suggest that it can be partially circumvented by constructing separable counterparts to otherwise non-separable statistics.⁷

For a sharp comparison of the directed and undirected protocols, we revisit the empirical illustration from [Leung \(2015\)](#) and analyze it under both protocols ([Appendix B](#)). The analysis shows that applying them to the same data may yield different results. As an additional contribution, our paper also advances the literature in risk-sharing arrangements in developing countries by providing first-hand evidence that indirect connections affect linking choices, while previous literature has focused mostly on documenting the number and characteristics of risk-sharing partners.⁸

Network formation models have proved difficult to estimate in the presence of externalities. Most of the existing tools were developed for directed networks and expect two distinct reports per dyad ([Leung, 2015](#); [Mele, 2017](#); [Badev, 2021](#)). On the other hand, the available models of undirected network formation rely on complete information and achieve set identification ([Miyachi, 2016](#); [Sheng, 2020](#); [De Paula et al., 2018](#)). The procedure we propose is computationally efficient, providing a convenient alternative to complete-information models. As such it can prove useful in a variety of applications where links are undirected for conceptual and/or practical reasons. From a conceptual viewpoint, in many instances it is legitimate to assume that link formation requires the consent of both parties. For example,

⁷[Ridder and Sheng \(2025\)](#) relax separability in the context of directed network games to incorporate spillover effects from friends in common, which entails a more complex econometric framework. They also outline how their approach could be adapted to undirected networks under bilateral link formation, although a full econometric treatment of that case is deferred to future work.

⁸An exception is [Krishnan and Sciubba \(2009\)](#), who identify the common features of all equilibrium configurations in a model with negative network externalities and test these predictions against data on labor exchange arrangements in Ethiopia.

link formation is ‘naturally’ interpreted as bilateral when data represent risk-sharing, trade deals, co-authorship among researchers, communication flows, and industrial executive linkages (Banerjee et al., 2013; Büchel, Ehrlich, Puga, and Viladecans-Marsal, 2020; Lalanne and Seabright, 2022). In these cases the practitioner may want to draw undirected links on the basis of multiple (possibly discordant) survey reports (Section 5.1). From a practical viewpoint, many data sources contain no information on linking intentions and only a single link outcome per pair. This mainly occurs when data originate from administrative sources (rather than individual surveys): for example, communication records retrieved from digital social networks, exchange data from online marketplaces, import-export shipment registries, scientific publication records and patent repositories only report ‘successful’ matches (Gaulier and Zignago, 2010; Hitsch, Hortasu, and Ariely, 2010; Ductor, Fafchamps, Goyal, and Van der Leij, 2014; Bailey, Cao, Kuchler, and Stroebel, 2018). In all these situations, the toolbox developed for directed networks is inadequate, and our estimator provides a useful alternative.

The paper is organized as follows. Section 2 introduces the theoretical setting. Section 3 presents the estimation method. Section 4 describes a simulation exercise. Section 5 applies the estimation method to risk-sharing data from rural Tanzania. Section 6 concludes. Appendix A discusses the smoothing of discrete variables. Appendix B revisits the self-reported dyadic data used in Leung (2015) to compare different models of link formation. All proofs are relegated to Appendix C.

2 The Model

2.1 The game

Let $N = \{1, 2, \dots, n\}$ be a set of agents who play to form an undirected network. For agent i , let $X_i = [X_i^1, \dots, X_i^q]$ be a vector of individual attributes of dimension $[1 \times q]$ and $X = \{X_1, \dots, X_n\}$ denote the set of these vectors.

Assumption 1 (Discrete X). *For every $i \in N$, X_i has finite support and for any x in the support $Pr(X_i = x)$ is bounded away from zero uniformly in n .*

Let $\epsilon_i = [\epsilon_{i,1}, \dots, \epsilon_{i,i-1}, 0, \epsilon_{i,i+1}, \dots, \epsilon_{i,n}]$ be a $[1 \times n]$ vector of shocks of agent i with all other agents (ϵ_{ij} does not necessarily equal ϵ_{ji}), which are stochastically independent from X . ϵ denotes the collection ϵ_i over all $i \in N$.

Assumption 2 (i.i.d. Shocks). *$\{\epsilon_{ij} \mid i, j \in N, i \neq j\}$ are independently drawn from the standard normal distribution.⁹*

Thus, shocks are assumed to be uncorrelated across and within individuals. The set of attributes vectors X is common knowledge, while the shocks are private information, i.e. only i knows ϵ_i .

Agents play a simultaneous-move game of link formation, where everyone announces independently the links they wish to form. The action of agent i is represented by a binary vector of length n , where the j th entry ($j \neq i$) equals 1 if i proposes j to form a link and 0 otherwise.¹⁰ The actions of all agents stacked on top of each other, denoted S , can be interpreted as an adjacency matrix of link proposals:

$$S = \begin{bmatrix} 0 & S_{1,2} & \dots & S_{1,n} \\ S_{2,1} & 0 & \dots & S_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n-1} & 0 \end{bmatrix} \quad (1)$$

These link proposals give rise to a network G . We consider two alternative rules by which G is formed. In the ‘bilateral rule’ an undirected link is formed if and only if both sides propose to one another. Formally, $G_{ij} = G_{ji} = S_{ij} \cdot S_{ji}$. The interpretation is that pairs of agents need bilateral consent in order to form an undirected link between them. In the

⁹The standard normal distribution is chosen here for the sake of convenience, but our results hold for other full-support distributions.

¹⁰Since an agent cannot form a link with herself, the i th entry always equals 0.

‘unilateral rule’, an undirected link is formed if and only if at least one side proposes to the other. Formally, $G_{ij} = G_{ji} = S_{ij} + S_{ji} - S_{ij} \cdot S_{ji}$. The interpretation is that agents may unilaterally form undirected links with others. Note that the issue of transforming proposals into links only arises when links are undirected, as when links are directed it is straightforward to set $G_{ij} = S_{ij}$ and $G_{ji} = S_{ji}$. For concreteness, we assume a bilateral rule throughout the theoretical discussion (the unilateral rule is discussed in Appendix B).

Given network G , agent i ’s utility is given by:

$$u_i(X, G; \theta_0) = \sum_{j \neq i} G_{ij} \cdot (v_{ij}(X, G_{-i}; \theta_0) + \epsilon_{ij}) \quad (2)$$

where G_{-i} indicates G with the i^{th} row and column deleted (‘leave-own-out network’), and $\theta_0 \in \Theta$ is a $[p \times 1]$ vector of parameters from a compact set Θ . Estimating the parameters in θ_0 is the goal of the procedure described in Section 3.

Assumption 3 (Linearity, Separability and Anonymity). *The $v_{ij}(\cdot)$ function: (i) is linear in θ_0 i.e. can be written in the form $Z_{ij}\theta_0$, where Z_{ij} is a $[1 \times p]$ covariates vector satisfying $\|Z_{ij}\| < \bar{Z} < \infty$ for all $i, j \in N$ (with $\|\cdot\|$ denoting the Euclidean norm); (ii) depends on G only through G_{-i} ; (iii) is insensitive to permutations of agents’ labels.*

The most restrictive part of Assumption 3 is condition (ii) (“Separability”), which is borrowed from Leung (2015) and requires i ’s marginal utility from a link with j to be independent from i ’s other links. We discuss it further in Remark 3.

2.2 Equilibrium

Let i ’s (pure) strategy be a function from commonly observed attributes and privately observed shocks to an action: $S_i : (X, \epsilon_i) \rightarrow \{0, 1\}^n$ (henceforth we omit the dependency on X). Letting ϵ_{-i} and $S_{-i}(\epsilon_{-i})$ denote the collection of shocks vectors and strategies (respectively) of agents other than i , a Bayes Nash Equilibrium (BNE) is defined as a strategy profile

$[S_i(\epsilon_i), S_{-i}(\epsilon_{-i})]$ such that for all $i \in N$ and for all $S'_i(\epsilon_i)$:

$$\mathbb{E}_{\epsilon_{-i}} [u_i(X, G[S_i(\epsilon_i), S_{-i}(\epsilon_{-i})]; \theta_0)] \geq \mathbb{E}_{\epsilon_{-i}} [u_i(X, G[S'_i(\epsilon_i), S_{-i}(\epsilon_{-i})]; \theta_0)] \quad (3)$$

The notation $\mathbb{E}_{\epsilon_{-i}}[\cdot]$ is used to stress that the expectation is taken with respect to the randomness in ϵ_{-i} . Due to the separability assumption, in any BNE agents consider proposal decisions separately.¹¹ Hence, we can write $S_i(X, \epsilon_i) = [S_{ij}(X, \epsilon_{ij})]_{j \in N}$, where $S_{ij} : (X, \epsilon_{ij}) \rightarrow \{0, 1\}$. In addition, in any BNE, S_{ij} must prescribe i to propose to j whenever it strictly increases her expected utility and not to propose whenever it strictly reduces it. Formally:

$$S_{ij}(\epsilon_{ij}) = \begin{cases} 1 & \text{if } \mathbb{E}_{\epsilon_{ji}}[S_{ji}(\epsilon_{ji})] \cdot (\mathbb{E}_{\epsilon_{-i}}[v_{ij}(X, G_{-i}[S_{-i}(\epsilon_{-i})]; \theta_0)] + \epsilon_{ij}) > 0 \\ 0 & \text{if } \mathbb{E}_{\epsilon_{ji}}[S_{ji}(\epsilon_{ji})] \cdot (\mathbb{E}_{\epsilon_{-i}}[v_{ij}(X, G_{-i}[S_{-i}(\epsilon_{-i})]; \theta_0)] + \epsilon_{ij}) < 0 \end{cases} \quad (4)$$

Whenever proposing to j does not change i 's expected utility, proposing and not proposing are both best-replies. This shows that Bayes Nash equilibria do not exclude coordination failures. For instance, a pair $S_{ij}(\epsilon_{ij})$ and $S_{ji}(\epsilon_{ji})$ that prescribe i and j (respectively) not to propose for any ϵ_{ij} and ϵ_{ji} (respectively) may well be part of a BNE profile, *even if* both i and j stand to gain (in expectation) from forming a link. Since we are interested in modeling bilateral network formation, where pairs of agents are free to coordinate their actions, we wish to rule out such equilibria. We do so by restricting attention to *admissible* Bayes Nash equilibria, i.e. equilibria where no player uses a (weakly) dominated strategy. In any *admissible* BNE, S_{ij} must prescribe i to propose to j whenever, *assuming j proposes to i* , her expected utility from proposing is strictly positive, and not to propose if it is strictly

¹¹Conditional independence of linking decisions is a common tractability assumption in dyadic link formation models (Fafchamps and Gubert, 2005; Comola and Prina, 2021).

negative. Formally:

$$S_{ij}(\epsilon_{ij}) = \begin{cases} 1 & \text{if } \mathbb{E}_{\epsilon_{-i}}[v_{ij}(X, G_{-i}[S_{-i}(\epsilon_{-i})]; \theta_0)] + \epsilon_{ij} > 0 \\ 0 & \text{if } \mathbb{E}_{\epsilon_{-i}}[v_{ij}(X, G_{-i}[S_{-i}(\epsilon_{-i})]; \theta_0)] + \epsilon_{ij} < 0 \end{cases} \quad (5)$$

Given this decision rule, one may reformulate the equilibrium condition in terms of beliefs over proposal probabilities. To that end, let σ^{S-i} be a $[(n-1) \times n]$ matrix representing i 's beliefs about the probabilities that each agent $j \neq i$ proposes to another agent $k \neq j$ (including i herself). Given the decision rule in Equation (5), and letting Φ denote the CDF of the standard normal distribution, the ex-ante probability that i proposes to j is:

$$Pr(S_{ij} = 1 \mid X, \sigma^{S-i}) = Pr(\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid X, \sigma^{S-i}] + \epsilon_{ij} > 0) \quad (6)$$

$$= \Phi(\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid X, \sigma^{S-i}]) \quad (7)$$

Note that since ϵ_{ij} is drawn from a continuous distribution, it makes no difference whether i 's strategy prescribes proposing or not when $\mathbb{E}_{\epsilon_{-i}}[v_{ij}(X, G_{-i}[S_{-i}(\epsilon_{-i})]; \theta_0)] + \epsilon_{ij}$ is exactly zero. A belief matrix σ^S corresponds to an admissible BNE if and only if it satisfies the following equality for all i and j :

$$\sigma_{ij}^{S-i} = Pr(S_{ij} = 1 \mid X, \sigma^{S-i}) \quad (8)$$

The fact that $v_{ij}(\cdot)$ depends on G_{-i} , rather than S_{-i} allows conditioning its expected value on beliefs over *linking* probabilities rather than proposal probabilities. In addition, due to Assumption 2, the probability that a link exists is simply the product of the proposal probabilities of the two parties involved. This allows reformulating the equilibrium condition in terms of beliefs over linking probabilities. To that end, let σ^G denote a $[n \times n]$ matrix representing agents' common beliefs about linking probabilities among all pairs of agents, and σ^{G-i} denote the same matrix but with its i^{th} row and column deleted. A belief matrix

σ^G corresponds to an admissible BNE if and only if it satisfies the condition below for all i and j . We call such σ^G an “equilibrium belief”.

$$\sigma_{ij}^G = \underbrace{\Phi \left(\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid X, \sigma^{G_{-i}}] \right)}_{Pr(i \text{ proposes to } j)} \underbrace{\Phi \left(\mathbb{E}[v_{ji}(X, G_{-j}; \theta_0) \mid X, \sigma^{G_{-j}}] \right)}_{Pr(j \text{ proposes to } i)} \quad (9)$$

Given an equilibrium belief σ^G , a network G is said to be an “equilibrium” if the following holds for all i and j :

$$G_{ij} = \underbrace{\mathbb{1} \left\{ \mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid X, \sigma^{G_{-i}}] + \epsilon_{ij} > 0 \right\}}_{i \text{ proposes to } j} \underbrace{\mathbb{1} \left\{ \mathbb{E}[v_{ji}(X, G_{-j}; \theta_0) \mid X, \sigma^{G_{-j}}] + \epsilon_{ji} > 0 \right\}}_{j \text{ proposes to } i} \quad (10)$$

Note that due to admissibility, an equilibrium network G is one that satisfies the pairwise stability conditions *in expectation*: (i) if i and j are linked in G then the marginal expected utilities this link provides each agent is positive; (ii) if i and j are not linked in G then the marginal expected utility this link provides is negative for at least one of them. Hence, even though the solution concept we deploy is non-cooperative, in equilibrium no pair of players fails to coordinate on forming a link.

Following [Leung \(2015\)](#), from here on we restrict attention to symmetric equilibria. A symmetric equilibrium is an equilibrium in which all pairs of agents that are observationally equivalent have the same linking probabilities. Formally, an equilibrium σ^G is symmetric if for all $i, j \neq k, l \in N$:

$$(X_i = X_k \text{ and } X_j = X_l) \text{ or } (X_i = X_l \text{ and } X_j = X_k) \implies \sigma_{ij}^G = \sigma_{kl}^G \quad (11)$$

Figure 1 illustrates this definition. Agents in this network have a single binary attribute – being either black or white – depicted by the colors of the nodes. Beliefs are depicted by edges’ labels and their values by their color (i.e. all red beliefs equal each other, and all blue beliefs equal each other). All pairs consisting of two black agents have the same σ^G value

(red), and the same holds for pairs of white and black agents (blue) and pairs of two white agents (green). The described beliefs are therefore symmetric.

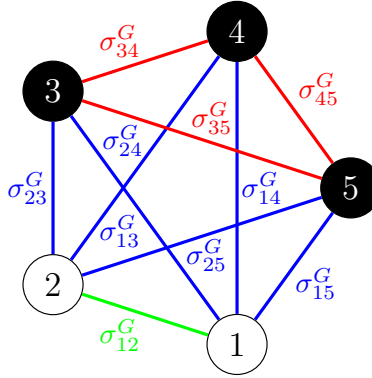


Figure 1: Example of a symmetric belief matrix

For given X and θ_0 , we let $\omega(X, \theta_0)$ denote the set of admissible and symmetric BNE. Proposition 1 establishes that $\omega(X, \theta_0)$ is non-empty.

Proposition 1 (Existence). *Under assumptions 1-3, there exists an admissible and symmetric BNE, i.e. $\omega(X, \theta_0) \neq \emptyset$.*

2.3 Example

Consider the case where 3 agents have one binary attribute X_i , and their utility function is as follows:

$$v_{ij}(X, G_{-i}; \theta_0) = \theta_1 + \theta_2 X_i + \theta_3 |X_i - X_j| + \theta_4 \frac{1}{n-1} \sum_{k \neq i} G_{jk} \quad (12)$$

with $\theta_0 = [-1, 1, -0.5, 1]'$. The term $|X_i - X_j|$ represents a measure of similarity between i and j . It thus accounts for homophily. The term $\frac{1}{n-1} \sum_{k \neq i} G_{jk}$ represents the (normalized) number of indirect connections (i.e. paths of length 2) that i gains by forming a link with j . It thus reflects externalities from the network topology.

Columns 1 and 2 in Table 1 present all possible ordered pairs in the 3-agent network. Columns 3 and 4 report the binary attributes of agents i and j respectively. Column 5

reports $|X_i - X_j|$. The third term in the utility function $\frac{1}{n-1} \sum_{k \neq i} G_{jk}$ depends on the network structure G . Its expected value therefore depends on the beliefs about the network structure σ^G .

Let us consider the set of beliefs reported in column 6. Column 7 uses these beliefs to compute $\frac{1}{n-1} \sum_{k \neq i} \sigma_{jk}^{G-i}$. Using θ_0 , columns 3, 5 and 7 and the functional form, we can now compute the expected value of v_{ij} for all pairs of agents. This is reported in column 8. Now, given that the ϵ_{ij} values are drawn independently from the standard normal distribution, the probability that i proposes to j (that is, that $\mathbb{E}[v_{ij}] + \epsilon_{ij} \geq 0$) is $\Phi(\mathbb{E}[v_{ij}])$. This is reported in column 9. Finally, the probability that a link exists in G is the product of the proposal probabilities of the two agents involved. This is reported in column 10.

1	2	3	4	5	6	7	8	9	10
i	j	X_i	X_j	$ X_i - X_j $	σ^G	$\frac{1}{n-1} \sum_{k \neq i} \sigma_{jk}^{G-i}$	$\mathbb{E}[v_{ij}]$	$\Phi(\mathbb{E}[v_{ij}])$	$\Phi(\mathbb{E}[v_{ij}]) \cdot \Phi(\mathbb{E}[v_{ji}])$
1	2	0	1	1	0.027	$0.5 \cdot 0.255$	-1.3725	0.0850	0.027
1	3	0	1	1	0.027	$0.5 \cdot 0.255$	-1.3725	0.0850	0.027
2	1	1	0	1	0.027	$0.5 \cdot 0.027$	-0.4865	0.3133	0.027
2	3	1	1	0	0.255	$0.5 \cdot 0.027$	0.0135	0.5054	0.255
3	1	1	0	1	0.027	$0.5 \cdot 0.027$	-0.4865	0.3133	0.027
3	2	1	1	0	0.255	$0.5 \cdot 0.027$	0.0135	0.5054	0.255

Table 1: Example

In this example, the condition $\sigma_{ij}^G = \Phi(\mathbb{E}[v_{ij}])\Phi(\mathbb{E}[v_{ji}])$ holds for all i and $j \neq i$. This means that the beliefs σ^G in column 6 are equilibrium beliefs. The condition that observationally equivalent pairs of agents (e.g. $\{1, 2\}$ and $\{1, 3\}$) have the same linking probabilities also holds. This means that the beliefs σ^G are symmetric.

2.4 Equilibrium Selection Mechanism

Since linking decisions are based on beliefs and the ϵ values are independent from one another, a given σ^G implies a unique probability distribution over networks. In particular, the probability that a given network G is formed under the belief matrix σ^G is given by the product of the probabilities of the linking statuses among all pairs of agents under that belief

matrix:

$$P(G|X, \sigma^G) = \prod_{i,j>i}^n \left[\left(P(G_{ij} = 1|X, \sigma^{G-i}) \right)^{G_{ij}} \cdot \left(P(G_{ij} = 0|X, \sigma^{G-i}) \right)^{1-G_{ij}} \right] \quad (13)$$

where $P(G_{ij} = 1|X, \sigma^{G-i})$ is given by the right hand side of equation 9. Conditional on beliefs the likelihood is therefore well defined. That is, it is a function that provides a unique output for any input (X, θ_0, σ^G) . For an econometric model to be “complete”, however, it is required that the likelihood be well defined *without conditioning on beliefs* (Tamer, 2003). If it were the case that for any X and θ_0 our model would have implied a *unique* admissible and symmetric BNE, then assuming it is played in the data would have sufficed to yield a complete model. As our model does not necessarily admit uniqueness of equilibria,¹² we now complete it by specifying an equilibrium selection mechanism.

Following Leung (2015), the idea is to assume the existence of a (randomly generated) publicly observed signal which all players use to coordinate on one particular admissible and symmetric BNE. Formally, the random variable corresponding to the public signal is denoted ν and defined as a random vector of arbitrary finite dimension such that $(X, \nu) \perp \epsilon$. The equilibrium selection mechanism is then a measurable function λ mapping (X, ν, θ_0) to $\omega(X, \theta_0)$.

Assumption 4 (Equilibrium Selection Mechanism). *There exist ν and λ defined as above such that:*

$$P(G|X) = \sum_{\sigma^G \in \omega(X, \theta_0)} P(\lambda(X, \nu, \theta_0) = \sigma^G|X) \cdot \prod_{i,j>i}^n \left[\left(P(G_{ij} = 1|X, \sigma^{G-i}) \right)^{G_{ij}} \times \left(P(G_{ij} = 0|X, \sigma^{G-i}) \right)^{1-G_{ij}} \right] \quad (14)$$

The randomness in ν implies that the mechanism may select multiple elements in $\omega(X, \theta_0)$ with strictly positive probabilities, even for a fixed X . An alternative, more substantial, as-

¹²In the example in Subsection 2.3 with $\theta_0 = [-1, 0, 0, 7]$, for instance, there exists a symmetric equilibrium in which $\sigma_{ij} = 0.037$ for all i and $j \neq i$, as well as one in which $\sigma_{ij} = 0.986$ for all i and $j \neq i$.

sumption that is commonly made when the econometrician observes many repetitions of the game (“many-market asymptotics”) is that the probability distribution over equilibria is degenerate. This guarantees that the equilibrium being played in all repetitions of the game is the same. As in [Leung \(2015\)](#), we are able to avoid this assumption and achieve point identification in a large-network case (“large market asymptotics”) thanks to the assumption that only *symmetric* equilibria are allowed to be selected. This will become clear in [Subsection 3.2](#), where we discuss ways to recover network-dependent regressors from the data.

3 Estimation

Suppose we observe a single network G , agents’ attributes X ,¹³ and that G is formed according to the model specified above, that is, the network results from all agents behaving optimally given the symmetric equilibrium belief σ^G and their realization of the error terms ϵ_i that we do not observe. Our goal is to estimate and conduct inference on the true parameter vector θ_0 . We now describe the estimation procedure.

3.1 Log-likelihood function

Let us denote by δ_{ij} a function that takes X_i, X_j and returns a vector of covariates of dimension $[1 \times (p - k)]$ (e.g. i ’s attributes and the distance between i and j ’s attributes, as in the example above). Denote by γ_{ij} a function that takes an adjacency matrix (possibly together with X) and returns a vector of covariates of dimension $[1 \times k]$ (e.g. the number of length-two paths i gains from linking with j , as in the example above). To facilitate an intercept, assume that δ_{ij} always returns 1 as a first element. We call the first type of covariates ‘exogenous’ as they do not depend on the network structure, and the second type

¹³Measurement error in the network topology is an important, yet largely unexplored issue that goes beyond the scope of this paper ([De Paula, 2017](#); [Advani and Malde, 2018](#); [Bramoullé, Djebbari, and Fortin, 2020](#)). Our estimator relies on the assumption that the network is measured in an accurate and complete manner, like other methods do ([Leung, 2015](#); [De Paula et al., 2018](#)).

‘endogenous’, as they do. Using this terminology, while $\gamma_{ij}(X, G_{-i})$ represents the endogenous covariates associated with i ’s linking with j , $\gamma_{ij}(X, \sigma^{G_{-i}})$ represents their conditional expectation. Whenever we write γ_{ij} (without specifying the input argument explicitly), it is taken to represent the latter.

By Assumption 3, $v_{ij}(\cdot)$ is a linear function of the exogenous and endogenous covariates:

$$v_{ij}(X, G_{-i}; \theta_0) = [\delta_{ij}, \gamma_{ij}(X, G_{-i})] \cdot \theta_0 \quad (15)$$

The expected value of v_{ij} conditional on X and the event that σ^G is the selected equilibrium is therefore:

$$\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid X, \sigma^{G_{-i}}] = [\delta_{ij}, \gamma_{ij}] \cdot \theta_0 \quad (16)$$

Equation (9) can thus be rewritten as:

$$P(G_{ij} = 1 \mid X, \sigma^G) = \Phi([\delta_{ij}, \gamma_{ij}]\theta_0) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta_0) \quad (17)$$

Since $\{\epsilon_{ij} \mid i, j \in N, i \neq j\}$ are drawn independently from one another, conditional on X and the event that σ^G is selected, the likelihood of observing a network G is:

$$L(\gamma, \theta) = \prod_{i,j>i}^n \left[\left(\Phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \right)^{G_{ij}} \times \left(1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \right)^{1-G_{ij}} \right] \quad (18)$$

where γ denotes the set of γ_{ij} for all i, j . Taking the log of this expression and dividing by

the number of observations we obtain the log-likelihood function:

$$l(\gamma, \theta) = \frac{2}{n(n-1)} \sum_{i,j>i}^n \left[\left(G_{ij} \cdot \log \left(\Phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \right) \right) \right. \\ \left. + \left((1 - G_{ij}) \cdot \log \left(1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \right) \right) \right] \quad (19)$$

Since γ is unobserved, we cannot directly proceed to maximize this function with respect to θ . Instead, we follow a two-step procedure:

(Step 1) Obtain an estimate of γ , denoted $\hat{\gamma}$ (Subsection 3.2);

(Step 2) Maximize $l(\hat{\gamma}, \theta)$ to obtain $\hat{\theta}$ (Subsection 3.3).

Remark 1. *If we rule out endogenous covariates, the model boils down to a bivariate probit with partial observability (Poirier, 1980), where a link is observed if and only if both partially-observed latent response variables representing proposals are positive.^{14,15} Comola and Fafchamps (2014) were the first to apply the partial observability framework to undirected network formation games without externalities, which correspond to a special case of our model.*

Remark 2. *Under uniqueness of equilibria, resorting to recovering γ from the data is not strictly necessary. Instead, one could find the unique equilibrium beliefs for any candidate θ that is being considered by the optimization algorithm and evaluate the endogenous covariates and log-likelihood function at these beliefs.¹⁶*

¹⁴As Poirier (1980) points out, identification in partial observability models may fail when the functional form of the latent equations is unrestricted and exogenous regressors have limited support (e.g., are binary). In our setting, imposing that the parameter vector θ is identical across the two equations resolves the under-identification issue.

¹⁵Note that in our setting the two latent response variables are partially observed, but the equilibrium link is observed accurately. This stands in contrast with situations where links are measured with error (Chandrasekhar and Lewis, 2012; Candelaria and Ura, 2023; Thirkettle, 2019).

¹⁶Under multiplicity, one could in principle calculate all equilibria for a candidate θ and compare their likelihood values, but this approach could be difficult to implement (Aguirregabiria and Mira, 2007).

3.2 Estimating Endogenous Regressors

Following Leung (2015), we estimate γ_{ij} by the empirical average of $\gamma_{kl}(X, G_{-k})$ among pairs of agents that are observationally equivalent to (i, j) . Formally:

$$\hat{\gamma}_{ij} \equiv \frac{\sum_{k,l \in N} \left[\gamma_{kl}(X, G_{-k}) \cdot \mathbb{1}\{(X_i = X_k \wedge X_j = X_l)\} \right]}{\sum_{k,l \in N} \left[\mathbb{1}\{(X_i = X_k \wedge X_j = X_l)\} \right]} \quad (20)$$

The consistency of this estimator generally depends on the endogenous covariates included in γ_{ij} . From here on we restrict attention to covariates of two specific forms.

Assumption 5. $\gamma_{ij}(\cdot)$ includes only covariates of the following forms:

$$\gamma_{ij}^1(X, G_{-i}) \equiv \frac{1}{n-1} \sum_{k \neq i} G_{jk} \cdot \mu(X_k) \quad (21)$$

$$\gamma_{ij}^2(X, G_{-i}) \equiv \frac{1}{(n-1)(n-2)} \sum_{k \neq i} \sum_{l \neq i} G_{jk} G_{jl} G_{kl} \quad (22)$$

$\gamma_{ij}^1(X, G_{-i})$ represents the fraction of agents $k \neq i$ which are friends with j , where each friendship is allowed to be weighted by some function of k 's attributes $\mu(X_k)$. $\gamma_{ij}^2(X, G_{-i})$ represents the fraction of agents $k, l \neq i$ which form triangles with j , capturing a measure of j 's clustering. These two statistics are natural and complementary candidates to capture the local network structure of the potential partner computed on the leave-one-out network: depending on the application, γ_{ij}^1 may proxy for benefits from 2-steps away friends or the amount of attention/resources j could devote to i (which has an opposite expected effect); and γ_{ij}^2 may proxy for trust, cohesion, or redundancy of information.

Proposition 2 (First stage). *Under assumptions 1-4, $\sup_{i,j \in N} |\hat{\gamma}_{ij}^z(X, \sigma^{G_{-i}}) - \gamma_{ij}^z(X, \sigma^{G_{-i}})| \xrightarrow{p} 0$ for both $z = 1$ and $z = 2$.*

Remark 3. *Uniform consistency of other endogenous regressors could potentially be established if they are appropriately normalized and satisfy “separability”, implying that it may*

depend on all walks in G besides those that pass through i . Two centrality measures of this type are “information centrality” (Stephenson and Zelen, 1989) and “targeting centrality” (Bramoullé and Genicot, 2024).^{17,18}

Beyond network statistics that satisfy separability “by nature”, it is possible to construct a separable counterpart to non-separable ones by computing them on the “leave-own-out” network. For instance, for any centrality measure $c_i(G)$ one could define its separable counterpart as $\tilde{c}_i(G) = \sum_{j \neq i} G_{ij} \cdot c_i(G_{-i} + ij)$. As an illustration, suppose $c_i(G)$ denotes i ’s diffusion centrality (Banerjee et al., 2013), then the interpretation of $\tilde{c}_i(G)$ is that i diffuses the message in period 1 and then never retransmits it again.

Lastly, we remark that our assumptions on $v_{ij}(\cdot)$ (Assumption 3) still allows capturing non-linear effects of (separable) endogenous statistics. For instance, including a quadratic transformation of γ_{ij}^1 could be used to test for decreasing marginal returns from j ’s friends. In other words, while separability limits the complexity of the statistics one could consider, there is still a lot to be gained by considering complex effects of “simple” statistics.

Remark 4. The first stage estimator described above leverages symmetry to rely on information from a single network. This broadens its applicability, since many network datasets depict a single network (Goyal, Van der Leij, and Moraga Gonzalez, 2006; Mele, 2017).¹⁹

Remark 5. Since the first stage estimator essentially divides the set of observations into bins of identical pairs of agents, we risk not having enough observations within each bin when the sample size is small, the number of attributes is high, and their support is large. This can be addressed by smoothing the agents’ attributes, discussed in Appendix A.

¹⁷ Information centrality assign weights for every path emanating from i and sum those weights up, such that $\gamma_{ij}^1(X, G_{-i})$ corresponds to the special case where the weights on paths of length three and above are set to zero. Brandes and Fleischer (2005) show that information centrality is equivalent to current-flow closeness centrality.

¹⁸Targeting centrality is defined through a diffusion process where messages spread in discrete time: each informed agent transmits to her friends with probability p . If a message is directed to agent j , her targeting centrality is the expected number of times she receives it, under the assumption that she never re-transmits.

¹⁹ Our estimation procedure also carries over to the case of multiple networks, as shown in Appendix B.

3.3 Estimating Preferences

Once $\hat{\gamma}$ is computed, plugging it into Equation (19) and maximizing with respect to θ yields our estimate $\hat{\theta}$ of θ_0 . The propositions below state its consistency and asymptotic normality.

Proposition 3 (Consistency). *Under assumptions 1-5 and standard regularity conditions, $\hat{\theta}$ is consistent for θ_0 .*

Proposition 4 (Asymptotic Normality). *Let I_p denote the $[p \times p]$ identity matrix and $\Lambda \equiv V^{-1}\Psi V^{-1}$ (see Equations 92 and 112 for their definitions). Under assumptions 1-5:*

$$\sqrt{\frac{1}{2}n(n-1)}\Lambda^{-1/2}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I_p) \quad (23)$$

4 Simulation

We now present a simulation exercise designed to evaluate the asymptotic performance of the estimator in networks of increasing size (from $n = 40$ to $n = 350$). We first describe the data generating process, then report the estimation results.

4.1 Data Generating Process

For a given number of agents n with a two-dimensional attribute vector X_i , we posit a data generating process of the form:

$$X_i^1 \sim U\{0, 1\} \quad (24)$$

$$X_i^2 \sim U\{0, 1, \dots, 9\} \quad (25)$$

$$\epsilon_{ij} \sim N(0, 1) \quad (26)$$

$$v_{ij} = \theta_1 + \theta_2 X_i^1 + \theta_3 X_i^2 + \theta_4 \mathbb{1}\{X_i^1 = X_j^1\} + \theta_5 |X_i^2 - X_j^2| + \theta_6 \gamma_{ij}^1(X, G_{-i}) + \theta_7 \gamma_{ij}^2(G_{-i}) \quad (27)$$

$$\theta_0 = [-2.8, 1, 0.5, 1, -0.1, 1]' \quad (28)$$

with the weights in $\gamma_{ij}^1(X, G_{-i})$ set to 1 for all attribute vectors (i.e. no weighting). θ_0 is set so that the utility function is not dominated by its deterministic component, i.e. so that proposal decisions are sensitive to ϵ_{ij} .

The data generating process consists of three steps. First, draw the attributes X_i^1 and X_i^2 for all i . Second, find a corresponding symmetric equilibrium σ^G . This step uses an algorithm that starts from a randomly drawn belief matrix, computes the corresponding linking probabilities, and updates beliefs accordingly until convergence is achieved (see Algorithm 1).²⁰ Third, draw the ϵ_{ij} values and construct a network G such that a link exists if and only if the realization of ϵ_{ij} and ϵ_{ji} are such that $\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) | X, \sigma^{G_{-i}}] + \epsilon_{ij} > 0$ and $\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) | X, \sigma^{G_{-i}}] + \epsilon_{ji} > 0$.

Algorithm 1: Search Algorithm

- 1 Generate a random belief matrix σ^G
 - 2 Calculate the matrix of linking probabilities L , given σ^G , X and θ_0 :
 - 3 $L_{ij} = L_{ji} = \Phi(\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) | X, \sigma^{G_{-i}}]) \cdot \Phi(\mathbb{E}[v_{ji}(X, G_{-j}; \theta_0) | X, \sigma^{G_{-j}}])$
 - 4 If $\sigma^G \neq L$:
 - 5 Re-assign $\sigma^G = L$ and go back to line 2
 - 6 Else:
 - 7 Return σ^G
-

For each $n \in \{40, 150, 350\}$ we generate 300 networks according to the procedure above. The networks that result from this process exhibit many commonly observed characteristics of real-world networks: the average geodesic distance between connected agents is low (≈ 2); the clustering coefficient is high compared to the linking probability of a comparable Poisson random network (≈ 0.38 vs. ≈ 0.17); and the degree distribution is positively skewed. The

²⁰While the algorithm is not guaranteed to converge, whenever it does, it converges to a symmetric equilibrium (see Rabinovich, Naroditskiy, Gerding, and Jennings (2013)). In practice, we declare “convergence” when the distance between all elements of the beliefs matrix (σ^G) and all corresponding elements of the linking probabilities matrix (denoted L in Algorithm 1) falls below some “convergence error”, set to $1 \cdot 10^{-6}$. This criterion was satisfied in all iterations of the simulation exercise reported here.

average degrees are approximately 6.9, 26.6 and 61.6 for $n \in \{40, 150, 350\}$ respectively.

4.2 Simulation Results

In the estimation step, for each simulation draw we use the realized network G and the agents' attributes X (but not the error terms and beliefs) to estimate $\gamma(X, \sigma^G)$ (as explained in Section 3.2). We then maximize Equation (19) by replacing γ with $\hat{\gamma}$ to obtain $\hat{\theta}$.

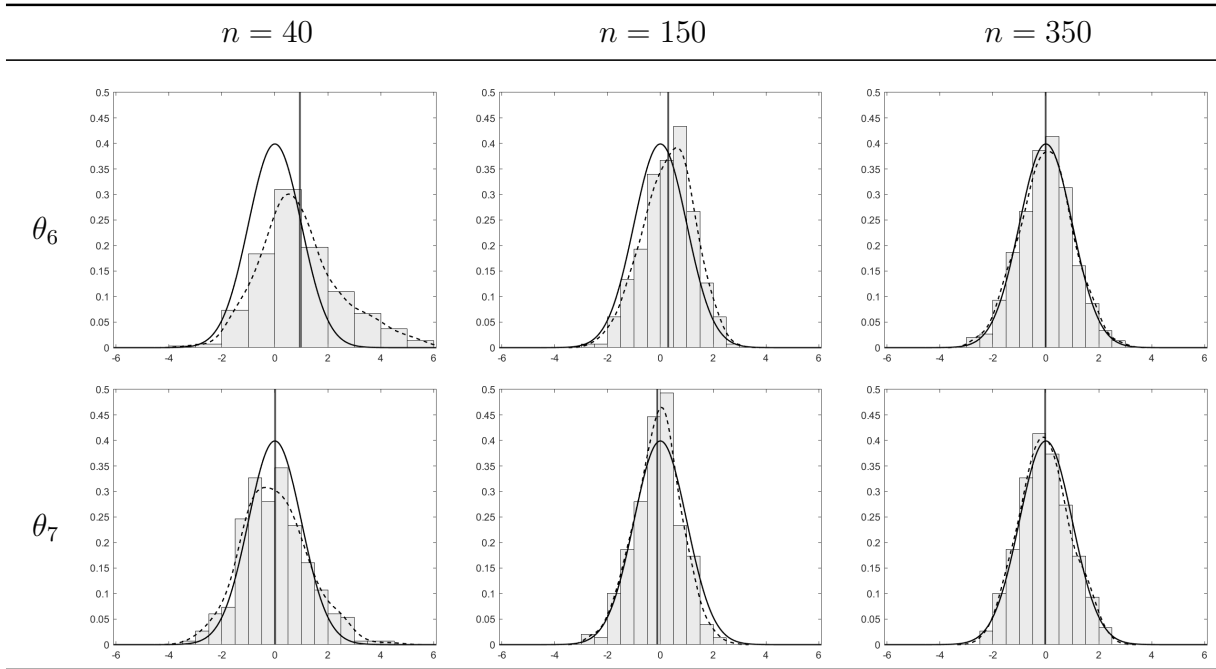


Table 2: Simulation results

Note: The table reports histograms of $\sqrt{\frac{1}{2}n(n-1)\Lambda^{-1/2}}(\hat{\theta} - \theta_0)$ against true standard normal distributions. The vertical lines in each sub-figure represents the mean value of the (centered and normalized) estimated coefficients across all iterations.

Table 2 presents density histograms of $\sqrt{\frac{1}{2}n(n-1)\Lambda^{-1/2}}(\hat{\theta} - \theta_0)$ for the estimated coefficients corresponding to the two endogenous regressors. The vertical lines show the means of the estimated coefficients across all iterations. The dashed curves are smoothed estimations of the histograms. The full curves are true standard normal distributions. As implied by Proposition 4, we see that as n increases the dashed curves converge to the full curves. At $n = 350$, the fit becomes very tight.

5 Empirical Illustration

5.1 Data Description

We use data on the risk sharing network of Nyakatoke, a small village in the Buboka rural district of Tanzania.²¹ Rural villages are an appropriate setting to study network formation, because the population can be entirely surveyed and several confounding effects (such as spatial and informational barriers) can be reasonably ruled out. The village of Nyakatoke consists of 119 households which have been interviewed in five regular intervals from February to December 2000. The data contain a rich set of information on households' demographics, wealth, income sources and income shocks, transfers and risk-sharing links.²²

During the first survey round all respondents were asked *'Can you give a list of people from inside or outside of Nyakatoke, who you can personally rely on for help and/or that can rely on you for help in cash, kind or labour?'*²³ The phrasing of this survey question was intended to capture undirected links of mutual assistance, and qualitative interviews and pilot tests suggested that respondents have understood it that way.²⁴ In what follows, we assume that these survey responses represent undirected bilateral agreements of mutual help which could be activated if one of the partners is struck by an income shock. This is in line both with the survey design and with theoretical work on the voluntary nature of risk-sharing arrangements (Bloch et al., 2008; Jackson et al., 2012).²⁵ In Appendix B we

²¹These data have been the object of numerous articles (De Weerd and Dercon, 2006; De Weerd and Fafchamps, 2011; Vandenbossche and Demuynck, 2013; Comola and Fafchamps, 2014).

²²At the time of the study, the village is isolated, densely inhabited, and relatively poor (consumption for adult equivalent is less than 2 US\$ a week, and most of the income comes from agricultural activities).

²³Respondents could list as many names as they wanted. They could also mention partners who live outside the village (this occurs in 34% of all declared partners). Since we have no information on the attributes of households outside the village we omit them from the analysis.

²⁴This phrasing was first piloted in the Philippines (Fafchamps and Lund, 2003) and subsequently adopted in the Nyakatoke survey, because respondents understand it and are willing to answer. Other survey questions on directed flows were tried during the pilots, for instance drawing a distinction between people which respondents would help and people which respondents would seek help from. But respondents were confused by this distinction, which they perceived as non-existent, and complained they are asked the same question twice. See also Comola and Fafchamps (2014).

²⁵In case of discordant reports, we assume that an undirected link exists whenever it is declared by at least one of the households involved. This is the most common stand in the empirical literature on risk-sharing links, and it is equivalent to assuming that all observed discordances are due to under-reporting.

revisit the trust network data analysed by Leung (2015), and show that our method could accommodate different network generation processes and yield different conclusions.

The resulting risk-sharing network of Nyakatoke consists of 490 links among $(119 \cdot 118)/2 = 7021$ household dyads. This network displays a mean geodesic distance of 2.5 steps and an average degree of 8.2. No household is isolated, and the network exhibits all the empirical regularities of large social networks.²⁶

5.2 Main Results

We now illustrate the estimation procedure described in Section 3 using the Nyakatoke data. We take the household as a unit of observation ($n = 119$) and include as covariates: a constant, three types of homophily regressors, and two types of endogenous regressors. The homophily regressors are binary variables that take the value 1 if i and j belong to the same family,²⁷ same clan,²⁸ or same religion²⁹. These exogenous covariates were identified by previous literature as strong predictors of risk-sharing link formation in developing countries. The endogenous regressors γ_{ij}^1 and γ_{ij}^2 are defined in accordance with equations (21) and (22), and represent the percentage of agents who are friends with j and the percentage of triangles around j , respectively.³⁰

The results are reported in Table 3. Column 1 presents a specification without endogenous regressors, for reference. Column 2 includes the endogenous regressors γ_{ij}^1 and γ_{ij}^2 , and Column 3 presents the marginal effects corresponding to the specification in column 2. Standard errors are computed according the expression given in Proposition 4 with the true

²⁶The Nyakatoke network has a unique component covering the entire population, the diameter is in the order of $\log(n)$ and the clustering coefficient is 7 times larger than in a randomly generated Poisson network with similar characteristics.

²⁷Two households i and j are said to belong to the same family if there is some blood relation between at least one of the members of i and at least one of the members of j .

²⁸There are 26 clans in Nyakatoke. 10 of them have only one household.

²⁹There are three religions in Nyakatoke: Roman Catholic (49 households), Lutheran (46 households) and Muslim (24 households).

³⁰To improve readability we omit the weight $\mu(X_k)$ for γ_{ij}^1 and we rescale both regressor into percentages. Thus, these regressors are defined as $\gamma_{ij}^1 = \frac{100}{n-1} \sum_{k \neq i} G_{jk}$ (percentage of agents who are friend with j) and $\gamma_{ij}^2 = \frac{100}{(n-1)(n-2)} \sum_{k \neq i} \sum_{l \neq i} G_{jk} G_{jl} G_{kl}$ (percentage of triangles around j).

parameters replaced by their estimates.

	coefficients		mfxf
	(1)	(2)	(3)
Same family	0.93*** (0.06)	0.94*** (0.06)	0.33*** (0.03)
Same clan	0.15*** (0.05)	0.14** (0.06)	0.04** (0.02)
Same religion	0.15*** (0.04)	0.19*** (0.04)	0.06*** (0.01)
γ_{ij}^1		0.19*** (0.03)	0.05*** (0.01)
γ_{ij}^2		-0.64 (1.07)	-0.17 (0.29)
Constant	-0.81*** (0.03)	-2.11*** (0.13)	
# observations	7021	7021	

Notes: Column 3 reports the marginal effects for column 2. Standard errors in parentheses. *p<10%, **p<5%, and ***p<1%

Table 3: Estimated coefficients.

Table 3 provides evidence for the existence of network externalities. The coefficient of γ_{ij}^1 is positive and significant, suggesting that agents benefit from having a financial partner with many other partners.³¹ For the average pair i and j , an increment of one percentage point in the expected number of j 's friends is associated with an increase of 5 percentage points in i 's probability of proposing to j . The second endogenous regressor γ_{ij}^2 , which represent the percentage of triangles around j , does not appear significant. The signs of the other estimates conform with our expectations. The constant appears negative, reflecting the idea that maintaining links is costly. The coefficients of the homophily regressors are all positive, in line with the large evidence that similarity between agents makes them more desirable to each other.

In Appendix B we illustrate the use of our estimation protocol in the context of self-

³¹ In the context of risk-sharing arrangements, the externalities from friends of friends can be in principle positive or negative. A friend with many other friends may dilute her attention and resources, but on the other hand she is likely to display greater financial resilience in case of need.

reported (and possibly discordant) network data. In particular, we modify our estimator to accommodate for a unilateral link formation rule, and we show that it yields different results from the directed unilateral estimator by [Leung \(2015\)](#).

6 Concluding remarks

Data on network interactions are becoming more available to economists. The recent enthusiasm for data from digital interaction platforms ([Vosoughi, Roy, and Aral, 2018](#); [Blumenstock, 2018](#)) has refueled the research interest about how non-digital links are formed, and how they respond to strategic incentives. Models of link formation with network externalities are at the frontier of the econometric research, facing difficulties related to dimensionality and equilibria multiplicity ([Graham, 2015](#); [Chandrasekhar, 2016](#); [De Paula, 2017](#)). Our paper fills a void in the literature by proposing a method to estimate network externalities in a simultaneous-move game of undirected link formation. This method is naturally suited for bilateral link formation models, but can also be applied to unilateral models where only the undirected link outcome (rather than the proposals) is observed. We provide existence, consistency and asymptotic normality results for the proposed estimator and test its asymptotic performance through a simulation exercise. In the context of bilateral link formation, this procedure provides a simpler alternative to methods exploiting pairwise stability under complete information ([De Paula et al., 2018](#); [Sheng, 2020](#)). Importantly, it allows to make inference about various aspects of agents' preferences over network topology when data on a single (and possibly large) network are available. For instance, our method could be paired with data issued from a randomized experiment, allowing the researcher to disentangle endogenous network externalities from other exogenous factors (e.g., agents randomly allocated treatment status).³²

We illustrate the method using data on risk-sharing in a Tanzanian village named Nyaka-

³²The assumption that the attributes of others are observable suits well the case of a medium-sized community where randomization is implemented through a public lottery.

token. Risk-sharing links are commonly assumed to be mutually agreed upon and provide an intriguing case for the role of externalities from indirect connections. Results confirm that the network architecture has an explanatory value: households seem to take into consideration the share of indirect friends they stand to gain when making linking decisions.

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Appendices

A Smoothing

A practical concern that may arise with respect to the first stage estimator is that in a finite sample the number of observations with identical attributes may be too small to allow for a meaningful estimation. This happens in particular when the sample size is small, the number of attributes is high and their support is large. [Li and Racine \(2007\)](#) suggest overcoming this problem by smoothing the variables. To define a “smoothed version” of our first stage estimator let X_i^s be the s th component of the X_i vector and define the following functions:

$$t_s(X_i^s, X_j^s, X_k^s, X_l^s) \equiv \begin{cases} 0 & \text{if } (X_i^s = X_k^s \wedge X_j^s = X_l^s) \\ 1 & \text{otherwise} \end{cases} \quad (29)$$

$$T(X_i^d, X_j^d, X_k^d, X_l^d, \lambda) \equiv \prod_s \lambda^{t_s(X_i^d, X_j^d, X_k^d, X_l^d, \lambda)} \quad (30)$$

where $\lambda \in [0, 1]$. The smoothed first stage estimator (for any endogenous variable γ_{ij}) is then defined as follows:

$$\hat{\gamma}_{ij}^G \equiv \frac{\sum_{k,l \in N} \gamma_{kl}(X, G_{-k}) \cdot T(X_i^d, X_j^d, X_k^d, X_l^d, \lambda)}{\sum_{k,l \in N} T(X_i^d, X_j^d, X_k^d, X_l^d, \lambda)} \quad (31)$$

Note that when $\lambda = 0$, $T(X_i^d, X_j^d, X_k^d, X_l^d, 0)$ takes the value 1 if ij and kl are observationally equivalent and 0 otherwise. (31) therefore reduces to the original estimator (20) and no smoothing occurs. On the other extreme, when $\lambda = 1$, $T(X_i^d, X_j^d, X_k^d, X_l^d, 1) = 1$ for all ij and kl , meaning that attributes are completely smoothed out. $\lambda \in (0, 1)$ corresponds to intermediate levels of smoothing.

B Auxiliary results

In what follows we provide an empirical demonstration using the Indian trust network data in Leung (2015), which originally appeared in Banerjee et al. (2013) and Jackson et al. (2012). We revisit Leung (2015)’s application by fitting and comparing three different link formation models with externalities, and show that they reach different conclusions. In particular, we compare our undirected bilateral model of Section 3 with two other estimation protocols: the first one (‘undirected unilateral model’) is a modification of our partial-observability protocol where link formation is assumed to be unilateral. The second (‘directed unilateral model’) is the model by Leung (2015) where survey reports are interpreted as directed proposals. These two alternative protocols are formally introduced below.

Undirected unilateral model

Our partial-observability framework could be modified to accommodate undirected links issued from a process of unilateral link formation. Given an undirected network G , if we believe that the underlying link formation process is unilateral, we may want to estimate a model where a link *does not* exist if and only if both agents *do not* propose to each other, that is:

$$G_{ij} = S_{ij} + S_{ji} - S_{ij} \cdot S_{ji}. \quad (32)$$

This implies rewriting Equation (17) as:

$$P(G_{ij} = 0 \mid X, \sigma^G) = [1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta_0)] \cdot [1 - \Phi([\delta_{ji}, \gamma_{ji}]\theta_0)] \quad (33)$$

and maximizing the associated log-likelihood function. This model is similar to the bilateral model discussed in Section 3 in that they both assume links to be undirected, and that link proposals are not directly observed.

Directed unilateral model

The model of unilateral link formation proposed by [Leung \(2015\)](#) maximizes the following log-likelihood function:

$$l(\gamma, \theta) = \frac{1}{n(n-1)} \sum_{i,j \neq i}^n \left[Z_{ij} \cdot \log \left(\Phi([\delta_{ij}, \gamma_{ij}]\theta) \right) + (1 - Z_{ij}) \cdot \log \left(1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta) \right) \right] \quad (34)$$

where Z denotes a directed network and γ is estimated in the first stage by [\(20\)](#). The standard errors are computed as described in the proof of [Proposition 4](#), with the log-likelihood function replaced by the one above.

The Indian trust data

The data used for this illustration were originally collected in 2006 across 75 villages in the Karnataka region of India and was intended to study how microfinance diffuses through village networks ([Banerjee et al., 2013](#)). It contains a comprehensive sets of individual and household characteristics, and information on a wide range of social interactions for a subset of sampled individuals (the overall sampling rate across villages is slightly below 50%).

Following [Leung \(2015\)](#) we restrict the analysis to a subset of 9 villages with non-negligible religious heterogeneity, and we identify social ties on the basis of the survey question ‘*Whom do you trust enough that if he or she needed to borrow Rs. 50 for a day you would lend it to him or her?*’. Differently from the case of Nyakatoke, this question is phrased in a directed manner and it was answered accordingly (e.g., i ’s report on link ij are not expected to coincide with j ’s). This gives us a suitable framework to discuss a setting where the interpretation of network data is ultimately at the discretion of the researcher depending on the characteristics and knowledge of the data application ([Ready and Power, 2021](#)).

In the first scenario, the researcher may believe that these survey reports depict undi-

rected links. This is indeed the case if she interprets the responses as proxies for latent social relationships rather than indicators of actual monetary transfers. She then chooses to build an undirected network matrix G based on these reports, and implicitly impute all discordance to mis-reporting.³³ This is the stand we take to estimate our undirected models, assuming that an undirected link exists whenever it is declared by at least one of the two parties involved. This is the most common stand in the applied network literature, and it is equivalent to assuming that all discordances are due to under-reporting. Alternative approaches, however, are possible and are discussed below. In the second scenario, the researcher may believe that survey responses depict directed links. For the survey question above this is a maintainable assumption if she is interested in the network of money flows across surveyed villagers. Thus, a directed link-formation model à la [Leung \(2015\)](#) is estimated, and self-reported links are interpreted as proposals.

Results

In [Table 4](#) below we reproduce the estimated coefficients for the second-stage endogenous regressors defined in analogy with the [Table 3](#) in [Leung \(2015\)](#). The first two columns report estimates from an undirected model when link formation is assumed bilateral (column 1) and unilateral (column 2) respectively. The third column reports estimates from the directed model by [Leung \(2015\)](#). The endogenous regressors of interest are: number of j 's friends, caste of j 's friends, religion of j 's friends (all rescaled by village size n).³⁴ The rest of the estimation procedure replicates the original paper.³⁵

Results from [Table 4](#) show that different models produce qualitatively different results.

³³Note that partial observability is conceptually distinct from measurement error related to mis-reporting, because all estimation methods presented here assume that the observed network is well measured (see footnote 12).

³⁴These endogenous regressors are suitable both for directed and undirected models. Note, however, that they take different forms depending on the model. Columns (1)-(2) include degree statistics computed on the undirected network while Column (3) include out-degree statistics (as in [Leung \(2015\)](#)). Our estimates in column (3) are not directly comparable to the ones in [Table \(3\)](#) of [Leung \(2015\)](#) because of an error in his original code, subsequently amended.

³⁵First-stage regressors include: age, gender, religion, caste, head of household, language. Second-stage homophylous regressors (non-reported) include: same religion, gender, language, caste, family.

Table 4: Application to the data used in [Leung \(2015\)](#)

	(1)	(2)	(3)
Model	undirected bilateral: $S_{ij}S_{ji}$	undirected unilateral: $S_{ij} + S_{ji} - S_{ij}S_{ji}$	directed unilateral: S_{ij}
number of j 's friends	52.4109*** (8.1185)	43.2457*** (5.7632)	26.0504*** (6.9725)
caste of j 's friends	-13.6293*** (5.6137)	-9.6498** (4.9925)	-11.5319*** (5.0957)
religion of j 's friends	-20.6551*** (9.3352)	-13.4883** (7.2454)	-2.8915 (7.5424)
Constant	-2.5287*** (0.1407)	-4.1715*** (0.1381)	-3.8536*** (0.0735)
# households	2031	2031	2031
# dyads	246345	246345	492690

Notes: Standard errors in parentheses. *p<10%, **p<5%, and ***p<1%.

Although the sign of the estimates remains constant across specifications, their magnitude and significance differ. Overall, we remark that the estimate for the number of j 's friends is significant across all specifications. However, its magnitude is comparable in the two partial-observability models (columns 1 and 2), while the estimate of column 3 is much smaller in size (i.e. approximately half). Unsurprisingly, results from the undirected unilateral model of column (2) lie between those of the undirected bilateral and the directed unilateral models. In fact, columns (2) and (3) both assume links are issued from a unilateral process. However, the undirected model in column (2) assumes that non-existing links are due to the refusal of both sides involved, while the directed model in column (3) assumes that proposals are observed. The estimate for caste of j 's friends is relatively stable across specifications, while friends' religion is much smaller and not significant in column (3).

Discussion

Administrative sources (e.g. registers of commercial or financial transactions, traceable interactions on digital platforms) usually contain one single link measurement per dyad, which

pave the way for the estimation of undirected models. The illustration above focuses on the (frequent) intermediate situation where link data contain two distinct reports per dyad, and the choice of the link formation model is somewhat discretionary based on the phrasing on the survey question and the data at hand. While the directed model may be attractive (because the estimation procedure is computationally simpler), our comparison shows that it may come at the cost of drawing different conclusions. This underscores our point that directed and undirected models are inherently different. While the interpretation of the Indian trust data remains an open question, our exercise confirms the insight from network theory that unilateral and bilateral link formation rules have profound implications on the resulting network structure and aggregate outcomes that can be achieved (Bala and Goyal, 2000; Charness and Jackson, 2007).

Lastly, if link data contain two distinct reports and the researcher fits an undirected model, assumptions must be made about the kind of misreporting driving measurement error.³⁶ If we denote R the directed matrix of binary reports (such that $R_{ij} = 1$ if i reports having a link with j), we frequently observe discrepancies of the form $R_{ij} \neq R_{ji}$. Such discrepancies are the rule rather than the exception even when link data depict supposedly mutual relationships such as risk-sharing, goods exchanges or friendship (Comola and Fafchamps, 2014, 2017). To estimate an undirected model the reports contained in R serve to build the undirected network matrix G , but they are *not* interpreted as a measure of the unobserved proposals S (see Section 2). In this situation, the most common stand is to impute discrepancies to under-reporting, that is, to assume that an undirected link exists whenever it is declared by at least one of the parties involved. In practice, this means that for estimation purposes one sets $G_{ij} = \max\{R_{ij}, R_{ji}\}$. This is the approach we follow throughout the paper, both for the main illustration and in this appendix. However, there are alternative ways to deal with discrepancies. For instance, one could also estimate the bilateral model by assuming over-reporting (i.e. $G_{ij} = \min\{R_{ij}, R_{ji}\}$), or by assuming that

³⁶Misreporting in survey data is a general concern that can be imputed to different factors such as unintentional errors, intentional omissions, data aggregation mistakes.

over-reporting and under-reporting are equally likely, as in Comola and Fafchamps (2014).

C Proofs

C.1 Proposition 1

Proof. Denote by Σ the set of all σ^G matrices such that:

1. $\forall i, j \in N, \sigma_{ij}^G \in [0, 1]$
2. $\forall i \in N, \sigma_{ii}^G = 0$
3. $\forall i, j \in N, \sigma_{ij}^G = \sigma_{ji}^G$
4. $\forall i, j, k, l \in N, (X_i = X_k \wedge X_j = X_l) \vee (X_i = X_l \wedge X_j = X_k) \implies \sigma_{ij}^G = \sigma_{kl}^G$

Denote by $\Gamma(\cdot)$ the function that maps belief matrices to linking probabilities:

$$\Gamma_{ij}(\sigma^G) \equiv \begin{cases} \Phi(\mathbb{E}[v_{ij}(X, G_{-i}; \theta_0) \mid \sigma^{G_{-i}}]) \cdot \Phi(\mathbb{E}[v_{ji}(X, G_{-j}; \theta_0) \mid \sigma^{G_{-j}}]) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \quad (35)$$

By 9, an equilibrium is a fixed point of $\Gamma(\cdot)$, i.e. a σ^G such that for all $i, j \in N$:

$$\Gamma_{ij}(\sigma^G) = \sigma_{ij}^G \quad (36)$$

To prove that such σ^G exists we verify the conditions of Brouwer's fixed point theorem.

$\Gamma(\cdot)$ **maps from Σ to Σ .** First, since Γ_{ij} is either the product of two probabilities or 0, $\Gamma_{ij} \in [0, 1]$ for all i, j . Second, by definition $\Gamma_{ii} = 0$ for all i . Third, since Γ_{ij} depends symmetrically on the expected utility of i from a link with j and of j from a link with i , $\Gamma_{ij} = \Gamma_{ji}$ for all i, j . Fourth, for any two agents i and k such that $X_i = X_k$, condition 4 above implies that for any third agent $j \neq i, k$ the input matrix must satisfy $\sigma_{ij}^G = \sigma_{kj}^G$. By conditions 2

and **3**, $\sigma_{ii}^G = \sigma_{kk}^G$ and $\sigma_{ik}^G = \sigma_{ki}^G$. The i th and k th rows and columns in σ^G therefore contain the same elements, implying that σ^{G-i} and σ^{G-k} are identical up to a permutation of labels. Anonymity of $v_{ij}(\cdot)$ thus implies that $\Gamma_{ij} = \Gamma_{kj}$ for all i, j, k . Applying the same logic for an agent l such that $X_j = X_l$, we obtain also that $\Gamma_{ij} = \Gamma_{kl}$ for all i, j, k, l .

Γ is continuous in σ^G . The expected utilities are continuous in σ^G , and $\Phi(\cdot)$ is a continuous function. Therefore Γ is continuous in σ^G .

Σ is a convex subset of $[0, 1]^{n \times n}$. Since any linear combination of any two matrices in Σ yields a matrix in Σ , it is a convex set.

Σ is a compact subset of $[0, 1]^{n \times n}$. The sets of values that each entry in the matrices in Σ can obtain are bounded (by 0 and 1) and closed (for off-diagonal elements because the boundaries 0 and 1 are included and for diagonal elements because they are singletons). The Cartesian product of bounded and closed sets is also bounded and closed, so Σ is bounded and closed. By the Heine-Borel theorem it follows that Σ is compact.

The existence of a symmetric Bayesian equilibrium thus follows from Brouwer's fixed point theorem. □

C.2 Proposition 2

Proof. Let \mathcal{X} be the (finite) support of dyad attributes (X_i, X_j) . For any $x \in \mathcal{X}$, define the set of pairs of agents with attributes x :

$$\mathcal{M}(x) \equiv \{(k, l) \in N \times N \mid k \neq l, (X_k, X_l) = x\} \tag{37}$$

By Assumption 1, $|\mathcal{M}(x)|$ goes to infinity as n goes to infinity, for any $x \in \mathcal{X}$.

By Assumption 4, any pair of observationally equivalent pairs (i.e. within the same $\mathcal{M}(x)$) share the same conditional mean for any symmetric endogenous statistic. Formally, for any $z \in \{1, 2\}$, $x \in \mathcal{X}$ and $(i, j) \in \mathcal{M}(x)$:

$$\mathbb{E}[\gamma_{ij}^z(X, G_{-i}) \mid X, \sigma^{G_{-i}}] = \mathbb{E}[\gamma_{kl}^z(X, G_{-k}) \mid (X_k, X_l) = x, X, \sigma^{G_{-i}}] \quad (38)$$

Thus, for any $z \in \{1, 2\}$, $x \in \mathcal{X}$ and $(i, j) \in \mathcal{M}(x)$:

$$\hat{\gamma}_{ij}^z(X, \sigma^{G_{-i}}) - \gamma_{ij}^z(X, \sigma^{G_{-i}}) = \frac{\sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}^z(X, G_{-k})}{|\mathcal{M}(x)|} - \gamma_{ij}^z(X, \sigma^{G_{-i}}) \quad (39)$$

$$= \underbrace{\frac{\sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}^z(X, G_{-k}) - \gamma_{kl}^z(X, \sigma^{G_{-i}})}{|\mathcal{M}(x)|}}_{\Delta_n^z(x)} \quad (40)$$

So our goal is to show that $\sup_{x \in \mathcal{X}} |\Delta_n^z(x)| \xrightarrow{p} 0$ for both $z = 1$ and $z = 2$. We do so by expressing $\Delta_n^z(x)$ as the sum of conditionally independent random variables and showing that $\text{Var}[\Delta_n^z(x) \mid X, \sigma^G]$ converges to zero uniformly over $x \in \mathcal{X}$ (note that since $\gamma_{kl}^z(X, \sigma^{G_{-k}}) = \mathbb{E}[\gamma_{kl}^z(X, G_{-k}) \mid X, \sigma^{G_{-k}}]$, $\mathbb{E}[\Delta_n^z(x) \mid X, \sigma^G] = 0$ is already evident from (40)).

Case 1: $z = 1$.

First, rewrite $\gamma_{kl}^1(X, G_{-k})$ as follows:

$$\gamma_{kl}^1(X, G_{-k}) = \frac{1}{n-1} \sum_{m \in N} G_{lm} \mu(X_m) - \frac{1}{n-1} G_{lk} \mu(X_k) \quad (41)$$

Define $K_l(x) \equiv \{k \in N \setminus \{l\} \mid (X_k, X_l) = x\}$ and note that $\sum_{(k,l) \in \mathcal{M}(x)} (\cdot) = \sum_{l \in N} \sum_{k \in K_l(x)} (\cdot)$. Summing (41) over $(k, l) \in \mathcal{M}(x)$ (for some fixed $x \in \mathcal{X}$) therefore yields:

$$\begin{aligned} \sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}^1(X, G_{-k}) &= \sum_{l \in N} \sum_{k \in K_l(x)} \frac{1}{n-1} \sum_{m \in N} G_{lm} \mu(X_m) - \sum_{l \in N} \sum_{k \in K_l(x)} \frac{1}{n-1} G_{lk} \mu(X_k) \quad (42) \\ &= \sum_{l \in N} \sum_{m \in N} \frac{1}{n-1} G_{lm} \mu(X_m) |K_l(x)| \end{aligned}$$

$$- \sum_{l \in N} \sum_{m \in N} \frac{1}{n-1} G_{lm} \mu(X_m) \mathbb{1}\{(X_m, X_l) = x\} \quad (43)$$

$$= \sum_{l \in N} \sum_{m \in N} G_{lm} \mu(X_m) \frac{|K_l(x)| - \mathbb{1}\{(X_m, X_l) = x\}}{n-1} \quad (44)$$

Define:

$$Y_{lm}(x) \equiv (G_{lm} - \sigma_{lm}^G) \mu(X_m) \frac{|K_l(x)| - \mathbb{1}\{(X_m, X_l) = x\}}{n-1} \quad (45)$$

So that

$$\Delta_n^1(x) = \frac{\sum_{l \in N} \sum_{m \in N} Y_{lm}(x)}{|\mathcal{M}(x)|} \quad (46)$$

Since $Y_{lm}(x)$ terms are conditionally independent, we have:

$$\text{Var}[\Delta_n^1(x) \mid X, \sigma^G] = \frac{\sum_{l \in N} \sum_{m \in N} \text{Var}[Y_{lm}(x) \mid X, \sigma^G]}{|\mathcal{M}(x)|^2} \quad (47)$$

We now expand and bound $\text{Var}[Y_{lm}(x) \mid X, \sigma^G]$:

$$\text{Var}[Y_{lm}(x) \mid X, \sigma^G] = \mu^2(x) \left(\frac{|K_l(x)| - \mathbb{1}\{(X_m, X_l) = x\}}{n-1} \right)^2 \text{Var}[G_{lm} \mid X, \sigma^G] \quad (48)$$

Since X_m has finite support, $|\mu(X_m)|$ is uniformly bounded, i.e. there exists $\bar{\mu} < \infty$ such that $|\mu(X_m)| < \bar{\mu}$ for any $m \in N$. Since $0 \leq |K_l(x)| \leq n-1$, $\left| \frac{|K_l(x)| - \mathbb{1}\{(X_m, X_l) = x\}}{n-1} \right| \leq 1$. Since $G_{lm} \in \{0, 1\}$, $\text{Var}[G_{lm} \mid X, \sigma^G] \leq \frac{1}{4}$. Thus, letting $C = \bar{\mu}^2 \cdot 1^2 \cdot \frac{1}{4}$, we have:

$$\text{Var}[Y_{lm}(x) \mid X, \sigma^G] \leq C \quad (49)$$

Now consider $\text{Var}[\Delta_n^1(x) \mid X, \sigma^G]$ (line (47)). From (49), $\sum_{l \in N} \sum_{m \in N} \text{Var}[Y_{lm}(x) \mid X, \sigma^G] \leq n(n-1)C$. By Assumption 1, there exists $c > 0$ such that $|\mathcal{M}(x)| \geq n(n-1)c$ (for any

$x \in \mathcal{X}$). So:

$$\text{Var}[\Delta_n^1(x) \mid X, \sigma^G] \leq \frac{n(n-1)C}{(n(n-1)c)^2} \quad (50)$$

$$= \frac{C}{c^2} \cdot \frac{1}{n(n-1)} \quad (51)$$

which clearly converges to zero. Lastly, Chebyshev's inequality and finite support of \mathcal{X} give $\sup_{x \in \mathcal{X}} |\Delta_n^1(x)| \xrightarrow{p} 0$, as required.

Case 2: $z = 2$.

We follow the same proof structure as for $z = 1$. First, rewrite $\gamma_{kl}^2(X, G_{-k})$ as follows:

$$\gamma_{kl}^2(X, G_{-k}) = \frac{1}{(n-1)(n-2)} \left(\sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} - \sum_{a \in N} G_{la} G_{lk} G_{ak} - \sum_{b \in N} G_{lk} G_{lb} G_{kb} \right) \quad (52)$$

As in the $z = 1$ case, we define $K_l(x) \equiv \{k \in N \setminus \{l\} \mid (X_k, X_l) = x\}$ and note that $\sum_{(k,l) \in \mathcal{M}(x)} (\cdot) = \sum_{l \in N} \sum_{k \in K_l(x)} (\cdot)$. Summing (52) over $(k, l) \in \mathcal{M}(x)$ (for some fixed $x \in \mathcal{X}$) therefore yields:

$$\begin{aligned} \sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}^2(X, G_{-k}) &= \frac{1}{(n-1)(n-2)} \left(\sum_{l \in N} \sum_{k \in K_l(x)} \sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} \right. \\ &\quad \left. - \sum_{l \in N} \sum_{k \in K_l(x)} \sum_{a \in N} G_{la} G_{lk} G_{ak} - \sum_{l \in N} \sum_{k \in K_l(x)} \sum_{b \in N} G_{lk} G_{lb} G_{kb} \right) \quad (53) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{(n-1)(n-2)} \left(\sum_{l \in N} \sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} |K_l(x)| \right. \\ &\quad \left. - \sum_{l \in N} \sum_{b \in N} \sum_{a \in N} G_{la} G_{lb} G_{ab} \mathbb{1}\{(X_b, X_l) = x\} \right. \\ &\quad \left. - \sum_{l \in N} \sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} \mathbb{1}\{(X_a, X_l) = x\} \right) \quad (54) \end{aligned}$$

$$\begin{aligned} &= \sum_{l \in N} \sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} \\ &\quad \times \frac{|K_l(x)| - \mathbb{1}\{(X_a, X_l) = x\} - \mathbb{1}\{(X_b, X_l) = x\}}{(n-1)(n-2)} \quad (55) \end{aligned}$$

$$= \sum_{l \in N} \sum_{a \in N} \sum_{b \in N} G_{la} G_{lb} G_{ab} w_{lab}(x) \quad (56)$$

(Line (56) simply lets $w_{lab}(x)$ denote the weight that multiplies each $G_{la} G_{lb} G_{ab}$ in line (55).)

Define:

$$Y_{lab}(x) \equiv (G_{la} G_{lb} G_{ab} - \sigma_{la}^G \sigma_{lb}^G \sigma_{ab}^G) w_{lab}(x) \quad (57)$$

So that

$$\Delta_n^2(x) = \frac{\sum_{l,a,b \in N} Y_{lab}(x)}{|\mathcal{M}(x)|} \quad (58)$$

Unlike in the $z = 1$ case, the summands in this expression are not conditionally independent (since they share common links). The key is to decompose $\sum_{l,a,b \in N} Y_{lab}(x)$ into a sum of conditionally independent terms plus a remainder and deal with each separately.

Define $Q_{uv} = G_{uv} - \sigma_{uv}^G$. For any triplet (l, a, b) we have:

$$\begin{aligned} G_{la} G_{lb} G_{ab} - \sigma_{la}^G \sigma_{lb}^G \sigma_{ab}^G &= \underbrace{\sigma_{lb}^G \sigma_{ab}^G Q_{la} + \sigma_{la}^G \sigma_{ab}^G Q_{lb} + \sigma_{la}^G \sigma_{lb}^G Q_{ab}}_{\text{first order terms}} \\ &\quad + \underbrace{\sigma_{ab}^G Q_{la} Q_{lb} + \sigma_{lb}^G Q_{la} Q_{ab} + \sigma_{la}^G Q_{lb} Q_{ab} + Q_{la} Q_{lb} Q_{ab}}_{\text{higher order terms}} \end{aligned} \quad (59)$$

Thus:

$$\sum_{l,a,b \in N} Y_{lab}(x) = \underbrace{\sum_{l,a,b \in N} w_{lab}(x) [\text{first order terms}]}_{L_x} + \underbrace{\sum_{l,a,b \in N} w_{lab}(x) [\text{higher order terms}]}_{R_x} \quad (60)$$

So we can decompose (58) as follows:

$$\Delta_n^2(x) = \frac{\sum_{l,a,b \in N} Y_{lab}(x)}{|\mathcal{M}(x)|} = \frac{L_x}{|\mathcal{M}(x)|} + \frac{R_x}{|\mathcal{M}(x)|} \quad (61)$$

We will show that the first part is an average of independent terms, the second is negligible, and both converge to zero in probability uniformly over $x \in \mathcal{X}$. To see that L_x is a sum of conditionally independent terms note that it can be written as

$$L_x = \sum_{u,v \in N} A_{uv}(x) Q_{uv} \quad (62)$$

where

$$\begin{aligned} A_{uv}(x) &\equiv \sum_{l,a,b \in N} w_{lab}(x) [\mathbb{1}\{(u,v) = (l,a)\} \sigma_{lb}^G \sigma_{ab}^G + \mathbb{1}\{(u,v) = (l,b)\} \sigma_{la}^G \sigma_{ab}^G \\ &\quad + \mathbb{1}\{(u,v) = (a,b)\} \sigma_{la}^G \sigma_{lb}^G] \end{aligned} \quad (63)$$

$$= \sum_{b \in N} w_{uvb}(x) \sigma_{ub}^G \sigma_{vb}^G + \sum_{a \in N} w_{uav}(x) \sigma_{ua}^G \sigma_{av}^G + \sum_{l \in N} w_{luv}(x) \sigma_{lu}^G \sigma_{lv}^G \quad (64)$$

We can therefore compute and bound the variance of $\frac{L_x}{|\mathcal{M}(x)|}$ as follows:

$$\text{Var} \left(\frac{L_x}{|\mathcal{M}(x)|} \mid X, \sigma^G \right) = \frac{1}{|\mathcal{M}(x)|^2} \sum_{u,v \in N} A_{uv}(x)^2 \text{Var}(G_{uv} \mid X, \sigma^G) \quad (65)$$

Similarly to the $z = 1$ case, $0 \leq |K_l(x)| \leq n-1$ implies $|w_{lab}(x)| \leq \frac{1}{n-2}$. Using (64) and noting that $\sigma_{ii}^G = 0$ for all $i \in N$, we then get $|A_{uv}(x)| \leq 3$. We also have $\text{Var}(G_{uv} \mid X, \sigma^G) \leq \frac{1}{4}$. The numerator in (65) is therefore $O(n^2)$. Like in the $z = 1$ case, the denominator in $O(n^4)$. Thus, $\left| \frac{L_x}{|\mathcal{M}(x)|} \right| \xrightarrow{p} 0$ uniformly over $x \in \mathcal{X}$.

With respect to R_x , observe that while its summands are not necessarily conditionally independent, the covariances of all distinct terms are zero. $\text{Var}(R_x \mid X, \sigma^G)$ can therefore be written as the sum of variances of n^3 quadratic terms plus n^3 cubic terms (with appropriate weights). These variances are bounded by constants. Due to the normalization term $(n-1)(n-2)$, a similar argument as above implies that their weights are $O\left(\frac{1}{n}\right)$. When taken out of the variance operator they become $O\left(\frac{1}{n^2}\right)$. Thus $\text{Var}(R_x \mid X, \sigma^G) = O\left(\frac{1}{n^2}\right) O(n^3) = O(n)$. As argued before, the denominator of $\text{Var} \left(\frac{R_x}{|\mathcal{M}(x)|} \mid X, \sigma^G \right) = \frac{\text{Var}(R_x \mid X, \sigma^G)}{|\mathcal{M}(x)|^2}$, on the other

hand, is $O(n^4)$. We therefore have $\left| \frac{R_x}{|\mathcal{M}(x)|} \right| \xrightarrow{p} 0$ uniformly over $x \in \mathcal{X}$, as required.

Lastly, since the absolute values of both components of $\Delta_n^2(x)$ converge in probability to zero uniformly over $x \in \mathcal{X}$, so does $|\Delta_n^2(x)|$. □

C.3 Proposition 3

Proof. To prove that $\hat{\theta}$ is consistent for θ we verify the conditions of Theorem 2.1 in [Newey and McFadden \(1994\)](#).

$\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G]$ is **uniquely maximized at θ_0** . We establish this by showing that $\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G] - \mathbb{E}[l(\gamma, \theta_0) \mid X, \sigma^G] \leq 0$ for all $\theta \in \Theta$, and that the left hand side equals zero only at $\theta = \theta_0$.

$$\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G] - \mathbb{E}[l(\gamma, \theta_0) \mid X, \sigma^G] = \mathbb{E} \left[\frac{\log(L(\gamma, \theta))}{\frac{1}{2}n(n-1)} - \frac{\log(L(\gamma, \theta_0))}{\frac{1}{2}n(n-1)} \right] \quad (66)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \mathbb{E} [\log(L(\gamma, \theta)) - \log(L(\gamma, \theta_0))] \quad (67)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \mathbb{E} \left[\log \left(\frac{L(\gamma, \theta)}{L(\gamma, \theta_0)} \right) \right] \quad (68)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i}^n \left[\Phi([\delta_{ij}, \gamma_{ij}]\theta_0) \Phi([\delta_{ji}, \gamma_{ji}]\theta_0) \times \log \left(\frac{\Phi([\delta_{ij}, \gamma_{ij}]\theta) \Phi([\delta_{ji}, \gamma_{ji}]\theta)}{\Phi([\delta_{ij}, \gamma_{ij}]\theta_0) \Phi([\delta_{ji}, \gamma_{ji}]\theta_0)} \right) + \right. \quad (69)$$

$$\left. (1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta_0) \Phi([\delta_{ji}, \gamma_{ji}]\theta_0)) \times \right.$$

$$\left. \log \left(\frac{1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta) \Phi([\delta_{ji}, \gamma_{ji}]\theta)}{1 - \Phi([\delta_{ij}, \gamma_{ij}]\theta_0) \Phi([\delta_{ji}, \gamma_{ji}]\theta_0)} \right) \right]$$

$$\leq \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i}^n \left[\log(1) \right] \quad (70)$$

$$= 0 \quad (71)$$

where line 70 is obtained by applying Jensen's inequality.

This establishes that θ_0 maximizes $\mathbb{E}[l(\gamma, \theta)]$. It remains to show that it is its *unique* maximizer. To see that it is, consider line 69. Since $\Phi(\cdot)$ is strictly positive, the only way 69 would equal 0 is if the fractions inside the logs evaluate to 1, but this only happens when $\theta = \theta_0$. Thus, θ_0 is the unique maximizer of $\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G]$.

Θ is compact. True by assumption.

$\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G]$ is continuous in θ and $l(\hat{\gamma}, \theta)$ converges uniformly in probability to $\mathbb{E}[l(\gamma, \theta) \mid X, \sigma^G]$. We show that this is true by verifying the conditions of Lemma 2.4 in Newey and McFadden (1994). First, Θ is compact, by assumption. Second, $l(\gamma, \theta)$ is continuous in θ because $\Phi(\cdot)$ and $\log(\cdot)$ are continuous. Third, we need to show that there exists a function $d(G, \delta, \hat{\gamma})$ such that $|l(\hat{\gamma}, \theta)| \leq d(G, \delta, \hat{\gamma})$ and $\mathbb{E}[d(G, \delta, \hat{\gamma})] < \infty$. We start by considering the absolute value of the first part of the log-likelihood function:

$$|\log(\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\theta)\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\theta))| \quad (72)$$

$$= |\log(\Phi(0)\Phi(0)) + \frac{\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}([\delta_{ij}, \hat{\gamma}_{ij}]\theta - 0) + \frac{\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}([\delta_{ij}, \hat{\gamma}_{ij}]\theta - 0)| \quad (73)$$

$$\leq 2 + \frac{\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})} |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| + \frac{\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})} |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| \quad (74)$$

$$\leq 2 + (1 + \|[\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta}\|) |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| + (1 + \|[\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta}\|) |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| \quad (75)$$

$$\leq 2 + (1 + \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\tilde{\theta}\|) \cdot \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\theta\| + (1 + \|[\delta_{ji}, \hat{\gamma}_{ji}]\| \cdot \|\tilde{\theta}\|) \cdot \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\theta\| \quad (76)$$

Where line 73 is true by the mean value theorem (recall that the derivative of $\log(\Phi(v)\Phi(u))$ w.r.t v is $\frac{\phi(v)}{\Phi(v)}$ and w.r.t u is $\frac{\phi(u)}{\Phi(u)}$), line 74 is true by the triangular inequality, line 75 is true because $\frac{\phi(v)}{\Phi(v)} \leq 1 + |v|$ for all v , and line 76 is true by the Cauchy-Schwartz inequality.

Consider now the absolute value of the second part of the log-likelihood function:

$$|\log(1 - \Phi([\delta_{ij}, \hat{\gamma}_{ij}]\theta)\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\theta))| \quad (77)$$

$$\begin{aligned} &= |\log(1 - \Phi(0)\Phi(0)) + \frac{-\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})}{1 - \Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})}([\delta_{ij}, \hat{\gamma}_{ij}]\theta - 0) \\ &\quad + \frac{-\phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{1 - \Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}([\delta_{ji}, \hat{\gamma}_{ji}]\theta - 0)| \end{aligned} \quad (78)$$

$$\begin{aligned} &\leq 2 + \frac{\phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})}{1 - \Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})\Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})} \cdot |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| \\ &\quad + \frac{\phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})}{1 - \Phi([\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta})\Phi([\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta})} \cdot |[\delta_{ji}, \hat{\gamma}_{ji}]\theta| \end{aligned} \quad (79)$$

$$\leq 2 + (1 + |[\delta_{ij}, \hat{\gamma}_{ij}]\tilde{\theta}|) \cdot |[\delta_{ij}, \hat{\gamma}_{ij}]\theta| + (1 + |[\delta_{ji}, \hat{\gamma}_{ji}]\tilde{\theta}|) \cdot |[\delta_{ji}, \hat{\gamma}_{ji}]\theta| \quad (80)$$

$$\leq 2 + (1 + \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\tilde{\theta}\|) \cdot \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\theta\| + (1 + \|[\delta_{ji}, \hat{\gamma}_{ji}]\| \cdot \|\tilde{\theta}\|) \cdot \|[\delta_{ji}, \hat{\gamma}_{ji}]\| \cdot \|\theta\| \quad (81)$$

Where line 80 is true because $\frac{\phi(v)\Phi(u)}{1-\Phi(v)\Phi(u)} \leq \frac{\phi(v)}{1-\Phi(v)} = \frac{\phi(v)}{\Phi(-v)} \leq 1 + |v|$.

Letting $\theta_m = \sup_{\theta \in \Theta} \|\theta\|$, 76 and 81 imply that $|l(\theta, \hat{\sigma}^G)|$ is bounded from above by:

$$2 + (1 + \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\theta_m\|) \cdot \|[\delta_{ij}, \hat{\gamma}_{ij}]\| \cdot \|\theta_m\| + (1 + \|[\delta_{ji}, \hat{\gamma}_{ji}]\| \cdot \|\theta_m\|) \cdot \|[\delta_{ji}, \hat{\gamma}_{ji}]\| \cdot \|\theta_m\| \quad (82)$$

and a sufficient condition for the expected value of this function to be finite is that $\mathbb{E} [[\delta_{ij}, \hat{\gamma}_{ij}][\delta_{ij}, \hat{\gamma}_{ij}]']$ and $\mathbb{E} [[\delta_{ji}, \hat{\gamma}_{ji}][\delta_{ji}, \hat{\gamma}_{ji}]']$ exist and are finite. \square

C.4 Proposition 4

Proof. Denote the score of the log-likelihood by S and its ij th summand by S_{ij} :

$$S(\gamma, \theta) \equiv \nabla_{\theta} l(\gamma, \theta) \quad (83)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta) \quad (84)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} \frac{q_{ij}(G_{ij} - m_{ij})}{m_{ij}(1 - m_{ij})} \quad (85)$$

Where:

$$m_{ij} \equiv \Phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \quad (86)$$

$$q_{ij} \equiv \phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot [\delta_{ij}, \gamma_{ij}] \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) + \phi([\delta_{ji}, \gamma_{ji}]\theta) \cdot [\delta_{ji}, \gamma_{ji}] \cdot \Phi([\delta_{ij}, \gamma_{ij}]\theta) \quad (87)$$

Let $\hat{\gamma}$ denote the set of $\hat{\gamma}_{ij}$ for all i, j . By first order condition:

$$S(\hat{\gamma}, \hat{\theta}) = 0 \quad (88)$$

By the mean value theorem, there exists a θ^* between $\hat{\theta}$ and θ_0 such that:

$$S(\hat{\gamma}, \hat{\theta}) = S(\hat{\gamma}, \theta_0) + \nabla_{\theta} S(\hat{\gamma}, \theta^*)(\hat{\theta} - \theta_0) \quad (89)$$

Combining 88 and 89, and solving for $(\hat{\theta} - \theta_0)$ gives:

$$\hat{\theta} - \theta_0 = -(\nabla_{\theta} S(\hat{\gamma}, \theta^*))^{-1} S(\hat{\gamma}, \theta_0) \quad (90)$$

Since $\hat{\gamma}$ is uniformly consistent (Proposition 2), and given that θ^* lies between $\hat{\theta}$ and θ_0 , Lemma 3 in Leung (2015) implies:

$$\nabla_{\theta} S(\hat{\gamma}, \theta^*) - \mathbb{E}[\nabla_{\theta} S(\gamma, \theta_0) \mid X, \sigma^G] \xrightarrow{p} 0 \quad (91)$$

Denote the expected value of the hessian, by V and its ij th summand by V_{ij} :

$$V(\gamma, \theta) \equiv \mathbb{E}[\nabla_{\theta} S(\gamma, \theta) \mid X, \sigma^G] \quad (92)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} V_{ij}(\gamma_{ij}, \gamma_{ji}, \theta) \quad (93)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} \frac{-q_{ij}q'_{ij}}{m_{ij}(1-m_{ij})} \quad (94)$$

We can thus rewrite 90 as:

$$\hat{\theta} - \theta_0 = - (V(\gamma, \theta_0) + o_p(1))^{-1} S(\hat{\gamma}, \theta_0) \quad (95)$$

By adding and subtracting $\mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G]$ we obtain:

$$\hat{\theta} - \theta_0 = - (V(\gamma, \theta_0) + o_p(1))^{-1} \left(\underbrace{S(\hat{\gamma}, \theta_0) - \mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G]}_a + \underbrace{\mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G]}_b \right) \quad (96)$$

where the conditional expectations do not integrate over the randomness in $\hat{\gamma}$.³⁷ By a second order Taylor expansion of $S(\hat{\gamma}, \theta_0)$:

$$\begin{aligned} S(\hat{\gamma}, \theta_0) &= S(\gamma, \theta_0) + \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [\nabla_{\gamma_{ij}} S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ij} - \gamma_{ij}) \\ &\quad + \nabla_{\gamma_{ji}} S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ji} - \gamma_{ji})] + R \end{aligned} \quad (97)$$

where the remainder, R , is defined as:

$$\begin{aligned} R &= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [(\hat{\gamma}_{ij} - \gamma_{ij})' \cdot \nabla_{\gamma_{ij}\gamma_{ij}} S_{ij}(\tilde{\gamma}_{ij}, \tilde{\gamma}_{ji}, \theta_0) \cdot (\hat{\gamma}_{ij} - \gamma_{ij}) \\ &\quad + (\hat{\gamma}_{ij} - \gamma_{ij})' \cdot \nabla_{\gamma_{ij}\gamma_{ji}} S_{ij}(\tilde{\gamma}_{ij}, \tilde{\gamma}_{ji}, \theta_0) \cdot (\hat{\gamma}_{ji} - \gamma_{ji}) \\ &\quad + (\hat{\gamma}_{ji} - \gamma_{ji})' \cdot \nabla_{\gamma_{ji}\gamma_{ji}} S_{ij}(\tilde{\gamma}_{ij}, \tilde{\gamma}_{ji}, \theta_0) \cdot (\hat{\gamma}_{ji} - \gamma_{ji})] \end{aligned} \quad (98)$$

and $\tilde{\gamma}_{ij}$ lies between $\hat{\gamma}_{ij}$ and γ_{ij} for all $i, j \in N$. To see that R is $o_p(1)$ note that, by Proposition 2, $\hat{\gamma}_{ji}$ is consistent for γ_{ji} . Since $\tilde{\gamma}_{ij}$ is in between the two, it is also consistent for γ_{ji} . By continuity in $\tilde{\gamma}_{ij}$ and $\hat{\gamma}_{ij}$, each of the derivatives in (98) converge to their respective derivatives evaluated at $(\gamma_{ij}, \gamma_{ji}, \theta_0)$, which, by part (i) of Assumption 3, are $O_p(1)$. This establishes $R = o_p(1)$.

³⁷Said otherwise, if we let $f(\gamma) = \mathbb{E}[S(\gamma, \theta_0) \mid X, \sigma^G]$ for some fixed function γ , then $\mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G] = f(\hat{\gamma})$.

By a second-order Taylor expansion of $\mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G]$:

$$\begin{aligned} \mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G] &= \underbrace{\mathbb{E}[S(\gamma, \theta_0) \mid X, \sigma^G]}_0 + \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} \mathbb{E}[\nabla_{\gamma_{ij}} S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \\ &\quad \cdot (\hat{\gamma}_{ij} - \gamma_{ij}) + \nabla_{\gamma_{ji}} S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ji} - \gamma_{ji}) \mid X, \sigma^G] + o_p(1) \end{aligned} \quad (99)$$

and the remainder is $o_p(1)$ by the same argument as above. Since, by the law of large numbers, the middle part of 97 converges to the middle part of 99:

$$a = S(\gamma, \theta_0) + o_p(1) \quad (100)$$

Denote the expected value of $\nabla_{\gamma_{ij}} S(\gamma, \theta)$ by M and its ij th summand by M_{ij} :

$$M(\gamma, \theta) \equiv \mathbb{E}[\nabla_{\gamma_{ij}} S(\gamma, \theta) \mid X, \sigma^G] \quad (101)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta) \quad (102)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} \frac{-q_{ij} p'_{ij}}{m_{ij}(1-m_{ij})} \quad (103)$$

Where:

$$p_{ij} \equiv \phi([\delta_{ij}, \gamma_{ij}]\theta) \cdot \theta^\gamma \cdot \Phi([\delta_{ji}, \gamma_{ji}]\theta) \quad (104)$$

and θ^γ denotes the elements in θ that correspond to endogenous regressors γ .

Using this notation, we can rewrite 99 as:

$$\mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G] = \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ij} - \gamma_{ij}) \quad (105)$$

$$+ M_{ji}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ji} - \gamma_{ji})] + o_p(1)$$

$$= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j \neq i} [M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\hat{\gamma}_{ij} - \gamma_{ij})] + o_p(1) \quad (106)$$

By Lemma 1, we can replace $\hat{\gamma}_{ij}$ in 106 by $\gamma_{ij}(X, G_{-i})$, which we denote here by α_{ij} :

$$\begin{aligned} \mathbb{E}[S(\hat{\gamma}, \theta_0) \mid X, \sigma^G] &= \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ij} - \gamma_{ij}) \\ &\quad + M_{ji}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ji} - \gamma_{ji})] + o_p(1) \end{aligned} \quad (107)$$

We now rewrite 96 using our replacements for a and b :

$$\begin{aligned} \hat{\theta} - \theta_0 &= -(V(\gamma, \theta_0) + o_p(1))^{-1} \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \\ &\quad + M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ij} - \gamma_{ij}) + M_{ji}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ji} - \gamma_{ji}) + o_p(1)] \end{aligned} \quad (108)$$

Denote the ij th summand in this equation by W_{ij} :

$$W_{ij} \equiv S_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) + M_{ij}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ij} - \gamma_{ij}) + M_{ji}(\gamma_{ij}, \gamma_{ji}, \theta_0) \cdot (\alpha_{ji} - \gamma_{ji}) \quad (109)$$

Using this notation and multiplying through by $\sqrt{\frac{1}{2}n(n-1)}$:

$$\begin{aligned} \sqrt{\frac{1}{2}n(n-1)}(\hat{\theta} - \theta_0) &= -(V(\gamma, \theta_0) + o_p(1))^{-1} \\ &\quad \cdot \sqrt{\frac{1}{2}n(n-1)} \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} [W_{ij} + o_p(1)] \end{aligned} \quad (110)$$

Since the summands under the summation sign are conditionally independent (because conditional on X and σ^G , the variation in G_{ij} comes only from ϵ_{ij} and ϵ_{ji} , which are all assumed to be i.i.d.), we can now apply a central limit theorem:

$$\sqrt{\frac{1}{2}n(n-1)}[V^{-1}\Psi V^{-1}]^{-1/2}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I_p) \quad (111)$$

Where:

$$\Psi \equiv \frac{1}{\frac{1}{2}n(n-1)} \sum_{i,j>i} W_{ij}W'_{ij} \quad (112)$$

□

C.5 Lemma 1

Lemma 1. *For any specification of $\gamma_{ij}(\cdot)$ (regardless of Assumption 5):*

$$\sum_{i,j \in N} \hat{\gamma}_{ij} = \sum_{i,j \in N} \gamma_{ij}(X, G_{-i}) \quad (113)$$

Proof. Using the notation introduced in Proposition 2 and letting $x \in \mathcal{X}$ be such that $(i, j) \in \mathcal{M}(x)$ we can rewrite $\hat{\gamma}_{ij}$ as follows:

$$\hat{\gamma}_{ij} = \frac{\sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}(X, G_{-k})}{|\mathcal{M}(x)|} \quad (114)$$

Summing over $i, j \in N$, we get:

$$\sum_{i,j \in N} \hat{\gamma}_{ij} = \sum_{x \in \mathcal{X}} |\mathcal{M}(x)| \frac{\sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}(X, G_{-k})}{|\mathcal{M}(x)|} \quad (115)$$

$$= \sum_{x \in \mathcal{X}} \sum_{(k,l) \in \mathcal{M}(x)} \gamma_{kl}(X, G_{-k}) \quad (116)$$

$$= \sum_{i,j \in N} \gamma_{ij}(X, G_{-i}) \quad (117)$$

□